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Universally Prestarlike Functions of Complex Order

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Abstract

Universally prestarlike functions of order $\alpha \leq 1$ in the slit domain $\Lambda = \mathcal{C} \setminus [1, \infty)$ have been recently introduced by S. Ruscheweyh.This notion generalizes the corresponding one for functions in the unit disk Δ(and other circular domains in $\mathcal{C}$). In this paper, we introducing Universally prestarlike functions of complex order.The motivation of this paper is to give the Fekete-Szegö inequality and fractional derivative for such functions.

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1 Introduction

Let $H(\Omega)$ denote the set of all analytic functions defined in a domain Ω. For domain Ω containing the origin $H_0(\Omega)$ stands for the set of all function $f \in H(\Omega)$ with $f(0) = 1$. We also use the notation $H_1(\Omega) = \{zf : f \in H_0(\Omega)\}$. In the special case when Ω is the open unit disk $\Delta = \{z \in \mathcal{C} : |z| < 1\}$, we use the abbreviation H, H_0 and H_1 respectively for $H(\Omega), H_0(\Omega)$ and $H_1(\Omega)$. A function $f \in H_1$ is called starlike of order α with $(0 \le \alpha < 1)$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in \Delta) \tag{1}$$

and the set of all such functions is denoted by S_α. The convolution or Hadamard Product of two functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

A function $f \in H_1$ is called prestarlike of order α if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \in S_\alpha \tag{2}$$

The set of all such functions is denoted by $\mathcal{R}_\alpha$. The notion of prestarlike functions has been extended from the unit disk to other disk and half planes containing the origin by Ruscheweyh and Salinas[7-9]. Let Ω be one such disk or half plane.Then there are two unique parameters $\gamma \in \mathcal{C} \setminus \{0\}$ and $\rho \in [0, 1]$ such that

$$\Omega_{\gamma,\rho} = \{w_{\gamma,\rho}(z) : z \in \Delta\} \tag{3}$$

where, $w_{\gamma,\rho}(z) = \dfrac{\gamma z}{1 - \rho z}$. Note that $1 \notin \Omega_{\gamma,\rho}$ iff $|\gamma + \rho| \le 1$.

Definition 1.1 (7-9) *Let $\alpha \le 1$, and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ, ρ). A function $f \in H_1(\Omega_{\gamma,\rho})$ is called prestarlike of order α in $\Omega_{\gamma,\rho}$ if*

$$f_{\gamma,\rho}(z) = \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha. \tag{4}$$

The set of all such functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.

Let $F(z) = \sum_{k=0}^{\infty} a_k z^k = \int_0^1 \frac{d\mu(t)}{1 - tz}$ where $a_k = \int_0^1 t^k d\mu(t)$, $\mu(t)$ is a probability measure on $[0, 1]$. Let T denote the set of all such functions F which are analytic in the slit domain $\Lambda = \mathcal{C} \setminus [1, \infty)$.(the slit being along the positive real axis)

Definition 1.2 (9) *Let $\alpha \leq 1$. A function $f \in H_1(\Lambda)$ is called universally prestarlike of order α if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ with $|\gamma + \rho| \leq 1$. The set of all such functions is denoted by $\mathcal{R}^u_\alpha$.*

We define the class of universally prestarlike functions of complex order as follows.

Definition 1.3 *Let $\alpha \leq 1$ and $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ, which satisfies $\phi(0) = 1$, $\phi'(0) > 0$ and which maps the unit disc Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class $\mathcal{R}^u_{\alpha,b}(\phi)$ consists of all analytic function $f \in H_1(\Lambda)$ satisfying*

$$1 + \frac{1}{b}\left(\frac{D^{3-2\alpha}f(z)}{D^{2-2\alpha}f(z)} - 1\right) \prec \phi(z). \tag{5}$$

where $\prec$ denotes the subordination,where $(D^\beta f)(z) = \frac{z}{(1-z)^\beta} \star f$, for $\beta \geq 0$.In particular, for $\beta = n \in \mathrm{N}$. we have $D^{n+1}f = \frac{z}{n!}(z^{n-1}f)^{(n)}$.

Moreover,we let $\mathcal{R}^u_\alpha(A, B, b)(b \neq 0, complex)$ denote the class $\mathcal{R}^u_{\alpha,b}(\phi)$ where

$$\phi(z) = \frac{1 + Az}{1 + Bz} \qquad (-1 \leq B < A \leq 1).$$

The class $\mathcal{R}^u_\alpha(A, B, b)$ and therefore the class $\mathcal{R}^u_{\alpha,b}(\phi)$, specialize to several well-known classes of univalent functions for suitable choices of A,B and b.Some of the classes are listed below
1. $\mathcal{R}^u_{\frac{1}{2}}(1, -1, 1)$ is the class S^* of starlike univalent functions[1][2][4].
2. $\mathcal{R}^u_{\frac{1}{2}}(1, -1, b)$ is the class of starlike functions of complex order introduced by Wiatrowski[13].
3. $\mathcal{R}^u_{\frac{1}{2}}(1, -1, 1 - \beta), (0 \leq \beta < 1)$is the class of starlike functions of order β. This class was introduced by Robertson[6].

To Prove our main results, we need the following Theorem and Lemmas

Theorem 1.4 (9) *Let $0 \leq \alpha \leq 1$ and $f \in H_1(\Lambda)$. Then $f \in \mathcal{R}^u_\alpha$ if and only if*

$$\frac{D^{3-2\alpha}f}{D^{2-2\alpha}f} \in T. \tag{6}$$

This admits an explicit representation of the function in $\mathcal{R}^u_\alpha$. If $f \in H_0$ has all its Taylor coefficients at the origin different from zero we write $f^{(-1)}$ for the (possibly formal but) unique solution of $f * f^{(-1)} = \frac{1}{1-z}$.

Lemma 1.5 (3) *If $P_1(z) = 1 + c_1z + c_2z^2 + \dots$ is an analytic function with positive real part in Δ, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v + 2, & v \leq 0 \\ 2, & 0 \leq v \leq 1 \\ 4v + 2, & v \geq 1 \end{cases}$$

when $v < 0$, or $v > 1$, the equality holds if and only if $P_1(z)$is $\dfrac{1+z}{1-z}$ or one of its rotations.when $0 < v < 1$, then the equality holds if and only if $P_1(z)$is $\dfrac{1+z^2}{1-z^2}$ or one of its rotations. If $v = 0$, the equality holds if and only if $P_1(z) = \left(\dfrac{1}{2} + \dfrac{\lambda}{2}\right)\dfrac{1+z}{1-z} + \left(\dfrac{1}{2} - \dfrac{\lambda}{2}\right)\dfrac{1-z}{1+z}$. $0 \leq \lambda \leq 1$ or one of its rotations. If $v = 1$, the equality holds if and only if $P_1(z)$ is the reciprocal of one of the function for which the equality holds in the case of $v = 0$. Also the above upper bound can be improved as follows when $0 < v < 1$

$$|c_2 - vc_1^2| + v|c_1|^2 \leq 2, \qquad (0 < v \leq \frac{1}{2}).$$

$$|c_2 - vc_1^2| + (1 - v)|c_1|^2 \leq 2, \qquad (\frac{1}{2} < v \leq 1).$$

Lemma 1.6 (3) *If $P_1(z) = 1 + c_1z + c_2z^2 + \dots$ is an analytic function with positive real part in Δ, then $|c_2 - vc_1^2| \leq 2max\{1, |2v - 1|\}$ the inequality is sharp for the function $P_1(z) = \dfrac{1+z}{1-z}$.*

In this section, we obtain the Fekete-Szegö inequality for functions in the class $\mathcal{R}^u_{\alpha,b}(\phi)$.

2 Fekete-Szegö inequality

Theorem 2.1 *Let $\phi(z) = 1 + B_1z + B_2z^2 + \dots$, If $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \mathcal{R}^u_{\alpha,b}(\phi)$ then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{b}{3-2\alpha}\left[B_2 + (2-2\alpha)bB_1^2 - (3-2\alpha)bB_1^2\mu\right], & \mu \leq \sigma_1 \\ \\ \dfrac{bB_1}{3-2\alpha}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \\ \dfrac{b}{3-2\alpha}\left[-B_2 - (2-2\alpha)bB_1^2 + (3-2\alpha)bB_1^2\mu\right], & \mu \geq \sigma_2, \end{cases}$$

where $\sigma_1 = \dfrac{(B_2 - B_1) + (2-2\alpha)bB_1^2}{(3-2\alpha)bB_1^2}$, $\sigma_2 = \dfrac{(B_2 + B_1) + (2-2\alpha)bB_1^2}{(3-2\alpha)bB_1^2}$ *the result is sharp.*

P r o o f. If $f \in \mathcal{R}^u_{\alpha,b}$, then there is a schwartz function $w(z)$, analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$ in Δ such that $1 + \dfrac{1}{b}\left(\dfrac{D^{3-2\alpha}f}{D^{2-2\alpha}f} - 1\right) = \phi(w(z))$. Define the function $P_1(z)$ by $P_1(z) = \dfrac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \ldots$. Since $w(z)$ is a schwartz function, we see that $ReP_1(z) > 0$ and $P_1(0) = 1$. Define the function $P(z) = 1 + \dfrac{1}{b}\left(\dfrac{D^{3-2\alpha}f}{D^{2-2\alpha}f} - 1\right) = 1 + b_1 z + b_2 z^2 + \ldots$ Now, $P(z) = \phi\left(\dfrac{P_1(z)-1}{P_1(z)+1}\right)$.
where,

$$\frac{P_1(z)-1}{P_1(z)+1} = \frac{c_1 z + c_2 z^2 + \ldots}{2 + c_1 z + c_2 z^2 + \ldots} = \frac{1}{2}\left[c_1 z + z^2[c_2 - \frac{c_1^2}{2}] + z^3[c_3 - c_1 c_2 + \frac{c_1^3}{4}] + \ldots\right]$$

Hence,on simplification we get,

$$P(z) = 1 + \frac{B_1 c_1 z}{2} + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots$$

Therefore,

$$1 + b_1 z + b_2 z^2 + \ldots = 1 + \frac{B_1 c_1 z}{2} + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \ldots$$

Equating the like coefficients we get,

$$b_1 = \frac{B_1 c_1}{2} \tag{7}$$

$$b_2 = \frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4} \tag{8}$$

Now, $1 + \dfrac{1}{b}\left(\dfrac{D^{3-2\alpha}f}{D^{2-2\alpha}f} - 1\right)$ becomes,

$$1+\frac{1}{b}\left[\mathcal{C}'(\alpha,2)a_2 - \mathcal{C}(\alpha,2)a_2\right]z+\frac{1}{b}\left[\mathcal{C}'(\alpha,3)a_3 - \mathcal{C}(\alpha,2)\mathcal{C}'(\alpha,2)a_2^2 - \mathcal{C}(\alpha,3)a_3 + (\mathcal{C}(\alpha,2)a_2)^2\right]z^2+\ldots$$

$$= 1 + b_1 z + b_2 z^2 + \ldots$$

Equating the coefficients of 'z' and z^2 respectively and simplifying we get,

$$a_2 = bb_1 \quad ; \quad a_3 = \frac{bb_2 + (2-2\alpha)b_2 b_1^2}{3-2\alpha} \tag{9}$$

Applying the equations (7) and (8) in (9) we get,

$$a_2 = \frac{bB_1c_1}{2} \quad ; \quad a_3 = \frac{1}{3-2\alpha}\left[\frac{bB_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{bB_2c_1^2}{4} + (2-2\alpha)b_2\frac{B_1^2c_1^2}{4}\right].$$

Now,

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{3-2\alpha}\left[\frac{bB_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{bB_2c_1^2}{4} + (2-2\alpha)b_2\frac{B_1^2c_1^2}{4}\right] - \mu\frac{b_2B_1^2c_1^2}{4} \\ &= \frac{bB_1}{2(3-2\alpha)}\left[c_2 - c_1^2\left[\frac{1}{2} - \frac{B_2}{2B_1} - (2-2\alpha)\frac{bB_1}{2} + (3-2\alpha)\mu\frac{bB_1}{2}\right]\right] \\ &= \frac{bB_1}{2(3-2\alpha)}\left[c_2 - c_1^2 v\right]. \end{aligned}$$

where,

$$v = \left[\frac{1}{2} - \frac{B_2}{2B_1} - (2-2\alpha)\frac{bB_1}{2} + (3-2\alpha)\mu\frac{bB_1}{2}\right]$$

Now by an application of Lemma (1.1) if $\mu \le \sigma_1$

$$|a_3 - \mu a_2^2| \le \frac{b}{3-2\alpha}\left[B_2 + (2-2\alpha)bB_1^2 - (3-2\alpha)bB_1^2\mu\right]$$

Now, if $\sigma_1 \le \mu \le \sigma_2$

$$|a_3 - \mu a_2^2| \le \frac{bB_1}{3-2\alpha}.$$

Now, if $\mu \ge \sigma_2$

$$|a_3 - \mu a_2^2| \le \frac{b}{3-2\alpha}\left[-B_2 - (2-2\alpha)bB_1^2 + (3-2\alpha)bB_1^2\mu\right],$$

If $\mu = \sigma_1$, then the equality holds in the lemma (1.1) if and only if $P_1(z) = \left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z}$ $0 \le \lambda \le 1$ or one of its rotations. If $\mu = \sigma_2$, then $\frac{1}{P_1(z)} = \frac{1}{\left(\frac{1}{2} + \frac{\lambda}{2}\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{\lambda}{2}\right)\frac{1-z}{1+z}}$.

If $\sigma_1 < \mu < \sigma_2$ $P_1(z) = \frac{1+\lambda z^2}{1-\lambda z^2}$. To show that the bounds are sharp, we define the function $K_\alpha^{\phi_n}$ $(n = 2, 3, \dots)$ by

$$\frac{D^{3-2\alpha}K_\alpha^{\phi_n}}{D^{3-2\alpha}K_\alpha^{\phi_n}} = \phi(z^{n-1})$$

$K_\alpha^{\phi_n}(0)=0$, $(K_\alpha^{\phi_n})'(0)=1$ and function F_α^λ and G_α^λ $(0\le\lambda\le1)$ by

$$\frac{\left(D^{3-2\alpha}F_\alpha^\lambda\right)(z)}{\left(D^{2-2\alpha}F_\alpha^\lambda\right)(z)}=\phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right)$$

$F_\alpha^\lambda(0)=0$, $(F_\alpha^\lambda)'(0)=1$ and similarly

$$\frac{\left(D^{3-2\alpha}G_\alpha^\lambda\right)(z)}{\left(D^{2-2\alpha}G_\alpha^\lambda\right)(z)}=\phi\left(\frac{z(z+\lambda)}{1+\lambda z}\right)$$

$G_\alpha^\lambda(0)=0$, $(G_\alpha^\lambda)'(0)=1$. Clearly, the functions $K_\alpha^{\phi_n}, F_\alpha^\lambda, G_\alpha^\lambda\in\mathcal{R}_{\alpha,b}^u$. Also we write $K_\alpha^\phi:=K_\alpha^{\phi_2}$. If $\mu<\sigma_1$ or $\mu<\sigma_2$, then the equality holds if and only if f is K_α^ϕ or one of its rotations. When $\sigma_1<\mu<\sigma_2$, then the equality holds if and only if f is $K_\alpha^{\phi_3}$ or one of its rotations. If $\mu=\sigma_1$, then the equality holds if and only if f is F_α^λ or one of its rotations If $\mu=\sigma_2$ then the equality holds if and only if f is G_α^λ or one of its rotations. Hence the result.

Theorem 2.2 *If $\sigma_1\le\mu\le\sigma_2$ then in view of Lemma(1.1) Theorem (2.1) can be improved. Let σ_3 be given by*

$$\sigma_3=\frac{B_2+(2-2\alpha)bB_1^2}{(3-2\alpha)bB_1^2}$$

If $\sigma_1\le\mu\le\sigma_3$. then,

$$|a_3-\mu a_2^2|+\left(\frac{(3-2\alpha)b\mu B_1^2-[(B_2-B_1)+(2-2\alpha)bB_1^2]}{(3-2\alpha)bB_1^2}\right)|a_2^2|\le\frac{bB_1}{3-2\alpha}$$

If $\sigma_2\le\mu\le\sigma_3$. then,

$$|a_3-\mu a_2^2|+\left(\frac{-(3-2\alpha)\mu bB_1^2+[B_2+B_1+(2-2\alpha)bB_1^2]}{(3-2\alpha)bB_1^2}\right)|a_2^2|\le\frac{bB_1}{3-2\alpha}$$

P r o o f. For $\sigma_1\le\mu\le\sigma_3$. we have,

$$\begin{aligned}|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2^2| &= \frac{bB_1}{2(3-2\alpha)}|c_2-vc_1^2|+\left(\mu-\frac{[(B_2-B_1)+(2-2\alpha)bB_1^2]}{(3-2\alpha)bB_1^2}\right)\frac{b^2B_1^2|c_1|^2}{4}\\ &= \frac{bB_1}{(3-2\alpha)}\left[\frac{1}{2}|c_2-vc_1^2|+\frac{1}{2}v|c_1|^2\right]\\ &= \frac{bB_1}{(3-2\alpha)}\left[\frac{1}{2}\left[|c_2-vc_1^2|+v|c_1|^2\right]\right]\end{aligned}$$

Now, by using Lemma (1.1) we get $|a_3-\mu a_2^2|+(\mu-\sigma_1)|a_2^2|\le\dfrac{bB_1}{(3-2\alpha)}$. Now, For $\sigma_2\le\mu\le\sigma_3$. we have,

$$\begin{aligned}|a_3-\mu a_2^2|+(\sigma_2-\mu)|a_2^2| &= \frac{bB_1}{2(3-2\alpha)}|c_2-vc_1^2|+\left(\frac{B_2+B_1+(2-2\alpha)bB_1^2}{(3-2\alpha)bB_1^2}-\mu\right)\frac{b^2B_1^2|c_1|^2}{4}\\ &= \frac{bB_1}{(3-2\alpha)}\left(\frac{1}{2}\left[|c_2-vc_1^2|+(1-v)|c_1|^2\right]\right)\end{aligned}$$

Now, by using Lemma (1.1) we get $|a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2^2| \leq \dfrac{bB_1}{(3-2\alpha)}$.
Hence the result.

Corollary 2.3 *If $b = 1$ in theorem (2.1) we get our earlier result viz., theorem (3.1) of [11].*

Corollary 2.4 *If $b = 1$ in theorem (2.2) we get our earlier result viz., Remark (3.1) of[11].*

Theorem 2.5 *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots,$ If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{R}_{\alpha,b}^{u}(\phi)$ then*

$$|a_3 - \mu a_2^2| \leq 2max\left\{1; \left|\frac{B_2}{B_1} + ((2-2\alpha) - (3-2\alpha)\mu)bB_1\right|\right\}$$

The result is sharp.

P r o o f. By following the same way of the above theorem and applying the Lemma (1.2) we get the required result.

Corollary 2.6 *If $b = 1, \alpha = \dfrac{1}{2}$ in theorem (2.2) we get Fekete-Szegö inequality for functions in the class starlike functions of complex order[5].*

3 Fractional Derivative

We begin with the following definition:

Definition 3.1 (10) *Let f be analytic in a simply connected region of the z-plane containing the origin. The fractional derivative of f of order λ is defined by*

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \qquad (0 < \lambda < 1) \tag{10}$$

where the multiplicity of $(z-\zeta)^{\lambda}$ is removed by requiring that $log(z-\zeta)$ is real for $z - \zeta > 0$. Using the above definition and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava introduced the operator $\Omega^{\lambda} : \mathcal{A} \to \mathcal{A}$ for λ any positive real number $\neq 2, 3, 4, \ldots$ defined by

$$(\Omega^{\lambda} f)(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \tag{11}$$

and $\mathcal{A} = H_1(\Delta)$. *The class* $(\mathcal{R}^u_{\alpha,b})^\lambda(\phi)$ *consists of function* $f \in \mathcal{A}$ *for which* $\Omega^\lambda f \in (\mathcal{R}^u_{\alpha,b})(\phi)$. *Note that* $(\mathcal{R}^u_{\alpha,b})^\lambda(\phi)$ *is the special case of the class* $(\mathcal{R}^u_{\alpha,b})^g(\phi)$ *when*

$$g(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n \tag{12}$$

Let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ $(g_n > 0)$, g *be analytic in* Δ *and* $f * g \neq 0$. *Since* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}^u_{\alpha,b})^g(\phi)$ *if and only if*

$$(f * g)(z) = z + \sum_{n=2}^{\infty} g_n a_n z^n \in (\mathcal{R}^u_{\alpha,b})(\phi), \tag{13}$$

we obtain the coefficient estimate for functions in the class $(\mathcal{R}^u_{\alpha,b})^g(\phi)$, *from the corresponding estimate for functions in the class* $(\mathcal{R}^u_{\alpha,b})(\phi)$

Theorem 3.2 *Let the function* ϕ *given by* $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$, *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in (\mathcal{R}^u_{\alpha,b})^g(\phi)$, *then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{b}{g_3(3-2\alpha)}\left(B_2 + bB_1^2(2-2\alpha) - \frac{(3-2\alpha)\mu g_3 b B_1^2}{g_2^2}\right), & \mu \leq \sigma_1 \\[2ex] \frac{bB_1}{g_3(3-2\alpha)}, & \sigma_1 \leq \mu \leq \sigma_2 \\[2ex] \frac{b}{g_3(3-2\alpha)}\left(-B_2 - bB_1^2(2-2\alpha) + \frac{(3-2\alpha)\mu g_3 b B_1^2}{g_2^2}\right), & \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{g_2^2}{g_3}\left[\frac{(B_2 - B_1) + (2-2\alpha)bB_1^2}{(3-2\alpha)bB_1^2}\right], \tag{14}$$

$$\sigma_2 = \frac{g_2^2}{g_3}\left[\frac{(B_2 + B_1) + (2-2\alpha)bB_1^2}{(3-2\alpha)bB_1^2}\right] \tag{15}$$

the result is sharp.

P r o o f. The same computation in theorem (2.1) will be done for $f * g$ we get the required result.

Corollary 3.3 *If* $b = 1$ *in theorem (3.1) we get our earlier result viz.,theorem (2.1)[12].*

Corollary 3.4 *If* $b = 1$ *and*

$$g(z) = (\Omega^\lambda f)(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n$$

in theorem (3.1) we get our earlier result viz.,corollary (2.3)[12].

Corollary 3.5 *If* $b = 1$ *and*

$$g(z) = z + \sum_{n=2}^{\infty} n^m z^n, \quad m \in \mathcal{N}_o = \{0\} \cup \mathcal{N}$$

in theorem (3.1) we get our earlier result viz.,corollary (2.4)[12].

References

[1] P. L. Duren, Univalent Functions,*Springer-Verlag,* (1983).

[2] A. W. Goodman, Univalent Functions,*Marine publishing company Inc.,Tampa Florida,* (1983).

[3] W. Ma, D. Minda, A unified treatment of some special classes of univalent functions, *Proc. Conf. on Complex Analysis, Z. (Eds: F. Li, L. Ren, L. Yang, S. Zhang), Int. Press* (1994), 157-169.

[4] Ch. R. Pommerenke.,Univalent Functions,*Vandenhoeck, ruprecht in Göttingen,* (1975).

[5] V. Ravichandran, Y. Polatoglu, M. Bolcol, A. Sen, Certain Subclasses of starlike and convex functions of complex order, *Hacet. J.Math. Stat.,* **34**, (2005), 9-15.

[6] M. S. Robertson, On the theory of univalent functions, *Ann. Math.,* **37**, (1936), 374-408.

[7] S. Ruscheweyh, Some properties of prestarlike and universally prestarlike functions, *Intern. Conf. Geometric Function Theory, J. Analysis.,* **15**, (2007), 247-254.

[8] S. Ruscheweyh, L. Salinas, Universally prestarlike functions as convolution multipiers,*Mathematische Zeitschrift.,* (2009), 607-617.

[9] S. Ruscheweyh, L. Salinas, T. Sugawa, Completely monotone sequences and universally prestarlike functions, *Israel J. Math.,* **171,** (2009), 285-304.

[10] H. M. Srivastava, S. Owa, An application of the fractional derivative, *Math.japan.,* bf 29, (1984), 383-389.

[11] T.N. Shanmugam, J. Lourthu Mary, Fekete-Szego inequality for universally prestarlike functions, *Fract. Calc. Appl. Anal.,* **13. No.4**, (2010), 385-394.

[12] Tirunelveli Nellaiappan Shanmugam, Joseph Lourthu Mary. A Note on Universally Prestarlike Functions, *Stud.Univ.Babeş-Bolyai Math.,* **57, No.1,** (2012), 53-60.

[13] P. Wiatrowski, The coeffecient of a certain family of holomorphic functions, *zest. Nauk. Math. przyord. serII. Zeszyt.,* **39**, (1971), 57-85.

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