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# Hypergeometric functions in the geometric function theory

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Dedicated to Professor H.M. Srivastava on the occasion of his 65th birthday

**Abstract**

In this lecture, uses and influences of hypergeometric functions (both Kummer's and confluent hypergeometric functions) in the study of geometric function theory and its generalizations are discussed, as a survey of the author's work.


*Keywords:* Hypergeometric functions; Gaussian; Hohlov operator; Generalized

## 1. Introduction

The univalent function theory is a fascinating area of research with continued interest in recent times also. This is classified under the broader area of geometric function theory due to the interplay between analysis and geometry. The Bieberbach's conjecture which remained open for a long time has been positively settled by de Branges in the year 1984. The study based on this problem yielded so many new results. But, on the other hand, interest on hypergeometric function theory was developed after the surprise use of hypergeometric functions in the proof of Bieberbach's conjecture by de Branges. It is known that Gaussian hypergeometric function $F(a,b;c;z) = {}_2F_1(a,b;c;z)$, defined by

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n, \tag{1.1}$$

is the solution of the second order homogenous differential equation

$$z(1-z)w''(z) + [c-(a+b+1)z]w'(z) - abw(z) = 0, \tag{1.2}$$

where $(a)_n$ is the Pochhammer symbol (or ascending factorial) defined by

$$(a)_n = \begin{cases} 1 & \text{for } n = 0 \\ a(a+1)\cdots(a+n-1) & \text{for } n = 1,2,3\ldots \end{cases} \tag{1.3}$$

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and where $a, b, c$ are complex numbers with $c \neq 0, -1, -2, \ldots$ Some of the basic properties of the Gaussian hypergeometric functions are

$$F(a,b;c;z) = F(b,a;c;z), \tag{1.4}$$

$$cF'(a,b;c;z) = abF(a+1,b+1;c+1;z), \tag{1.5}$$

$$F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z), \tag{1.6}$$

$$F(a,b;c;z) = (1-z)^{-a}F\left(c,c-b;c;\frac{z}{(z-1)}\right), \tag{1.7}$$

$$F(a,b;b;z) = \frac{z}{(1-z)^a}. \tag{1.8}$$

Also, If $\Re(c) > \Re(b) > 0$, then there is a probability measure $\mu(t)$ on $[0,1]$ given by

$$\mathrm{d}\mu(t) = \left[\frac{\Gamma(c)t^{b-1}(1-t)^{b-1}}{\Gamma(b)\Gamma(c-b)}\right]\mathrm{d}t \tag{1.9}$$

such that,

$$F(a,b;c;z) = \int_0^1 (1-tz)^{-a}\,\mathrm{d}\mu(t).$$

Also, the Confluent (or Kummer's) hypergeometric function $\Phi(a,\ c;\ z) = {}_1F_1(a,\ c;\ z)$ defined by

$$\Phi(a,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n(1)_n}z^n, \tag{1.10}$$

where $(a)_n$ is the Pochhammer symbol as defined in (1.3), is the solution of the second order differential equation

$$zw''(z) + [c-z]w'(z) - aw(z) = 0. \tag{1.11}$$

Some of the basic properties of $\Phi$ are

$$c\Phi'(a,b;c;z) = a\Phi(a+1;c+1;z), \tag{1.12}$$

$$\Phi(a;c;z) = \mathrm{e}^z\Phi(c-a;c;z), \tag{1.13}$$

$$\Phi(a;a;z) = \mathrm{e}^z. \tag{1.14}$$

Also, If $\Re(c) > \Re(a) > 0$, then

$$\Phi(a;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\mathrm{e}^{tz}\,\mathrm{d}t = \int_0^1 \mathrm{e}^{tz}\,\mathrm{d}\mu(t), \tag{1.15}$$

where $\mu(t)$ is given by

$$\mathrm{d}\mu(t) = \left[\frac{\Gamma(c)t^{a-1}(1-t)^{c-a-1}}{\Gamma(a)\Gamma(c-a)}\right]\mathrm{d}t \tag{1.16}$$

is a probability measure on $[0,1]$. It is to be noted that both Gaussian and Kummers hypergeometric functions are analytic in the open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathscr{A}$ be the class of all functions $f$, whose Maclaurin's series is of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.17}$$

which are *analytic* in $\Delta := \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathscr{S}$ be the subclass of $\mathscr{A}$, consisting of univalent functions. The two important subclasses of $\mathscr{S}$ are the class of starlike functions, $S^*$ and convex functions, $K$.

A function $f \in \mathscr{A}$ is starlike of order $\alpha, 0 \leqslant \alpha < 1$, if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in \Delta. \tag{1.18}$$

This class is denoted by $S^*(\alpha)$ where $S^*(0) = S^*$, the class of functions that are starlike.

A function $f \in \mathscr{A}$ is said to be in the class $K(\alpha)$, the class of functions which are convex univalent of order $\alpha$, that is if $zf'(z) \in S^*(\alpha)$. Clearly $K(0) = K$, the class of convex univalent functions.

If $f \in \mathscr{S}$ and $\frac{f(z)f'(z)}{z} \neq 0,\ z \in \Delta$, and

$$\Re\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)\right\} > 0, \quad z \in \Delta,$$

and $\alpha$ is real, then $f$ is called an $\alpha$-convex function, denoted by $M(\alpha)$. In fact this class of functions unifies the classes $K$ and $\mathscr{S}^*$. Note that $M(0) = \mathscr{S}^*$ and $M(1) = K$.

A function $f \in \mathscr{A}$ is said to be uniformly convex in $\Delta$ if it has the property that for every circular arc $\gamma$ contained in the open unit disc $\Delta$, with center $\zeta$ also in $\Delta$, the image arc $f(\gamma)$ is a convex arc. This class was introduced by Goodman [6]. In another paper he [7] introduced the uniform starlike functions.

The Class $UCV$ describes geometrically the domain of values of the expression $1 + \frac{zf''(z)}{f'(z)},\ z \in \Delta$ to lie in a parabolic region $\Omega = \{\omega \in \mathbb{C} : (\mathrm{Im}(\omega))^2 < 2\Re(\omega) - 1\}$.

Rønning [17], and Ma and Minda [13] gave a one variable characterization for $f \in UCV$, as

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in \Delta. \tag{1.19}$$

Using the alexander relation, Rønning [17] defined a new class called $S_p$ consisting of functions $f \in \mathscr{A}$ satisfying

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in \Delta. \tag{1.20}$$

Kanas and Wiśniowska [9] defined the class $k - UCV$ as

$$k - UCV := \left\{f \in \mathscr{A} : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right|,\ (0 \leqslant k < \infty)\right\}. \tag{1.21}$$

Note that the class $k - UCV$ is an extension of the class $UCV$ studied by Goodman [4,5]. In another paper, Kanas and Wiśniowska [10] extended the class $S_p$ as

$$k - ST := \left\{f \in \mathscr{A} : \Re\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right|,\ (0 \leqslant k < \infty)\right\}. \tag{1.22}$$

The various properties of the classes $k - UCV$ and $k - ST$ were extensively studied by Kanas and Srivastava [11].

Bharathi et al. [1] extended the class $k - UCV$ to $UCV(\alpha, \beta)$ with $k = \alpha \geqslant 0$, and $0 \leqslant \beta < 1$, as

$$UCV(\alpha, \beta) := \left\{f \in \mathscr{A} : \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \alpha\left|\frac{zf''(z)}{f'(z)}\right| + \beta,\ z \in \Delta\right\}. \tag{1.23}$$

They gave a sufficient condition for a function $f \in \mathscr{A}$ to be in $UCV(\alpha, \beta)$ in terms of the coefficient of the function. The class $S_p(\alpha, \beta)$ was also discussed in [1], as

$$S_p(\alpha, \beta) := \left\{f \in \mathscr{A} : \Re\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \alpha\left|\frac{zf'(z)}{f(z)} - 1\right| + \beta,\ z \in \Delta\right\}. \tag{1.24}$$

Let $\tau \in \mathbb{C} \setminus \{0\},\ -1 \leqslant B < A \leqslant 1$. A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{R}^\tau(A, B)$ if it satisfies the inequality

$$\left|\frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]}\right| < 1 \quad (z \in \Delta). \tag{1.25}$$

The class $\mathscr{R}^{\tau}(A,B)$ was introduced by Dixit and Pal [3]. For the choices of $\tau = e^{-i\eta}\cos\eta$ $(-\frac{\pi}{2} < \eta < \frac{\pi}{2})$, $A = 1-2\gamma$ $(0 \leqslant \gamma < 1)$ and $B=-1$, the class $\mathscr{R}^{\tau}(A,B)$ reduces to the class $\mathscr{R}_{\eta}(\gamma)$ studied by Ponnusamy and Rønning [16], where

$$\mathscr{R}_{\eta}(\gamma) := \left\{f \in \mathscr{A} : \Re(e^{i\eta}(f'(z)-\gamma)) > 0\left(z \in \Delta; -\frac{\pi}{2} < \eta < \frac{\pi}{2}; 0 \leqslant \gamma < 1\right)\right\}. \tag{1.26}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then the Hadamard product or convolution of $f(z)$ and $g(z)$ [18], written as $(f*g)(z)$ is defined by

$$(f*g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{1.27}$$

For $f \in \mathscr{A}$, we recall the operator $I_{a,b;c}(f)$ of Hohlov [8] which maps $\mathscr{A}$ into itself defined by

$$[I_{a,b;c}(f)](z) = zF(a,b;c;z) * f(z) \tag{1.28}$$

where $*$ denotes the Hadamard's convolution.

## 2. The uses of hypergeometric functions

In this section, we illustrate the use of hypergeometric functions in ascertaining the invariance of the classes of functions under integral operators.

Shanmugam [20] introduced an integral operator combining the Ruscheweyh integral operator and the Mocanu integral operator as

$$F(z) = A_g(f)(z) = \left\{\frac{c+1/\alpha}{g^c(z)}\int_0^z g^{c-1}(t)g'(t)f^{\frac{1}{\alpha}}(t)\,dt\right\}^{\alpha}, \tag{2.1}$$

where $\alpha > 0$. Certain interesting results concerning the above integral operator have been studied by Shanmugam [20]. Refer also [14,19]. They are given as theorems without proofs.

**Theorem 1.** *Suppose $g \in \mathscr{A}$ and $\frac{g(z)g'(z)}{z} \neq 0$ in $\Delta$. Let $\alpha$ and $c$ be positive real numbers satisfying $(c+1)\alpha > 1 > (c-1)\alpha$ and $\beta$ be any real number in the interval $[\beta_0, 0)$ where $\beta_0 = \max\{\frac{1-(c+1)\alpha}{2}, -c\alpha, \frac{c\alpha-(\alpha+1)}{2c\alpha}\}$. Further, let $g(z)$ satisfy*

$$\Re\left\{c\frac{zg'(z)}{g(z)}\right\} \geqslant c + \frac{\delta}{\alpha} \tag{2.2}$$

*and*

$$\Re\left\{(c+1)\left(\frac{zg'(z)}{g(z)}\right) - \left(1+\frac{zg''(z)}{g'(z)}\right)\right\} \leqslant c + \frac{\beta}{\alpha}, \tag{2.3}$$

*where*

$$\delta = \delta\left(\frac{1}{\alpha};\beta;c\right) \tag{2.4}$$

*is given by*

$$\delta\left(\frac{1}{\alpha};\beta;c\right) = \alpha\left\{\frac{\frac{1}{\alpha}+c}{F\left(1,\frac{2(1-\beta)}{\alpha};c+1+\frac{1}{\alpha};1/2\right)} - c\right\}, \tag{2.5}$$

*where $F(a,b;c;z)$ is the Gaussian hypergeometric function. Then $F(z) = A_g(f)$ defined in Eq. (2.1) is $\alpha$- convex (in the sense of Mocanu) in $\Delta$ whenever $f$ is $\alpha$- convex (in the sense of Mocanu) in $\Delta$.*

Next theorem says under certain conditions on the function $g(z)$, $\alpha$ and $c$, $F(z)$ given by the Eq. (2.1) is a $\alpha$-close-to-convex function.

**Theorem 2.** *Suppose* $g \in \mathcal{A}$ *and* $\frac{g(z)g'(z)}{z} \neq 0$ *in* $\Delta$. *If there exists a* $\alpha \geqslant 1$, $c > 0$ $(c-1)\alpha < 1$ *and* $\beta \in \mathbb{R}$ *such that* $\beta \in [\beta_0, 0)$ *where* $\beta_0 = \max\{\frac{1-(c+1)\alpha}{2}, -c\alpha, \frac{c\alpha-(\alpha+1)}{2c\alpha}\}$ *and*

$$\Re\left\{c\frac{zg'(z)}{g(z)}\right\} \geqslant c + \frac{\beta}{\alpha} \tag{2.6}$$

*and*

$$0 \leqslant \Re\left\{(c+1)\frac{zg'(z)}{g(z)} - \left(\frac{1+zg''(z)}{g'(z)}\right)\right\} \leqslant c + \frac{\delta}{\alpha}, \tag{2.7}$$

*where*

$$\delta = \delta\left(\frac{1}{\alpha};\beta;c\right) \tag{2.8}$$

*is given by* (2.5), *then* $F(z)$ *is given by Eq.* (2.1), *is a* $\alpha$-*close-to-convex function whenever* $f(z)$ *is a* $\alpha$-*close-to-convex function.*

## 3. Influences of hypergeometric functions

In this section, we discuss how hypergeometric function influence certain studies in the geometric function theory. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ and $\beta_1, \beta_2, \cdots, \beta_q$ be complex numbers with $\beta_j \neq 0, -1, -2, \ldots, (j = 1, 2, \ldots, q)$. Then the generalized hypergeometric function ${}_pF_q(z)$ is defined by

$$ {}_pF_q(z) = {}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_q)_n(1)_n}, \quad p \leqslant q+1 \quad \text{and} \quad p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{3.1}$$

where $(\lambda)_n$ is the *Pochhammer* symbol as defined by (1.3).

Now, we recall a sufficiently adequate special case of a convolution operator which was introduced earlier by Dziok and Srivastava [4] by means of the Hadamard product (or convolution) involving generalized hypergeometric functions.

We observe that the series in (3.1) converges absolutely in the entire complex plane for $p < q+1$, and in the unit disc for $p \leqslant q+1$.

Thus, we have, $f \in \mathcal{A}$; $p \leqslant q+1$; $z \in U$

$$I\begin{pmatrix}\alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q\end{pmatrix} f(z) := z_pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q, z) * f(z) \tag{3.2}$$

so that, for a function of the form (1.17)

$$I\begin{pmatrix}\alpha_1, \ldots, \alpha_p \\ \beta_1 \ldots, \beta_q\end{pmatrix} f(z) = z + \sum_{n=2}^{\infty} B_n a_n z^n, \tag{3.3}$$

where, for convenience,

$$B_n := \frac{(\alpha_1)_{n-1} \cdots (\alpha_p)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_1)_{n-1}(1)_{n-1}} \quad (n \in \mathbb{N} \setminus \{1\}). \tag{3.4}$$

Several mapping properties of the class $k - UCV$ and $k - ST$ using the generalized hypergeometric functions was studied recently by Gangadharan et al. [5]. We list some of the results, again without proofs.

## 4. Generalized hypergeometric functions

Next, we define the classes $S^*_\lambda$ and $C_\lambda$ (cf. e.g. [11, p.142]) by

$$S^*_\lambda := \left\{ f \in S : \left|\frac{zf'(z)}{f(z)} - 1\right| < \lambda \ (z \in \mathcal{U}, \lambda > 0)\right\}$$

and

$$C_\lambda := \left\{ f \in S : \left| \frac{zf''(z)}{f'(z)} \right| < \lambda \ (z \in \mathcal{U}, \lambda > 0) \right\}.$$

Obviously $f \in C_\lambda \iff zf' \in S^*_\lambda (\lambda > 0)$ which is analogous to the Alexander equivalence (see, for details [12]).

**Lemma 1** (see [5]). *Let $f \in \mathcal{A}$ be of the form (1.17) and if*

$$\sum_{n=2}^{\infty} (\lambda + n - 1)|a_n| \leqslant \lambda, \quad \lambda > 0, \tag{4.1}$$

*then $f \in S^*_\lambda$.*

**Lemma 2** (see [5]). *Let $f \in \mathcal{A}$ be of the form (1.17). If*

$$\sum_{n=2}^{\infty} n(\lambda + n - 1)|a_n| \leqslant \lambda \quad (\lambda > 0) \tag{4.2}$$

*holds true, then $f \in C_\lambda$.*

**Lemma 3.** *Let $0 \leqslant k < \infty$, and let $f \in \mathcal{A}$ be of the form (1.17). If $f \in k - UCV$, then*

$$|a_n| \leqslant \frac{(P_1)_{n-1}}{n!}, \quad n \in \mathbb{N} \setminus \{1\}, \tag{4.3}$$

*where $P_1 = P_1(k)$ is the coefficient of $z$ in the function*

$$p_k(z) = 1 + \sum_{n=1}^{\infty} P_n(k) z^n, \tag{4.4}$$

*which is the extremal function for the class $P(p_k)$ related to the class $k - UCV$ by the range of the expression*

$$1 + \frac{zf''(z)}{f'(z)} \quad (z \in \Delta),$$

*where $P_1 = P_1(k)$ is given, as above, by (4.4).*

**Theorem 3.** *Suppose that $\alpha_i \in \mathbb{C} \setminus \{0\}$ $(j = 1, 2, 3, \ldots, p)$, $\Re(\beta_j) > 0$ $(j = 1, 2, 3, \ldots, q)$ and (in the case $p = q + 1$)*

$$\Re\left( \sum_{j=1}^{q} \beta_j \right) > 1 + \sum_{j=1}^{p} |\alpha_j|.$$

*If $f \in \mathcal{R}^\tau(A, B)$, and for some $k$ $(0 \leqslant k < \infty)$, the hypergeometric inequality*

$${}_pF_q(1 + |\alpha_1|, \ldots, 1 + |\alpha_p|; 1 + \Re(\beta_1), \ldots, 1 + \Re(\beta_q); 1) < \frac{\Re(\beta_1), \ldots, \Re(\beta_q)}{(k+2)|\alpha_1 \cdots \alpha_p|},$$

*holds true, then*

$$I\begin{pmatrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{pmatrix} f \in k - UCV.$$

**Theorem 4.** *Suppose that $\alpha_i \in \mathbb{C} \setminus \{0\}$ $(j = 1, 2, 3, \ldots, p)$, $\Re(\beta_j) > 0$ $(j = 1, 2, 3, \ldots, q)$ and (in the case $p = q + 1$)*

$$\Re\left( \sum_{j=1}^{q} \beta_j \right) > \sum_{i=1}^{p} |\alpha_i|.$$

*If* $f \in \mathscr{R}^{\tau}(A,B)$ *and the hypergeometric inequality*

$$\left.\begin{array}{l}(1+k)\quad {}_pF_q(|\alpha_1|,\ldots,|\alpha_p|;\Re(\beta_1),\ldots,\quad\Re(\beta_q);1)-\\ k\quad {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,|\alpha_p|,1;\Re(\beta_1),\ldots,\quad\Re(\beta_q),2;1)\end{array}\right\} < \frac{1}{(A-B)|\tau|}+1$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix}f \in k-ST.$$

**Theorem 5.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j=1,2,3,\ldots,p), \Re(\beta_j)>0$ $(j=1,2,3,\ldots,q)$ *and* (*in the case* $p=q+1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + \sum_{j=1}^{p}|\alpha_j|,$$

*where* $P_1 = P_1(k)$ *is given, as above, by* (4.4). *If* $f \in k-UCV$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality*

$${}_{p+1}F_{q+1}(1+|\alpha_1|,\ldots,1+|\alpha_p|,P_1+1;1+\Re(\beta_1),\ldots,1+\Re(\beta_q),2;1) < \frac{\Re(\beta_1),\ldots,\Re(\beta_q)}{(k+2)|\alpha_1\cdots\alpha_p|P_1},$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix}f \in k-UCV.$$

**Theorem 6.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j=1,2,3,\ldots,p)$, $\Re(\beta_j)>0$ $(j=1,2,3,\ldots,q)$ *and* (*in the case* $p=q+1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{j=1}^{p}|\alpha_j|,$$

*where* $P_1 = P_1(k)$ *is given, as above, by* (4.4). *If* $f \in k-ST$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality*

$$\left.\begin{array}{l}\dfrac{(k+2)|\alpha_1\cdots\alpha_p|}{\Re(\beta_1)\cdots\Re(\beta_q)}{}_{p+1}F_{q+1}(1+|\alpha_1|,\ldots,1+|\alpha_p|,P_1+1;1+\Re(\beta_1),\ldots,1+\Re(\beta_q),2;1)+\\ {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,|\alpha_p|,P_1;\Re(\beta_1),\ldots,\Re(\beta_q),1;1)\end{array}\right\} < 2,$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix}f \in k-ST.$$

**Theorem 7.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j=1,2,3,\ldots,p)$, $\Re(\beta_j)>0$ $(j=1,2,3,\ldots,q)$ *and* (*in the case* $p=q+1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{j=1}^{p}|\alpha_j|,$$

*where* $P_1 = P_1(k)$ *given as by* (19). *If* $f \in k-UCV$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality*

$$\left.\begin{array}{l}\dfrac{(k+2)|\alpha_1\cdots\alpha_p|}{2\Re(\beta_1)\cdots\Re(\beta_q)}{}_{p+1}F_{q+1}(1+|\alpha_1|,\ldots,1+|\alpha_p|,P_1+1;1+\Re(\beta_1),\ldots,1+\Re(\beta_q),3;1)+\\ {}_{p+1}F_{q+1}(|\alpha_1|,\ldots,|\alpha_p|,P_1;\Re(\beta_1),\ldots,\Re(\beta_q),2;1)\end{array}\right\} < 2,$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix}f \in k - ST.$$

**Theorem 8.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1,2,3,\ldots,p)$, $\Re(\beta_j) > 0$ $(j = 1,2,3,\ldots,q)$ *and* (*in the case* $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{j=1}^{p}|\alpha_j|,$$

*where* $P_1 = P_1(k)$ *is given, as above, by* (4.4). *If* $f \in k - ST$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality*

$${}_{p+2}F_{q+2}(1+|\alpha_1|,\ldots,1+|\alpha_p|,P_1+1,3;+\Re(\beta_1),\ldots,1+\Re(\beta_q),2,2;1)+ < \frac{\Re(\beta_1)\cdots\Re(\beta_q)}{2P_1(k+2)|\alpha_1\cdots\alpha_p|},$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix}f \in k - UCV.$$

**Theorem 9.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1,2,3,\ldots,p)$, $\Re(\beta_j) > 0$ $(j = 1,2,3,\ldots,q)$ *and* (*in the case* $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 - 1 + \sum_{i=1}^{p}|\alpha_i|$$

*where* $P_1 = P_1(k)$ *is given, as above, by* (4.4). *If* $f \in k - UCV$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality*

$${}_{p+2}F_{q+2}(|\alpha_1|,\ldots,|\alpha_p|,P_1,\lambda+1;\Re(\beta_1),\ldots,\Re(\beta_q),\lambda,2;1) < 2\,(\lambda > 0),$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix}f \in S^*_\lambda.$$

**Theorem 10.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1,2,3,\ldots,p)$, $\Re(\beta_j) > 0$ $(j = 1,2,3,\ldots,q)$ *and* (*in the case* $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + \sum_{i=1}^{p}|\alpha_i|,$$

*where* $P_1 = P_1(k)$ *is given, as above, by* (4.4). *If* $f \in k - ST$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality*

$${}_{p+2}F_{q+2}(|\alpha_1|,\ldots,|\alpha_p|,P_1,\lambda+1;\Re(\beta_1),\ldots,\Re(\beta_q),\lambda,1;1) < 2 \quad (\lambda > 0), \tag{4.5}$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix}f \in S^*_\lambda.$$

**Theorem 11.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1,2,3,\ldots,p)$, $\Re(\beta_j) > 0$ $(j = 1,2,3,\ldots,q)$ *and* (*in the case* $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 - 1 + \sum_{i=1}^{p}|\alpha_i|,$$

*where* $P_1 = P_1(k)$ *is given, as above, by* (4.4). *If* $f \in k-UCV$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality* (4.5) *holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix} f \in C_\lambda.$$

**Theorem 12.** *Suppose that* $\alpha_i \in \mathbb{C}\setminus\{0\}$ $(j = 1,2,3,\ldots,p)$, $\Re(\beta_j) > 0$ $(j = 1,2,3,\ldots,q)$ *and* (*in the case* $p = q + 1$)

$$\Re\left(\sum_{j=1}^{q}\beta_j\right) > P_1 + 1 + \sum_{i=1}^{p}|\alpha_i|,$$

*where* $P_1 = P_1(k)$ *is given, as above, by* (4.4). *If* $f \in k-ST$, *and for some* $k$ $(0 \leqslant k < \infty)$, *the hypergeometric inequality*

$${}_{p+3}F_{q+3}(|\alpha_1|,\ldots,|\alpha_p|, P_1, \lambda+1, 2; \Re(\beta_1),\ldots,\Re(\beta_q), \lambda, 1, 1; 1) < 2 \quad (\lambda > 0),$$

*holds true, then*

$$I\begin{pmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{pmatrix} f \in C\lambda.$$

Recently, Shanmugam and Sivasubramanian [21] have obtained sufficient condition for the Hohlov operator to be in the class $S_p(\alpha,\beta)$. Some of the main results in the paper are put forth in the following section.

## 5. The CLASS *UCV(α,β)*

We state the following results which will be used to prove our main results.

**Theorem 13** [11, Theorem 2.3, p. 21]. *A function f(z) of the form* (1.17) *is in* $UCV(\alpha,\beta)$ *if it satisfies the condition*

$$\sum_{n=2}^{\infty} n(n(1+\alpha) - (\alpha+\beta))|a_n| \leqslant 1-\beta. \tag{5.1}$$

**Theorem 14** [1, Theorem 2.6, p. 23]. *A function f(z) of the form* (1.17) *is in* $S_p(\alpha,\beta)$ *if it satisfies the condition*

$$\sum_{n=2}^{\infty} (n(1+\alpha) - (\alpha+\beta))|a_n| \leqslant 1-\beta. \tag{5.2}$$

**Lemma 4** [1, Lemma 2, p. 767]

(i) *For* $a \neq 1, b \neq 1$ *and* $c \neq 1$ *with* $c > \max[0, a+b-1]$,

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}} = \frac{1}{(a-1)(b-1)}\left[\frac{\Gamma(c-a-b+1)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} - (c-1)\right].$$

(ii) *For a,b > 0 and c > a + b + 3*

$$\sum_{n=0}^{\infty}(n+1)^3\frac{(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{(a)_3(b)_3}{(c-a-b-3)_3}+6\frac{(a)_2(b)_2}{(c-a-b-2)_2}+7\frac{ab}{(c-a-b)}+1\right].$$

(iii) *For a,b > 0 and c > a + b + 2*

$$\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(a)_2(b)_2}{(c-a-b-2)_2}+3\frac{ab}{(c-a-b)}\right].$$

(iv) *For a,b > 0 and c > a + b + 1*

$$\sum_{n=0}^{\infty}(n+1)\frac{(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab}{(c-a-b-1)}\right].$$

**Lemma 5** [3]. *If $f\in\mathfrak{R}^{\tau}(A,B)$ is of form (1.17), then*

$$|a_n|\leqslant(A-B)\frac{|\tau|}{n},n\neq 1. \tag{5.3}$$

**Theorem 15.** *Let $f\in\mathfrak{R}^{\tau}(A,B)$, $a,b\in\mathbb{C}\setminus\{0\}$ and $c>|a|+|b|+1$. If*

$$\left.\begin{array}{l}\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(1-\beta)+(1+\alpha)\frac{(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3}\right]+\\ \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}(6+5\alpha-\beta)\right]+\\ \frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|a|)}\left[\frac{|ab|}{(c-|a|-|b|-3)}(7+4\alpha-3\beta)\right]\end{array}\right\}\leqslant 2(1-\beta)$$

*then, $I_{a,b;c}(f)\in UCV(\alpha,\beta)$.*

Taking $\alpha=1$, $\beta=0$ and $f\in\mathfrak{R}_{\eta}(\gamma)$ in Theorem 15, we get the following result of Kim and Ponnusamy [12].

**Corollary 1.** *Let $f\in\mathfrak{R}_{\eta}(\gamma)$, $a,b\in\mathbb{C}\setminus\{0\}$ and $c>|a|+|b|+1$.*

$$2(1-\gamma)\cos\eta\left\{\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[2\left(\frac{|ab|}{c-|a|-|b|-1}\right)+1\right]-1\right\}\leqslant 1 \tag{5.4}$$

*then, $I_{a,b;c}(f)\in UCV$.*

**Theorem 16.** *Let $f\in\mathscr{S}$, $a,b\in\mathbb{C}\setminus\{0\}$ and $c>|a|+|b|+3$. If*

$$\begin{aligned}&\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[(1-\beta)+(1+\alpha)\frac{(|a|)_3(|b|)_3}{(c-|a|-|b|-3)_3}+\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}(6+5\alpha-\beta)\right]\\&\quad+\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|a|)}\left[\frac{|ab|}{(c-|a|-|b|-3)}(7+4\alpha-3\beta)\right]\leqslant 2(1-\beta)\end{aligned} \tag{5.5}$$

*then, $I_{a,b;c}(f)\in UCV(\alpha,\beta)$.*

Taking $f(z)=\frac{z}{1-z}$ in Theorem 16, we have the following result obtained by Swaminathan [17].

**Corollary 2.** *If a,b > 0 and c > a + b + 2 then a sufficient condition for $zF(a,b;c;z)$ to be in $UCV(\alpha,\beta)$ is that*

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\left(\frac{3+2\alpha-\beta}{1-\beta}\right)\left(\frac{ab}{c-a-b-1}\right)+\left(\frac{1+\alpha}{1-\beta}\right)\left(\frac{(a)_2(b)_2}{(c-a-b)_2}\right)\right]\leqslant 2.$$

Taking $f(z)=\frac{z}{1-z}$, $\alpha=1$ in Theorem 16, we get the following result of Cho et al. [2].

**Corollary 3.** *If $a, b > 0$ and $c > a + b + 2$ then a sufficient condition for $zF(a,b;c;z)$ to be in $UCV(\beta)$ is that*

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\left(\frac{5-\beta}{1-\beta}\right)\left(\frac{ab}{c-a-b-1}\right)+\left(\frac{2}{1-\beta}\right)\left(\frac{(a)_2(b)_2}{(c-a-b)_2}\right)\right]\leqslant 2.$$

Taking $f(z) = \frac{z}{1-z}$, $\alpha = 1$, $\beta = 0$ in Theorem 16, we get the following result of Kim and Ponnusamy [12].

**Corollary 4.** *If $a, b \in \mathbb{C}\setminus\{0\}$ and $c > |a| + |b| + 1$, then a sufficient condition for $zF(a,b;c;z)$ to be in UCV is that*

$$\frac{\Gamma(c-|a|-|b|)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+5\left(\frac{|ab|}{c-|a|-|b|-1}\right)+2\left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|)_2}\right)\right]\leqslant 2.$$

## 6. The class $S_p(\alpha, \beta)$

**Theorem 17.** *Let $f \in \Re^{\tau}(A,B)$, $a, b \in \mathbb{C}\setminus\{0\}$ with $|a| \neq 1, |b| \neq 1$. If $c > \max\{0, |a| + |b|\}$, then a sufficient condition for $I_{a,b;c}(f)$ to be in the class $S_p(\alpha,\beta)$ is*

$$\left.\begin{array}{l}(1+\alpha)\left(\dfrac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right)-\\ \dfrac{\alpha+\beta}{(|a|-1)(|b|-1)}\left(\dfrac{\Gamma(c-|a|-|b|+1)\Gamma(c)}{\Gamma(c-|a|)\Gamma(c-|b|)}\right)+\\ (\alpha+\beta)\left[\dfrac{(c-1)}{(|a|-1)(|b|-1)}+1\right]\end{array}\right\}\leqslant\frac{1-\beta}{(A-B)|\tau|}.$$

**Theorem 18.** *Let $f \in R_{\eta}(\gamma)$, $a, b \in \mathbb{C}\setminus\{0\}$ with $|a| \neq 1, |b| \neq 1$. If $c > \max\{0, |a| + |b| - 1\}$, then a sufficient condition for $I_{a,b;c}(f)$ to be in the class $k - ST$ is*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{k(|ab|-c+1)}{(|a|-1)(|b|-1)}\right]\leqslant\frac{1}{2(1-\beta)\cos\eta}+k\left[\frac{(1-c)}{(|a|-1)(|b|-1)}\right]+1.$$

**Corollary 5.** *If $f \in R_{\eta}(\gamma)$, $\alpha = 1$ and $\beta = 0$ in Theorem 17, we get the sufficient condition for $I_{a,b;c}(f)$ to be in the class of $S_p$ obtained by Parvatham and Prabhakaran [15].*

**Theorem 19.** *Let $f \in S$, $a, b \in \mathbb{C}\setminus\{0\}$ and $c > |a| + |b| + 2$. If*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+\frac{(1+\alpha)}{(1-\beta)}\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}+\frac{(3+2\alpha-\beta)}{(1-\beta)}\frac{|ab|}{(c-|a|-|b|-1)}\right]\leqslant 2$$

*then, $I_{a,b;c}(f) \in S_p(\alpha,\beta)$.*

If $\alpha = k$ $(0 \leqslant k < \infty)$, $\beta = 0$ in Theorem 19, we get the following result of Kanas and Srivastava [11].

**Corollary 6.** *Let $f \in S$, $a, b \in \mathbb{C}\setminus\{0\}$ and $c > |a| + |b| + 2$. If*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+(1+\alpha)\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}+(3+2\alpha)\frac{|ab|}{(c-|a|-|b|-1)}\right]\leqslant 2$$

*then, $I_{a,b;c}(f) \in k - ST$.*

If $\alpha = 1, \beta = 0$ in Theorem 19, we get the following result obtained by Parvatham and Prabhakaran [15].

**Corollary 7.** *Let $f \in \mathscr{S}$, $a, b \in \mathbb{C}\setminus\{0\}$ and $c > |a| + |b| + 2$. If*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[1+2\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2}+5\frac{|ab|}{(c-|a|-|b|-1)}\right]\leqslant 2 \tag{6.1}$$

*then, $I_{a,b;c}(f) \in S_p$.*

Taking $f(z)=\frac{z}{1-z}$ in Theorem 19, we have the following result obtained by Swaminathan [23].

**Theorem 20.** *If a,b > 0 and c > a + b + 1, then a sufficient condition for zF(a,b;c;z) to be in $S_p(\alpha,\beta)$ is that*

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\left(\frac{1+\alpha}{1-\beta}\right)\left(\frac{ab}{c-a-b-1}\right)\right]\leqslant 2.$$

**Corollary 8.** *Taking $f(z)=\frac{z}{1-z}$, $\alpha=0$ in Theorem 19, we get the sufficient condition for $zF(a,b;c;z)$ to be in the class of starlike functions of order $\beta$ obtained by Silverman [22].*

**Corollary 9.** *Taking $f(z)=\frac{z}{1-z}$, $\alpha=0$ and $\beta=1$ in Theorem 19, we get the sufficient condition for $zF(a,b;c;z)$ to be in the class of parabolic starlike functions $S_p$ obtained by Parvatham and Prabhakaran [15].*

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