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A concentration theorem of expanders on Hadamard manifolds

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Abstract

In this note, we prove a concentration theorem of expanders. As a simple corollary, one can prove that expanding sequences of graphs do not admit coarse embeddings into Hadamard manifolds with bounded sectional curvatures.
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In [6], a concentration theorem of expanders for Banach spaces whose unit balls are uniformly embeddable into Hilbert spaces is proved. We borrow the method and prove the following theorem.

Theorem 1. *Let M be a Hadamard manifold with bounded sectional curvatures. And let (V, E) be a (k, λ)-regular expander. Then there exists $R > 0$ such that for any $f : V \to M$*

$$\frac{1}{|V|}\sum_{v\in V} d\big(f(v), m\big)^2 \leqslant R^2\big(\mathrm{Lip}(f)+1\big)^2,$$

where m is a point such that $\sum_{v\in V}\log_m(f(v)) = 0$.

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First, let us recall basic definitions of expanders (see [2] for more information and for references). Let (V, E) be a finite graph with the vertex set V and the edge set E. Denote the cardinality of V and E by $|V| = n$ and $|E| = m$. We also define an orientation on E. The differential $d : \ell_2(V) \to \ell_2(E)$ is defined by $d(f)(e) = f(e^+) - f(e^-)$ for all $f \in \ell_2(V)$ and $e = (e^+, e^-) \in E$ where e^+ and e^- are initial and end points of e respectively.

This differential d is an $m \times n$ matrix. The discrete Laplace operator $\Delta = d^*d$, where d^* is the adjoint operator of d. This definition does not depend on the choice of the orientation of E. Δ is self-adjoint and positive. Hence it has real nonnegative eigenvalues. Denote by $\lambda_1(V)$ the minimal positive eigenvalue of the discrete Laplace operator Δ of the graph (V, E).

Definition 2. A (k, λ)-*regular expander* is a graph (V, E) with a fixed degree k and $\lambda_1(V) \geqslant \lambda$. A sequence of graphs $(V_n, E_n)_{n\in\mathbb{N}}$ of a fixed degree k and with $|V_n| \to \infty$ is called an *expanding sequence of graphs* if there is a positive constant λ such that $\lambda_1(V_n) \geqslant \lambda$ for all $n \in \mathbb{N}$.

It is not an easy job to construct an expanding sequence of graphs. See [4,3,5] for explicit constructions.

Proposition 3 *(Concentration Property). Let (V, E) be a (k, λ)-expander, and let R be a Euclidean space (finite or infinite dimensional). For any $f : V \to R$*

$$\frac{1}{|V|}\sum_{v\in V}\|f(v) - m\|_R^2 \leqslant \frac{k}{2\lambda}\operatorname{Lip}(f)^2,$$

where $m = \frac{1}{|V|}\sum_{v\in V} f(v)$.

This simply implies that an expanding sequence of graphs cannot admit coarse embeddings into Euclidean spaces. And this is the property which catches the attention of noncommutative geometers. See [9] for the relation with the Coarse Baum–Connes Conjecture.

Let M be a Hadamard manifold, i.e. M is a simply-connected complete Riemannian manifold with nonpositive sectional curvatures. For any $m \in M$, the exponential map $\exp_m : T_mM \to M$ is a diffeomorphism and the inverse of exponential map $\log_m : M \to T_mM$ is a Lipschitz map with Lipschitz constant 1.

For any $m \in M$ and $r > 0$, define

$$\omega_m(r) = \sup\bigl\{d_M\bigl(\exp_m(x), \exp_m(y)\bigr) \bigm| d(x, y) \leqslant r,\ x, y \in T_mM\bigr\}.$$

By comparison theorem, we have the following lemma.

Lemma 4. *Let M be a Hadamard manifold with bounded sectional curvatures. Then there exists $\omega(r)$ such that for any $m \in M$, $\omega_m(r) < \omega(r)$ for all $r > 0$ and $\lim_{r\to 0}\omega(r) = 0$.*

The point m which appears in Theorem 1 does exist.

Lemma 5. *(See [1].) Let M be a Hadamard manifold and $m_1, \ldots, m_n \in M$ be n points. Then there exists $m \in M$ such that $\sum_{i=1}^n \log_m(m_i) = 0$.*

To prove the main theorem, the following simple inequalities are useful.

Lemma 6. *Let (V,E) be a (k,λ)-expander and $f: V \to \mathbb{R}_+$. If $|\{v \in V \mid f(v) \leqslant R_0\}| \geqslant \frac{|V|}{2}$ then*

$$\frac{1}{|V|}\sum_{v\in V} f(v)^2 \leqslant 2R_0^2 + \frac{3k}{2\lambda}\operatorname{Lip}(f)^2.$$

Proof. Define $V_1 = \{v \in V \mid f(v) \leqslant R_0\}$ and $V_2 = V \setminus V_1$ with $|V_1| \geqslant \frac{|V|}{2}$. Let $n = |V|$ and $V_1 = \{v_1, \dots, v_k\}$ and $V_2 = \{v_{k+1}, \dots, v_n\}$, $k \geqslant \frac{n}{2}$ and $k \geqslant n-k$. Let $m = \frac{1}{|V|}\sum_{v\in V} f(v)$.

$$\begin{aligned}
\frac{1}{|V|}\sum_{v\in V} f(v)^2 &= \frac{1}{|V|}\sum_{v\in V_1} f(v)^2 + \frac{1}{|V|}\sum_{v\in V_2} f(v)^2 \\
&\leqslant \frac{1}{|V|}|V_1|R_0^2 + \frac{1}{|V|}\sum_{i=1}^{n-k}\bigl(f(v_{k+i}) - m - f(v_i) + m + f(v_i)\bigr)^2 \\
&\leqslant \frac{1}{|V|}|V_1|R_0^2 + \frac{1}{|V|}\sum_{i=1}^{n-k}3\bigl(\bigl(f(v_{k+i}) - m\bigr)^2 + \bigl(f(v_i) - m\bigr)^2 + f(v_i)^2\bigr) \\
&\leqslant \frac{1}{|V|}|V_1|R_0^2 + \frac{1}{|V|}\sum_{v\in V}3\bigl(f(v) - m\bigr)^2 + \frac{1}{|V|}3\sum_{i=1}^{n-k} f(v_i)^2 \\
&\leqslant \frac{1}{|V|}|V_1|R_0^2 + \frac{1}{|V|}\sum_{v\in V}3\bigl(f(v) - m\bigr)^2 + \frac{1}{|V|}3|V_2|R_0^2 \\
&= 2R_0^2 + \frac{3k}{2\lambda}\operatorname{Lip}(f)^2,
\end{aligned}$$

where $\frac{1}{|V|}\sum_{v\in V} 3(f(v)-m)^2 \leqslant \frac{3k}{2\lambda}\operatorname{Lip}(f)^2$ by Proposition 3 and $2|V_2| \leqslant |V|$. □

Lemma 7. *Let $\{x_i \in \mathbb{R}^r \mid i = 1, \dots, n\}$ and $m = \frac{1}{n}\sum_{i=1}^n x_i$. For any $x \in \mathbb{R}^r$,*

$$\sum_{i=1}^n \|x_i - m\|^2 \leqslant 4\sum_{i=1}^n \|x_i - x\|^2.$$

Proof. The proof consists of two simply computations.

$$\begin{aligned}
n^2\sum_{i=1}^n \|x_i - m\|^2 = \sum_{i=1}^n \Biggl\|\sum_{j=1}^n (x_i - x_j)\Biggr\|^2 &\leqslant \sum_{i=1}^n n\sum_{j=1}^n \|x_i - x_j\|^2 \\
&\leqslant n\sum_{i=1}^n\sum_{j=1}^n \|x_i - x_j\|^2,
\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\|x_i-x_j\|^2&=\sum_{i=1}^{n}\sum_{j=1}^{n}\big\|x_i-x-(x_j-x)\big\|^2\\&\leqslant\sum_{i=1}^{n}\sum_{j=1}^{n}2\big(\|x_i-x\|^2+\|x_j-x\|^2\big)\\&=4n\sum_{i=1}^{n}\|x_i-x\|^2.\end{aligned}$$

Hence

$$\sum_{i=1}^{n}\|x_i-m\|^2\leqslant 4\sum_{i=1}^{n}\|x_i-x\|^2. \qquad \square$$

Let (V,E) be a (k,λ)-expander and let M be a Hadamard manifold. For $f:V\to M$, define

$$\mathrm{Lip}(f)=\max\left\{\frac{d(f(v),f(w))}{d(v,w)}\;\Big|\;v,w\in V\right\}.$$

For $v\in V$, define $\tilde f(v)$ the unique point on the unique geodesic ray starting from m and through $f(v)$ with $d(\tilde f(v),m)=\frac{d(f(v),m)}{\mathrm{Lip}(f)+1}$. Then $\sum_{v\in V}\log_m(\tilde f(v))=\frac{1}{\mathrm{Lip}(f)+1}\sum_{v\in V}\log_m(f(v))=0$. Define $\alpha_v:[0,1]\to M$ such that the image of α_v is the geodesic connecting m and $f(v)$ with $\alpha_v(0)=m$, $\alpha_v(1)=f(v)$ and $d(\alpha_v(t),m)=td(m,f(v))$. Since the distance function of M is convex, we have $d(\alpha_v(t),\alpha_w(t))\leqslant td(f(v),f(w))$ for any $v,w\in V$. Hence $\mathrm{Lip}(\tilde f)<1$.

Now let us prove the main theorem.

Proof of Theorem 1. We prove the theorem by contradiction. Choose $R>0$ large enough so that $R^2\geqslant\frac{Nk}{\lambda}$ and $\omega(\frac{4k}{\lambda}(\frac{8}{R})^2)<\frac{1}{9}$ for some integer $N>2011$ to be determined, where ω is defined in Lemma 4. Suppose that there exists a map $f:V\to M$ such that $\frac{1}{|V|}\sum_{v\in V}d(f(v),m)^2>R^2(\mathrm{Lip}(f)+1)^2$. Without loss of generality, we assume that $\frac{1}{|V|}\sum_{v\in V}d(f(v),m)^2>R^2$ with $\mathrm{Lip}(f)<1$. Choose $R_0>0$ such that there exists $v_0\in V$ such that

$$\big|\big\{v\in V\colon d\big(f(v),f(v_0)\big)\leqslant R_0\big\}\big|\geqslant\frac{3}{4}|V|.$$

We assume that R_0 attains its infimum with v_0 under this condition. Define $\|f\|:V\to\mathbb{R}_+$ by $\|f\|(v)=\|\log_m(f(v))-\log_m(f(v_0))\|$ for all $v\in V$. Then $\mathrm{Lip}(\|f\|)\leqslant\mathrm{Lip}(f)<1$ and $\|f\|(v)\leqslant d(f(v),f(v_0))$. So by Lemma 6 and Lemma 7

$$\begin{aligned}R^2&<\frac{1}{|V|}\sum_{v\in V}d\big(f(v),m\big)^2=\frac{1}{|V|}\sum_{v\in V}\big\|\log_m\big(f(v)\big)\big\|^2\\&\leqslant\frac{4}{|V|}\sum_{v\in V}\big\|\log_m\big(f(v)\big)-\log_m\big(f(v_0)\big)\big\|^2\leqslant 8R_0^2+\frac{6k}{\lambda}.\end{aligned}$$

Hence $R_0^2 > \frac{N-6}{8N} R^2 > \frac{N-6}{8}\frac{k}{\lambda}$. Define

$$\tilde{f}(v) = \begin{cases} \exp_{f(v_0)}\big(\frac{\log_{f(v_0)}(f(v))}{R_0}\big) & \text{if } \log_{f(v_0)}(f(v)) \leqslant R_0, \\ \exp_{f(v_0)}\big(\frac{\log_{f(v_0)}(f(v))}{\|\log_{f(v_0)}(f(v))\|}\big) & \text{if } \log_{f(v_0)}(f(v)) > R_0. \end{cases}$$

Then $\mathrm{Lip}(\tilde{f}) \leqslant \frac{2}{R_0}$. Define $g: V \to T_{f(v_0)}M$ by $g = \log_{f(v_0)} \circ \tilde{f}$. Hence $\mathrm{Lip}\, g \leqslant \frac{2}{R_0} \leqslant \sqrt{\frac{8N}{N-6}}\frac{2}{R} \leqslant \frac{8}{R}$. So by Proposition 3

$$\frac{1}{|V|}\sum_{v\in V}\|g(v) - m_g\|^2 \leqslant \frac{k}{2\lambda}\left(\frac{8}{R}\right)^2 = \delta,$$

where $m_g = \frac{1}{|V|}\sum_{v\in V} g(v)$. It follows that for $A = \{v \in V\colon \|g(v) - m_g\| \leqslant 4\delta\}$, we have $|A| \geqslant \frac{3}{4}|V|$. Hence, for $A_1 = A \cap \{v \in V\colon d(f(v), f(v_0)) \leqslant R_0\}$, we have $|A_1| \geqslant \frac{|V|}{2}$. Choose $v_1 \in A_1$. Note that for $v \in A_1$, we have $\|g(v) - g(v_1)\| \leqslant 8\delta$ and hence $d(f(v), f(v_1)) \leqslant \omega_{\exp_{f(v_0)}}(8\delta)R_0 \leqslant \omega(8\delta)R_0 \leqslant \frac{R_0}{9}$ by the assumption on R and Lemma 4. It follows that

$$\sum_{v\in V} d\big(f(v), f(v_1)\big)^2 \leqslant \left(\frac{2}{81}R_0^2 + \frac{3k}{2\lambda}\right)|V| \leqslant \left(\frac{2}{81} + \frac{12}{N-6}\right)R_0^2|V|.$$

Choose a sufficiently large N (for example $N = 4000$) so that $(\frac{2}{81} + \frac{12}{N-6})R_0^2|V| \leqslant \frac{1}{36}R_0^2|V|$. Therefore, we get

$$\left|\left\{v \in V\colon d\big(f(v), f(v_1)\big) > \frac{1}{3}R_0\right\}\right| < \frac{1}{4}|V|,$$

i.e.

$$\left|\left\{v \in V\colon d\big(f(v), f(v_1)\big) \leqslant \frac{1}{3}R_0\right\}\right| \geqslant \frac{3}{4}|V|,$$

which contradicts the minimality of R_0. □

Let X, Y be two metric spaces. $\phi: X \to Y$ is said to be a *coarse embedding* if there exist two non-decreasing maps $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that for all $x, y \in X$

$$\rho_1\big(d_X(x, y)\big) \leqslant d_Y\big(\phi(x), \phi(y)\big) \leqslant \rho_2\big(d_X(x, y)\big)$$

and $\lim_{t\to\infty} \rho_1(t) = \infty$.

A sequence of metric space (X_n) is said to be *equi-coarsely embeddable* into a metric space Y if there exist $\phi_n : X_n \to Y$ for all $n \in \mathbb{N}$ and two non-decreasing maps $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that for all $x_n, y_n \in X_n$

$$\rho_1\big(d_{X_n}(x_n, y_n)\big) \leqslant d_Y\big(\phi_n(x_n), \phi_n(y_n)\big) \leqslant \rho_2\big(d_{X_n}(x_n, y_n)\big)$$

and $\lim_{t\to\infty} \rho_1(t) = \infty$. The sequence (ϕ_n) is called an equi-coarse embedding of (X_n) into Y.

As a simple application of Theorem 1, we have the following corollary.

Corollary 8. *An expanding sequence of graphs does not admit an equi-coarse embedding into Hadamard manifolds with bounded sectional curvature.*

See [7,8] for the relation with the Coarse Geometric Novikov Conjecture in this case.

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