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Full Length Research Paper

Two non-standard finite difference schemes for the Timoshenko beam

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In this paper, we derive two non-standard finite difference schemes for the Timoshenko beam problem. The schemes are uniform with respect to the thickness small parameter of the beam, thus, no locking phenomenon occurs. Numerical results are presented.

Key words: Timoshenko beam, non-standard difference scheme, locking phenomenon.

INTRODUCTION

In this paper, we consider the following Timoshenko beam model. According to the Timoshenko beam theory, the in-plane bending of a clamped uniform beam of length *L*, cross section *A*, moment of inertia *I*, Young's modulus *E* and shear modulus *G*, subject to a distributed load $p(\bar{x})$ and a distributed moment $m(\bar{x})$, is governed by the following system of differential equations for $\bar{x} \in (0, L)$:

$$-\frac{dQ}{d\bar{x}} = p, \quad -EI\frac{d^2\theta}{d\bar{x}^2} - Q = 0, \qquad -\frac{Q}{\kappa GA} + \frac{dW}{d\bar{x}} - \theta = 0 ,$$

where $Q(\bar{x})$ is the shear force, $M(\bar{x})$ is the bending moment, $\theta(\bar{x})$ is the cross-sectional rotation, $W(\bar{x})$ is the transverse displacement and κ is the shear correction factor. The boundary conditions are,

$$W\ 0 = W\ L \ = 0;\ \theta\ 0 \ = \theta\ L \ = 0$$

We non-dimensionalize the problem by introducing the following change of variables:

$$x = \frac{\bar{x}}{L}, \quad w = \frac{W}{L}, \quad \sigma = \frac{QL^2}{EI}, \quad f = \frac{pL^3}{EI} ,$$

Then, the original problem is transformed to the following model problem. Find w, θ and σ such that in (0,1),

$$\begin{cases} -\sigma' = f \\ -\theta'' - \sigma = 0 \\ -\varepsilon^2\sigma + w' - \theta = 0 \end{cases} \tag{1}$$

or

$$\begin{cases} -\theta'' + \varepsilon^{-2}(\theta - w') = 0 \\ \varepsilon^{-2}(\theta' - w'') = f \end{cases} \tag{2}$$

together with the boundary conditions:

$$w(0) = w(1) = 0; \quad \theta(0) = \theta(1) = 0 \tag{3}$$

The parameter $\varepsilon^2 = EI/(\kappa GAL^2)$ is a constant proportional to the ratio of the thickness to length of the beam. In most realistic applications $\varepsilon \le 1$.

The numerical approximation of the Timoshenko beam has been frequently used as a starting point for a better understanding of the much more complex Reissner-

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Mindlin plate problem. When solving these problems with the standard Galerkin finite element methods or finite difference methods, some bad behaviors may occur such as the locking phenomenon (Arnold, 1981).

To construct the numerical scheme uniform with respect to the small parameter, Loula et al. (1987a), proposed a Petrov-Galerkin formulation, Arnold (1981), Cheng and Xue (2002), Cheng et al. (1997) and Loula et al. (1987b) used the reduced integration which is equivalent to the mixed formulation. Jou and Yang (2000) discussed the least-squares finite element method and Li (1990) applied the p and $h-p$ versions of the finite element method.

In this paper, we derive two non-standard finite difference schemes for the Timoshenko beam. Unlike the standard difference method, we do not use forward, backward or central difference to replace the derivative. We solve the differential equations in each subinterval (one element or two elements), then obtain the equations by some connecting conditions. Numerical experiments are presented for these two schemes.

THE NON-STANDARD DIFFERENCE SCHEME

For simplicity, we consider the uniform partition of (0, 1). Let $h=1/N$ for some positive integer N. Denote $x_i = ih,\ i=1,2,\ldots,N$ and $x_{i+\frac{1}{2}} = \frac{1}{2}\left(x_i + x_{i+1}\right)$

Scheme I

In element $\Delta_i = \left[x_{i-1}, x_i\right]$, we let $f \equiv f_{i-\frac{1}{2}} = f(x_{i-\frac{1}{2}})$ be a constant. Then, we solve Equations 1 or 2 in Δ_i.

$$\begin{cases} -\sigma' = f_{i-\frac{1}{2}} \\ -\theta'' - \sigma = 0 \\ -\varepsilon^2\sigma + w' - \theta = 0 \end{cases} \tag{4}$$

We can obtain the solution with four constants,

$$\begin{cases} \sigma(x) = c_1^{(i)} - f_{i-\frac{1}{2}}\left(x - x_{i-1}\right) \\ \theta(x) = -\frac{1}{2}c_1^{(i)}\left(x-x_{i-1}\right)^2 + c_2^{(i)}\left(x-x_{i-1}\right) + c_3^{(i)} + \frac{1}{6}f_{i-\frac{1}{2}}\left(x-x_{i-1}\right)^3 \\ w(x) = -\frac{1}{6}c_1^{(i)}\left(x-x_{i-1}\right)^3 + \frac{1}{2}c_2^{(i)}\left(x-x_{i-1}\right)^2 + \left(c_1^{(i)}\varepsilon^2 + c_3^{(i)}\right)\left(x-x_{i-1}\right) \\ \qquad + c_4^{(i)} - \frac{1}{2}\varepsilon^2 f_{i-\frac{1}{2}}\left(x-x_{i-1}\right)^2 + \frac{1}{24}f_{i-\frac{1}{2}}\left(x-x_{i-1}\right)^4 \end{cases} \tag{5}$$

Applying the boundary conditions $\theta(x_{i-1}) = \theta_{i-1},\quad \theta(x_i) = \theta_i$; and $w(x_{i-1}) = w_{i-1},\quad w(x_i) = w_i$; we can derive:

$$\begin{cases} c_1^{(i)} = \frac{12}{h(12\varepsilon^2 + h^2)}\left(w_i - w_{i-1} - \frac{1}{2}h(\theta_{i-1}+\theta_i)\right) + \frac{1}{2}hf_{i-\frac{1}{2}} \\ c_2^{(i)} = \frac{6}{12\varepsilon^2 + h^2}\left(w_i - w_{i-1} - \frac{1}{2}h(\theta_{i-1}+\theta_i)\right) + \frac{1}{h}(\theta_{i-1}+\theta_i) + \frac{1}{12}h^2 f_{i-\frac{1}{2}} \\ c_3^{(i)} = \theta_{i-1} \\ c_4^{(i)} = w_{i-1} \end{cases} \tag{6}$$

From the first continuity condition,

$$\lim_{x\to x_i^-}\theta'(x) = \lim_{x\to x_i^+}\theta'(x)$$

we obtain the equation:

$$-hc_1^{(i)} + c_2^{(i)} + \frac{1}{2}h^2 f_{i-\frac{1}{2}} = c_2^{(i+1)} \tag{7}$$

From the second continuity condition,

$$\lim_{x\to x_i^-} w'(x) = \lim_{x\to x_i^+} w'(x)$$

we get:

$$-\frac{1}{2}h^2c_1^{(i)} + hc_2^{(i)} + \varepsilon^2 c_1^{(i)} + c_3^{(i)} - h\varepsilon^2 f_{i-\frac{1}{2}} + \frac{1}{6}h^3 f_{i-\frac{1}{2}} = c_1^{(i+1)}\varepsilon^2 + c_3^{(i+1)} \tag{8}$$

Thus, we obtain the linear system from Equations 7 and 8 for $i=1,2,\ldots,N-1$,

$$\begin{cases} \frac{12}{12\varepsilon^2+h^2}\left(\frac{\theta_{i-1}+2\theta_i+\theta_{i+1}}{4} - \frac{w_{i+1}-w_{i-1}}{2h}\right) - \frac{\theta_{i-1}-2\theta_i+\theta_{i+1}}{h^2} = \frac{1}{12}h\left(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}\right) \\ \qquad \frac{12}{12\varepsilon^2+h^2}\left(\frac{\theta_{i+1}-\theta_{i-1}}{2h} - \frac{w_{i-1}-2w_i+w_{i+1}}{h^2}\right) = \frac{1}{2}\left(f_{i+\frac{1}{2}} + f_{i-\frac{1}{2}}\right) \end{cases} \tag{9}$$

and with the boundary conditions:

$$\theta_0 = \theta_N = 0, \qquad w_0 = w_N = 0 \tag{10}$$

The scheme of Equations 8 and 9 is similar to the one derive from the Petrov-Galerkin formulation of Loula et al. (1987a). It is also the scheme which we apply the difference method to the Equation 2 with shear dampening.

Scheme II

Let $\Delta^i = \; x_{i-1}, x_{i+1}$, we let $f \equiv \overline{f}_i = \frac{1}{2}\; f \;\; x_{i-\frac{1}{2}} \; + f \;\; x_{i+\frac{1}{2}}$ be a constant. Then, we solve the Equations 1 or 2 in Δ^i ,

$$\begin{cases} -\sigma' = \overline{f_i} \\ -\theta'' - \sigma' = 0 \\ -\varepsilon^2\sigma + w' - \theta = 0 \end{cases} \quad (11)$$

Again, we can obtain the solution with four constants,

$$\begin{cases} \sigma(x) = d_1^{(i)} - \overline{f_i} \;\; x - x_{i-1} \\ \theta(x) = -\frac{1}{2}d_1^{(i)} \;\; x - x_{i-1} \;\; {}^2 + d_2^{(i)} \;\; x - x_{i-1} \;\; + d_3^{(i)} + \frac{1}{6}\overline{f_i} \;\; x - x_{i-1} \;\; {}^3 \\ w(x) = -\frac{1}{6}d_1^{(i)} \;\; x - x_{i-1} \;\; {}^3 + \frac{1}{2}d_2^{(i)} \;\; x - x_{i-1} \;\; {}^2 + \;\; d_1^{(i)}\varepsilon^2 + d_3^{(i)} \;\; x - x_{i-1} \\ \qquad + d_4^{(i)} - \frac{1}{2}\varepsilon^2\overline{f_i} \;\; x - x_{i-1} \;\; {}^2 + \frac{1}{24}\overline{f_i} \;\; x - x_{i-1} \;\; {}^4 \end{cases} \quad (12)$$

Applying the boundary conditions $\theta \;\; x_{i-1} \; = \theta_{i-1}, \; \theta \;\; x_{i+1} \; = \theta_{i+1}$ and $w \;\; x_{i-1} \; = w_{i-1}, \; w \;\; x_{i+1} \; = w_{i+1}$, we can derive:

$$\begin{cases} d_1^{(i)} = \frac{3}{h \;\; 63\varepsilon^2 + 2h^2} \;\; w_{i+1} - w_{i-1} - h(\theta_{i-1} + \theta_{i+1}) \;\; + h\overline{f_i} \\ d_2^{(i)} = \frac{3}{6\varepsilon^2 + 2h^2} \;\; w_{i+1} - w_{i-1} - h(\theta_{i-1} + \theta_{i+1}) \;\; + \frac{1}{2h}(\theta_{i+1} - \theta_{i-1}) + \frac{1}{3}h^2\overline{f_i} \\ d_3^{(i)} = \theta_{i-1} \\ d_4^{(i)} = w_{i-1} \end{cases} \quad (13)$$

Then, from Equations 12 and 13, we have the relations:

$$\theta_i = \theta(x_i) = -\frac{1}{2}hd_1^{(i)} + hd_2^{(i)} + d_3^{(i)} + \frac{1}{6}h^3\overline{f_i}$$

$$w_i = w(x_i) = -\frac{1}{6}h^3\overline{f_i} + \frac{1}{2}hd_2^{(i)} + h(\varepsilon^2 d_1^{(i)} + d_3^{(i)}) + d_4^{(i)} - \frac{1}{2}\varepsilon^2 h^2\overline{f_i} + \frac{1}{24}h^4\overline{f_i}$$

Thus, from Equation 13 and continuity conditions, we obtain the system for $i = 1, 2, \ldots, N-1$:

$$\begin{cases} \frac{3}{3\varepsilon^2 + h^2}\left(\frac{\theta_{i-1} + \theta_{i+1}}{2} - \frac{w_{i+1} - w_{i-1}}{2h}\right) - \frac{\theta_{i-1} - 2\theta_i + \theta_{i+1}}{h^2} = 0 \\ \frac{12}{12\varepsilon^2 + h^2}\left(\frac{\theta_{i+1} - \theta_{i-1}}{2h} - \frac{w_{i-1} - 2w_i + w_{i+1}}{h^2}\right) = \overline{f_i} \end{cases} \quad (14)$$

with the boundary conditions:

$$\theta_0 = \theta_N = 0, \qquad w_0 = w_N = 0 \quad (15)$$

We can see that the second equation of Equation 14 is the same as the one in Equation 9, but the first equation of Equation 14 is slightly different with the first equation of Equation 9.

Next we give some error estimates. First, we have the following priori estimates.

Theorem 1

Let θ, w be the solutions of Equations 1 or 2 and 3, then there exists a constant C independent ε such that:

$$\|\theta\|_\infty + \|w\|_\infty \le C\|f\|_\infty \quad (16)$$

where $\|\cdot\|_\infty$ is the uniform norm on the interval (0, 1).

Proof

By the first equation of Equation 1, we have:

$$-\sigma(x) = -F(x) + C_1, \qquad F(x) = \int_0^x f(t)\,dt$$

By the second Equation of Equation 1 and the boundary conditions of Equation 3, we get:

$$\theta(x) = -\frac{1}{2}C_1x^2 + C_1x + G(x)$$

with

$$G(x) = \int_0^x \int_0^s F(t)\,dt \;\; ds - \int_0^1 \int_0^s F(t)\,dt \;\; ds$$

From the third equation of Equation 1, we obtain:

$$w(x) = -\frac{1}{6}C_1x^3 + \frac{1}{4}C_1x^2 + \varepsilon^2 C_1 x - \varepsilon^2\int_0^x F(t)\,dt + \int_0^x G(t)\,dt$$

Applying the conditions *w*(0) = *w*(1) = 0, we can derive:

$$C_1 = \frac{\varepsilon^2\int_0^1 F(t)\,dt - \int_0^1 G(t)\,dt}{\varepsilon^2 + \frac{1}{12}}$$

Thus for $0 < \varepsilon < 1$, we can verify:

Table 1. The case f = 1; h = 1=10.

		x = 0.1	x = 0.5	x = 0.8
	Scheme I :: $\theta(x)/w(x)$	0.0060/0.0116	0.0000/0.0339	–0.0080/0.0211
$\varepsilon = 0.5$	Scheme II:: $\theta(x)/w(x)$	0.0060/0.0116	0.0000/0.0339	–0.0080/0.0211
	Exact solution	0.0060/0.0116	0.0000/0.0339	–0.0080/0.0211
	Scheme I:: $\theta(x)/w(x)$	0.0060/0.0008	0.0000/0.0039	–0.0080/0.0019
$\varepsilon = 0.1$	Scheme II:: $\theta(x)/w(x)$	0.0060/0.0008	0.0000/0.0039	–0.0080/0.0019
	Exact solution	0.0060/0.0008	0.0000/0.0039	–0.0080/0.0019
	Scheme I :: $\theta(x)/w(x)$	0.0060/0.0003	0.0000/0.0026	–0.0080/0.0011
$\varepsilon = 0.01$	Scheme II:: $\theta(x)/w(x)$	0.0060/0.0003	0.0000/0.0026	–0.0080/0.0011
	Exact solution	0.0060/0.0003	0.0000/0.0026	–0.0080/0.0011

$$\|F\|_\infty \le C\|f\|_\infty, \qquad \|G\|_\infty \le C\|f\|_\infty, \qquad |C_1| \le C\|f\|_\infty$$

The proof is completed.

Denote the piecewise constant function $f_h(x)$ from *f*(*x*) by $f_h(x) = f(x_{i-\frac{1}{2}})$, $x_{i-1} < x < x_i$. Consider the equations:

$$\begin{cases} -\tilde{\theta}'' + \varepsilon^{-2}\left(\tilde{\theta} - \tilde{w}'\right) = 0 \\ \varepsilon^{-2}\left(\tilde{\theta}' - \tilde{w}''\right) = f_h \end{cases} \qquad (17)$$

Together with the boundary conditions,

$$\tilde{w}(0) = \tilde{w}(1) = 0, \qquad \tilde{\theta}(0) = \tilde{\theta}(1) = 0 \qquad (18)$$

we can see that, $\tilde{\theta}(x_i) = \tilde{\theta}_i$, $\tilde{w}(x_i) = \tilde{w}_i$, $i = 0,1,2,\ldots,N$ for the solutions of Equations 9 and 10.

As $\theta(x) - \tilde{\theta}(x)$ and $w(x) - \tilde{w}(x)$ are the solution of Equations 2 and 3 with the replacement *f*(*x*) by $f(x) - f_h(x)$, then we apply the Theorem 1 to obtain:

$$\max_i |\theta(x_i) - \theta_i| + \max_i |w(x_i) - w_i| \le C\|f - f_h\|_\infty$$

It is the uniform bounds with respect to the small parameter ε.

For Scheme **II**, we can not derive the similar bounds of errors but numerical experiments show its uniform bounds with respect to parameter.

NUMERICAL EXPERIMENTS

We first consider the uniform load case *f* = 1; 0 < *x* < 1. The exact solutions are,

$$\theta(x) = \frac{1}{12}x(1-x)(1-2x)$$

$$w(x) = \frac{1}{24}x^2(1-x)^2 + \frac{1}{2}\varepsilon^2 x(1-x)$$

For both schemes the solution should be exact as shown in Table 1.

We let $f(x) = 100(e^x + x)$ as the second example. We do not know the exact solution but we give the approximated solutions for different mesh and ε. We see that no locking phenomenon occurs (Tables 2 and 3).

Conclusion

We consider the numerical approximation of a Timoshenko beam with boundary feedback, and present the non-standard finite difference schemes with a uniform mesh without presenting the problem of the locking phenomenon. These schemes approximate the functions $\theta(x)$ and $w(x)$ under values of *x* and ε. We may easily see the approximate schemes in Table 1.

Table 2. The case $f(x)=100(e^x+x)$, ε = 0.1.

		x = 0.1	x = 0.5	x = 0.8
$h=1/10$	Scheme I: $\theta(x)/w(x)$	1.2214/0.1469	0.1989/0.8400	–1.7749//0.4508
	Scheme II: $\theta(x)/w(x)$	1.2210/0.1471	0.1957/0.8403	–1.7782/0.4508
$h=1/20$	Scheme I: $\theta(x)/w(x)$	1.2210/0.1468	0.1988/0.8398	–1.7745/0.4507
	Scheme II: $\theta(x)/w(x)$	1.2210/0.1469	0.1980/0.8398	–1.7753/0.4507
$h=1/40$	Scheme I : $\theta(x)/w(x)$	1.2210/0.1468	0.1988/0.8397	–1.7744/0.4507
	Scheme II: $\theta(x)/w(x)$	1.2209/0.1468	0.1988/0.8397	–1.7746/0.4507
$h=1/80$	Scheme I: $\theta(x)/w(x)$	1.2209/0.1468	0.1988/0.8397	–1.7739/0.4507
	Scheme II: $\theta(x)/w(x)$	1.2209/0.1468	0.1987/0.8397	–1.7739//0.4507

Table 3. The case $f(x)=100(e^x+x)$, $h=1/10$.

		x = 0.1	*x* = 0.5	*x* = 0.8
$\varepsilon=0.1$	Scheme I:: $\theta(x)/w(x)$	1.2214/0.1469	0.1989/0.8400	–1.7749//0.4508
	Scheme II:: $\theta(x)/w(x)$	1.2210/0.1471	0.1957/0.8403	–1.7782/0.4508
$\varepsilon=0.01$	Scheme I:: $\theta(x)/w(x)$	1.2001/0.0674	0.1397/0.5697	–1.8128/0.2519
	Scheme II:: $\theta(x)/w(x)$	1.2007/0.0674	0.1395/0.5700	–1.8141/0.2520
$\varepsilon=0.001$	Scheme I :: $\theta(x)/w(x)$	1.2001/0.0666	0.1391/0.5670	–1.8131/0.2499
	Scheme II:: $\theta(x)/w(x)$	1.2005/0.0666	0.1389/0.5673	–1.8145/0.2500
$\varepsilon=0.0001$	Scheme I :: $\theta(x)/w(x)$	1.1998/0.0666	0.1391/0.5670	–1.8132/0.2499
	Scheme II:: $\theta(x)/w(x)$	1.2005/0.0666	0.1389/0.5673	–1.8145/0.2500

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