BOUNDS FOR THE ZEROS OF POLYNOMIALS

W. M. Shah

(*Bemina College, India*)

A.Liman

(*College Srinagar, India*)

Received Sept. 3, 2002

Abstract

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n. *In this paper we prove a more general result which interalia improves upon the bounds of a class of polynomials. We also prove a result which includes some extensions and generalizations of Eneström-Kakeya theorem.*

Key words *polynomials, bounds, Eneström-Kakeya theorem*

AMS(2000)subject classification 30A06, 30A64

1 Introduction and Statement of Results

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, then by a classical result due to Cauchy[1] all the zeros of $P(z)$ lie in

$$|z| \leq 1 + M, \tag{1}$$

where

$$M = \max_{1 \leq j \leq n-1} |\frac{a_j}{a_n}|.$$

There are several improvements and generalizations of this result (for reference see [7] and [8]). As an improvement of this result, Joyal, Labelle and Rahman[6] proved the following

Theorem A. *If* $B = \max_{1 \leq j \leq n-1} |a_j|$, *then all the zeros of the polynomials* $P(z) = z^n +$

$\sum_{j=0}^{n-1} a_j z^j$ *are contained in the circle*

$$|z| \leq \frac{1}{2}\left[1 + |a_{n-1}| + \{(1 - |a_{n-1}|)^2 + 4B\}^{1/2}\right]. \tag{2}$$

Recently Dewan[2] obtained the following ring shaped region containing all the zeros of $P(z)$.

Theorem B. *Let* $P(z) = z^n + \sum_{j=0}^{n-1} a_j z^j$ *be a polynomial of degree* n *and* $B = \max_{0\leq j\leq n-1} |a_j|$, *then all the zeros* $P(z)$ *lie in the ring shaped region given by*

$$R_2 \leq |z| \leq R_1, \tag{3}$$

here

$$R_1 = \frac{1}{2}\left[1 + |a_{n-1}| + \{(1 - |a_{n-1}|)^2 + 4B\}^{1/2}\right],$$

$$R_2 = \frac{1}{2M_1^2}\left[-R_1^2|b|(M_1 - |a_0|) + \{R_1^4|b|^2(M_1 - |a_0|)^2 + 4|a_0|R_1^2 M_1^3\}^{1/2}\right],$$

where

$$M_1 = R_1^n[R_1 + 1 + 2|a_{n-1}| + (2n-3)B],$$

$$b = a_1 - a_0.$$

Mohammad[9] used Schwartz Lemma and obtained the following upper bound for the absolute values of the zeros of $P(z)$.

Theorem C. *All zeros of the polynomial* $P(z) = \sum_{j=0}^{n} a_j z^j$ *lie in* $|z| \leq \frac{M}{|a_n|}$, *if* $|a_n| \leq M$, *where*

$$M = \max_{|z|=1} |a_{n-1}z^{n-1} + \cdots + a_0| = \max_{|z|=1} |a_0 z^{n-1} + \cdots + a_{n-1}|.$$

The aim of this paper is to prove a more general result which not only improves upon the above results, but also a variety of interesting results can be deduced from it by a fairly uniform procedure. In fact, we prove

Theorem 1. *Let* $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ *be a polynomial of degree* n. *If*

$$\max_{|z|=R} |a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z| \leq M_1 \tag{4}$$

and

$$\max_{|z|=R} |a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z| \leq M_2, \tag{5}$$

then all the zeros of $P(z)$ *lie in the ring shaped region*

$$r_1 \leq |z| \leq r_2, \tag{6}$$

where

$$r_1 = \frac{1}{2M_1^2}\left[-R^2|a_1|(M_1 - |a_0|) + R\{(M_1 - |a_0|)^2R^2|a_1|^2 + 4M_1^3|a_0|\}^{1/2}\right], \tag{7}$$

$$r_2 = \frac{|a_{n-1}|}{2}\left\{\frac{1}{|a_n|} - \frac{1}{M_2}\right\} + \left[\frac{M_2}{|a_n|R^2} + \frac{|a_{n-1}|^2}{4}\left\{\frac{1}{|a_n|} - \frac{1}{M_2}\right\}^2\right]^{1/2}. \tag{8}$$

From Theorem 1, we immediately have the following

Corollary 1. *If $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ is a polynomial of degree n such that*

$$\max_{|z|=R} |a_nz^n + a_{n-1}z^{n-1} + \cdots a_1z| \leq |a_0|,$$

then $P(z)$ does not vanish in $|z| < R$.

Corollary 2. *If $P(z) = \sum_{j=0}^{n} a_jz^j$ is a polynomial of degree n such that*

$$\max_{|z|=R} |a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z| \leq |a_n|,$$

then $P(z)$ has all its zeros in $|z| < \frac{1}{R}$.

Corollary 3. *If $P(z) = a_0 + \sum_{j=2}^{n} a_jz^j$ is a polynomial of degree n such that*

$$\max_{|z|=R} |a_nz^n + \cdots + a_2z^2| \leq M_1,$$

then $P(z)$ does not vanish in

$$|z| < R\sqrt{\frac{|a_0|}{M_1}}.$$

Corollary 4. *If $P(z) = a_nz^n + \sum_{j=0}^{n-2} a_jz^j$ is a polynomial of degree n such that*

$$\max_{|z|=R} |a_0z^n + a_1z^{n-1} + \cdots + a_{n-2}z^2| \leq M_2,$$

then $P(z)$ has all its zeros in

$$|z| < \frac{1}{R}\sqrt{\frac{M_2}{|a_n|}}.$$

Next we use Theorem 1 to prove the following result, which is an extension of a theorem due to Mohammad[9].

Theorem 2. *If $P(z) = \sum_{j=0}^{n} a_jz^j$ and*

$$\max_{|z|=R} |a_0z^{n-1} + a_1z^{n-2} + \cdots + a_{n-1}| \leq M, \tag{9}$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leq \max\left\{\frac{M}{|a_n|}, \frac{1}{R}\right\}. \tag{10}$$

If we take $R = 1$ in Theorem 2, we get the following

Corollary 5. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *and*

$$\max_{|z|=1} |a_0 z^{n-1} + a_1 z^{n-2} + \cdots + a_{n-1}| = \max_{|z|=1} |a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0| \leq M,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leq \max\left\{\frac{M}{|a_n|}, 1\right\}. \tag{11}$$

For $|a_n| \leq M$, Corollary 5 reduces to the result of Mohammad[9].

Remark 1. Theorem 1 gives better bounds than Theorems A,B and C. To illustrate this, we consider the polynomial

$$P(z) = z^3 + a_2 z^2 + a_1 z + a_0,$$

where

$$|a_2| = 0.3, \qquad |a_1| = 0.7 \quad \text{and} \quad |a_0| = 0.7.$$

The following table gives various bounds

S.No	Theorems	Bounds
1	Theorem A	$\|z\| < 1.55$
2	Theorem B	$0.19 \leq \|z\| \leq 1.55$
3	Theorem C	$\|z\| < 1.7$
4	Theorem 1 with $R = 1$	$0.48 < \|z\| < 1.36$

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq \cdots \geq a_0 > 0,$$

then according to a famous result due to Enström and Kakeya, the polynomial $P(z)$ does not vanish in $|z| > 1$. In the literature [7,8], there exist some extensions and generalizations of Enström-Kakeya theorem. Govil and Rahman[4] generalized this theorem to the polynomial with complex coefficients. While refining the result of Govil and Rahman, Govil and Jain[3] proved the following result.

Theorem D. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial with complex coefficients such that for some* β,

$$|\arg a_k-\beta|\le\alpha<\pi/2,\qquad k=0,1,2,\cdots,n$$

and

$$|a_n|\ge|a_{n-1}|\ge\cdots\ge|a_1|\ge|a_0|,$$

then all the zeros of $P(z)$ *lie in the ring-shaped region given by*

$$R_3\le|z|\le R_2, \tag{12}$$

here

$$R_2=\frac{c}{2}\left\{\frac{1}{|a_n|}-\frac{1}{M_1}\right\}+\left\{\frac{c^2}{2}\left(\frac{1}{|a_n|}-\frac{1}{M_1}\right)^2+\frac{M_1}{|a_n|}\right\}^{1/2}$$

and

$$R_3=\frac{1}{2M_2^2}\left[-R_2^2|b|(M_2-|a_0|)+\{4|a_0|R_2^2M_2^3+R_2^4|b|^2(M_2-|a_0|)^2\}^{1/2}\right],$$

where

$$M_1=|a_n|R,$$
$$M_2=|a_n|R_2^2\left[R+R_2-\frac{|a_0|}{|a_n|}(\cos\alpha+\sin\alpha)\right],$$
$$C=|a_n-a_{n-1}|,$$
$$b=a_1-a_0,$$

and

$$R=\cos\alpha+\sin\alpha+\frac{2\sin\alpha}{|a_n|}\sum_{k=0}^{n-1}|a_k|.$$

We now use Theorem 1 to prove a more general result, which among other well known results include Theorem D as a special case.

Theorem 3. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree* n. *If*

$$\max_{|z|=R}|-a_nz^{n+1}+(a_n-a_{n-1})z^n+\cdots+(a_1-a_0)z|\le M_3, \tag{13}$$

and

$$\max_{|z|=R}|a_0z^{n+1}+(a_1-a_0)z^n+\cdots+(a_n-a_{n-1})z|\le M_4, \tag{14}$$

then all the zeros of $P(z)$ *lie in*

$$r_3\le|z|\le r_4, \tag{15}$$

where

$$r_3 = \frac{1}{2M_3^2}\left[R\{R^2|a_1-a_0|^2(M_3-|a_0|)^2+4M_3^3|a_0|\}^{1/2}-R^2|a_1-a_0|(M_3-|a_0|)\right], \tag{16}$$

and

$$r_4 = \frac{|a_n-a_{n-1}|}{2}\left(\frac{1}{|a_n|}-\frac{1}{M_4}\right)+\left\{\frac{|a_n-a_{n-1}|^2}{4}\left(\frac{1}{|a_n|}-\frac{1}{M_4}\right)^2+\frac{M_4}{|a_n|R^2}\right\}^{1/2}. \tag{17}$$

Suppose $P(z)=\sum_{j=0}^{n} a_j z^j$ satisfies the conditions of Enström-Kakeya theorem, that is, $a_n \geq a_{n-1} \geq \cdots \geq a_0 \geq 0$, then for $R=1$, we have

$$\begin{aligned}
&\max_{|z|=1}|-a_nz^{n+1}+(a_n-a_{n-1})z^n+\cdots+(a_1-a_0)z| \\
&\leq |a_n|+|a_n-a_{n-1}|+\cdots+|a_1-a_0| \\
&= a_n+a_n-a_{n-1}+\cdots+a_1-a_0 \\
&= 2a_n-a_0 = M_3,
\end{aligned}$$

and

$$\begin{aligned}
&\max_{|z|=1}|a_nz^{n+1}+(a_1-a_0)z^n+\cdots+(a_n-a_{n-1})z| \\
&\leq |a_0|+|a_1-a_0|+\cdots+|a_n-a_{n-1}| \\
&= a_0+a_1-a_0+\cdots+a_n-a_{n-1} \\
&= a_n = M_4.
\end{aligned}$$

Using these observations in (16) and (17), we have

$$r_3 = \frac{-(a_1-a_0)(a_n-a_0)+\{(a_1-a_0)^2(a_n-a_0)^2+a_0(2a_n-a_0)^3\}^{\frac{1}{2}}}{2(2a_n-a_0)^3}$$

and

$$r_4 = 1.$$

Consequently, we have the following

Corollary 6. *If $P(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_0 \geq 0,$$

then all the zeros of $P(z)$ lie in

$$\lambda \leq |z| \leq 1,$$

where

$$\lambda = \frac{-(a_1 - a_0)(a_n - a_0) + \{(a_1 - a_0)^2(a_n - a_0)^2 + a_0(2a_n - a_0)^3\}^{\frac{1}{2}}}{(2a_n - a_0)^3}.$$

We also show that Theorem D follows from Theorem 3 as a special case. For this, suppose that $P(z) = \sum_{j=0}^{n} a_j z^j$ satisfies the conditions of Theorem D. Since (for reference see [5])

$$|a_j - a_{j-1}| \le \{(|a_j| - |a_{j-1}|)\cos\alpha + (|a_j| + |a_{j-1}|)\sin\alpha\},$$

we have from (14) for $R = 1$

$$\begin{aligned}
&\max_{|z|=1} |a_0 z^{n+1} + (a_1 - a_0)z^n + \cdots + (a_n - a_{n-1})z| \\
&\le |a_0| + \sum_{j=1}^{n} |a_j - a_{j-1}| \\
&\le |a_0| + \sum_{j=1}^{n}(|a_j| - |a_{j-1}|)\cos\alpha + \sum_{j=1}^{n}(|a_j| + |a_{j-1}|)\sin\alpha \\
&= |a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0|(\cos\alpha + \sin\alpha - 1) \\
&\le |a_n|(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \\
&= |a_n| r = M_4,
\end{aligned}$$

say, where

$$r = \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{j=0}^{n-1} |a_j|.$$

From (17), we have with $R = 1$,

$$r_4 = \frac{|a_n - a_{n-1}|}{2}\left(\frac{1}{|a_n|} - \frac{1}{M_4}\right) + \left\{\frac{|a_n - a_{n-1}|^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{M_4}\right)^2 + \frac{M_4}{|a_n|}\right\}^{\frac{1}{2}}.$$

Clearly $r_4 \ge 1$ and by similar argument as above it follows that

$$\begin{aligned}
&\max_{|z|=r_4} |-a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z| \\
&\le a_n r_4^{n+1} + r_4^n \sum_{j=1}^{n} |a_j - a_{j-1}| \\
&\le |a_n| r_4^n \left\{ r_4 + r - \frac{|a_0|}{|a_n|}(\cos\alpha + \sin\alpha) \right\} = M_3,
\end{aligned}$$

say. Now, from (17) for R=1, we get $r_4 = R_2$ and from (16) for $R = R_2$, we get $r_3 = R_3$. Consequently it follows by Theorem 3 that all the zeros of $P(z)$ lie in

$$R_3 \le |z| \le R_2,$$

which is equivalently the conclusion of Theorem B. Many other such results can easily fallow from Theorem 3 by a foirly similarly procedure.

2 Lemmas

For the proofs of these theorems we need the following lemmas. The first lemma is due to Govil, Rahman and Schmeisser[5].

Lemma 1. *If $f(z)$ is analytic in $|z| \le 1, f(0) = a, |a| < 1, f'(0) = b, |f(z)| \le 1$ on $|z| = 1$, then for $|z| \le 1$,*

$$|f(z)| \le \frac{(1-|a_1|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}.$$

From Lemma 1, one can easily deduce the following

Lemma 2. *If $f(z)$ is analytic in $|z| \le R, f(0) = 0, f'(0) = b$, and $|f(z)| \le M$ for $|z| = R$, then*

$$|f(z)| \le \frac{M|z|}{R^2} \cdot \frac{M|z| + R^2|b|}{M + |b||z|} \qquad \text{for} \quad |z| \le R.$$

3 Proofs of the Theorems

Proof of Theorem 1. We have

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + a_n z^n,$$

so that

$$|P(z)| \ge |a_0| - |Q(z)|, \tag{18}$$

where

$$Q(z) = a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

Clearly, $Q(0) = 0, Q'(0) = a_1$ and by (4) $|Q(z)| \le M_1$ for $|z| = R$.

Therefore, it follows by Lemma 2

$$|Q(z)| \le \frac{M_1|z| + R^2|a_1|}{M + |a_1||z|} \cdot \frac{M_1|z|}{R^2}, \qquad \text{for} \quad |z| = R. \tag{19}$$

Using (19) in (18), we have

$$\begin{aligned} |P(z)| &\ge |a_0| - \frac{M_1|z| + R^2|a_1|}{M_1 + |a_1||z|} \cdot \frac{M_1|z|}{R^2} \\ &= \frac{|a_0|R^2M_1 + R^2|a_0||a_1||z| - M_1^2|z|^2 - M_1R^2|a_1||z|}{R^2(M_1 + |a_1||z|)} \\ &> 0, \end{aligned}$$

if

$$M_1^2|z|^2 + (M_1 - |a_0|)|a_1|R^2|z| - |a_0|R^2M_1 < 0.$$

That is, $|Q(z)| > 0$, if

$$|z| < \frac{R\{(M_1 - |a_0|)^2|a_1|^2R^2 + 4|a_0|M_1^3\}^{1/2} - (M_1 - |a_0||a_1|R^2)}{2M_1^2} = r_1,$$

by (7), which implies that all zeros of $P(z)$ lie in

$$|z| \geq r_1. \tag{20}$$

Again, let

$$G(z) = z^nP(1/z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n$$

so that

$$|G(z)| \geq |a_n| - |H(z)|, \tag{21}$$

where

$$H(z) = a_{n-1}z + a_{n-2}z^2 + \cdots + a_0z^n.$$

Clearly, $H(0) = 0, H'(0) = a_{n-1}$, and by (5) $|H(z)| \leq M_2$ for $|z| = R$. By using Lemma 2, it follows that

$$|H(z)| \leq \frac{M_2|z|}{R^2} \cdot \frac{M_2|z| + R^2|a_{n-1}|}{M_2 + |a_{n-1}||z|}, \qquad \text{for } |z| \leq R. \tag{22}$$

Combining inequalities (21) and (22), we have

$$|G(z)| \geq \frac{M_2R^2|a_n| + R^2|a_n||a_{n-1}||z| - M_2^2|z|^2 - M_2R^2|a_{n-1}||z|}{R^2(M_2 + |a_{n-1}||z|)} > 0,$$

if

$$M_2^2|z|^2 + R^2|a_{n-1}|(M_2 - |a_n|)|z| - M_2R^2|a_n| < 0,$$

which gives

$$|G(z)| > 0$$

if

$$|z| < \frac{R\{4M_2^3|a_n| + |a_{n-1}|^2R^2(M_2 - |a_n|)^2\}^{1/2} - R^2|a_{n-1}|(M_2 - |a_n|)}{2M_2^2}.$$

This shows that all the zeros of $G(z)$ lie in

$$|z| \geq \frac{R\{4M_2^3|a_n| + |a_{n-1}|^2R^2(M_2 - |a_n|)^2\}^{1/2} - R^2|a_{n-1}|(M_2 - |a_n|)}{2M_2^2}.$$

Since $P(z) = z^nG(1/z)$, it follows that all the zeros of $P(z)$ lie in

$$|z| \leq \frac{2M_2^2}{R\{4M_2^3|a_n| + |a_{n-1}|^2R^2(M_2 - |a_n|)^2\}^{1/2} - R^2|a_{n-1}|(M_2 - |a_n|)}.$$

A simple calculation shows that

$$\frac{2M_2^2}{R\{4M_2^3|a_n| + |a_{n-1}|^2R^2(M_2 - |a_n|)^2\}^{1/2} - R^2|a_{n-1}|(M_2 - |a_n|)}$$
$$= \frac{R\{4M_2^3|a_n| + |a_{n-1}|R^2(M_2 - |a_n|)^2\}^{1/2} + |a_{n-1}|R^2(M_2 - |a_n|)}{2|a_n|M_2R^2}$$
$$= \frac{|a_{n-1}|}{2}\left\{\frac{1}{|a_n|} - \frac{1}{M_2}\right\} + \left[\frac{M_2}{|a_n|R^2} + \frac{|a_{n-1}|^2}{4}\left\{\frac{1}{|a_n|} - \frac{1}{M_2}\right\}^2\right]^{1/2}$$
$$= r_2,$$

by (8). This shows that all the zeros of $P(z)$ lie in

$$|z| \le r_2. \tag{23}$$

Combining (20) and (23), the desired result follows.

Proof of Theorem 2. We have from (9) and (5)

$$\max_{|z|=R} |a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z| \le MR = M_2,$$

say. Replacing M_2 by MR in (8), we obtain

$$r_2 = \frac{|a_{n-1}|}{2}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right) + \left(\frac{M}{|a_n|R} + \frac{|a_{n-1}|^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right)^2\right]^{1/2}. \tag{24}$$

Suppose, first that $|a_n| \le MR$, then

$$\frac{1}{|a_n|} - \frac{1}{MR} \ge 0.$$

Since $|a_{n-1}| \le M$, therefore, we have

$$M\frac{|a_{n-1}|}{|a_n|}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right) \le \frac{M^2}{|a_n|}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right).$$

Equivalently,

$$\frac{M}{|a_n|R} \le \frac{M^2}{|a_n|^2} - \frac{M|a_{n-1}|}{|a_n|}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right). \tag{25}$$

Adding $\frac{|a_{n-1}|^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right)^2$ both sides of (25), we get

$$\frac{M}{|a_n|R} + \frac{|a_{n-1}|^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right)^2 \le \left[\frac{M}{|a_n|} - \frac{|a_{n-1}|}{2}\left\{\frac{1}{|a_n|} - \frac{1}{MR}\right\}\right]^2,$$

or equivalently

$$\frac{|a_{n-1}|}{2}\left\{\frac{1}{|a_n|} - \frac{1}{MR}\right\} + \left[\frac{M}{|a_n|R} + \frac{|a_{n-1}|^2}{4}\left(\frac{1}{|a_n|} - \frac{1}{MR}\right)^2\right]^{1/2} \le \frac{M}{|a_n|}.$$

Therefore, in this case we have with the help of (24), that

$$r_2 \leq \frac{M}{|a_n|}.$$

Hence, by Theorem 1, it follows that all the zeros of $P(z)$ lie in

$$|z| \leq \frac{M}{|a_n|}. \tag{26}$$

Again, if $|a_n| > MR$, then it easily follows from (9) that

$$|a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z| < |a_n| \qquad \text{for } |z| = R. \tag{27}$$

By using Rouche's theorem, it follows from (27) that $G(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ does not vanish in $|z| < R$ which in otherwords implies that the polynomial $P(z) = z^n G(1/z)$ does not vanish in $|z| > 1/R$. Thus in this case all that zeros of $P(z)$ lie in

$$|z| \leq 1/R. \tag{28}$$

From (26) and (28) it follows that all the zeros of $P(z)$ lie in

$$|z| \leq \max\left\{\frac{M}{|a_n|}, \frac{1}{R}\right\}.$$

This proves Theorem 2 completely.

Proof of Theorem 3. Consider the polynomial

$$F(z) = (1-z)P(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0. \tag{29}$$

Applying Theorem 1 to $F(z)$ which is a polynomial of degree $n+1$ and noting that every zero of $P(z)$ is also a zero of $F(z)$, the proof of theorem 3 follows.

References

[1] Cauchy, A.L., Exercises de Mathematique, In: Cuvres, 9:2(1829), 122.

[2] Dewan, K.K., On the Location of Zeros of Polynomials, Mathematics Student, 50(1982), 170-175.

[3] Govil, N,K. and Jain, V.K., On the Enström-Kakey Theorem II, J. Approx. Theory, 22(1978), 1-10.

[4] Govil, N,K. and Rahman, Q.I., On the Eneström-Kekeya Theorem, Tohokv Math. J., 20(1968), 126-136.

[5] Govil, N,K., Rahman, Q.I. and Schmesser, G., On the Derivative of a Polynomial, Illinois. Math. Joxs, 23(1979), 319-329.

[6] Joyal, A., Labelle, G. and I-Rahman, Q., On the Location of Zeros of Polynomials, Canad. Math. Bull., 10(1967), 53-63.

[7] Mardon, M., Geometry of Polynomials, 2^{nd} ed. Mathematical Survey's No.3 , Amer. Math. Soc., Providence, RI, 1966.

[8] Milovanovic, G.V., Mitrinovic, D.S. and Rassias, Th.M., Topics in Polynomial Extremal Problems, Inequalities, Zeros.

[9] Mohammad, Q.G., On the Zeros of Polynomials, Amer. Math. Monthly., 72(1965), 631-633.

W. M. Shah

Department of Mathematics

Bemina College, Srinagar,

Kashmir, 190012,

India

A. Liman

Department of Mathematics

Regional Engineering,

College Srinagar, Kashmir

190006, India