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On Eneström–Kakeya theorem and related analytic functions

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Abstract. We prove some extensions of the classical results concerning the Eneström–Kakeya theorem and related analytic functions. Besides several consequences, our results considerably improve the bounds by relaxing and weakening the hypothesis in some cases.

Keywords. Polynomials; zeros; Eneström–Kakeya theorem; analytic functions.

1. Introduction

The following elegant result due to Eneström–Kakeya (p. 136 of [9]) (see also [10]) is well-known in the theory of distribution of the zeros of polynomials.

Theorem A. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0, \tag{1}$$

then $P(z)$ has all its zeros in $|z| \leq 1$.

In the literature [1–10], there exist some extensions and generalizations of this theorem. Joyal *et al* [8] extended this theorem to the polynomials whose coefficients are monotonic but not necessarily non-negative. Whereas, Govil and Rahman [7] extended it to a polynomial with complex coefficients by proving the following result:

Theorem B. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients such that for some real β,*

$$|\arg\, a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad 0 \leq j \leq n$$

and

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then $P(z)$ has all its zeros in the disk

$$|z| \leq (\sin\alpha + \cos\alpha) + \frac{2\sin\alpha}{|a_n|}\sum_{j=0}^{n-1}|a_j|.$$

Recently Aziz and Zargar [4] relaxed the hypothesis of Theorem A and proved the following extension of Eneström–Kakeya theorem.

Theorem C. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n such that for some $k \geq 1$,*

$$ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0, \tag{2}$$

then all the zeros of $P(z)$ lie in $|z + k - 1| \leq k$.

Aziz and Mohammad [2] extended Eneström–Kakeya theorem to the class of analytic functions $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$, with its coefficients a_j satisfying a relation analogous to (1) and proved the following theorem.

Theorem D. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$ be analytic in $|z| \leq t$. If $a_j > 0$ and $a_{j-1} - ta_j \geq 0$, $j = 1, 2, 3, \ldots$ then $f(z)$ does not vanish in $|z| < t$.*

Recently Aziz and Shah [3] relaxed the hypothesis of Theorem D and proved the following:

Theorem E. *Let $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$ be analytic in $|z| \leq t$, such that for some $k \geq 1$,*

$$ka_0 \geq ta_1 \geq t^2 a_2 \geq \ldots,$$

then $f(z)$ does not vanish in $\left|z - \left(\frac{k-1}{2k-1}\right) t\right| < \frac{kt}{2k-1}$.

In this paper we wish to weaken the hypothesis of Theorems C and E by considering a larger class of polynomials and obtain some extensions of the classical results concerning the Eneström–Kakeya theorem and related analytic functions. Besides, we considerably improve the bounds in some cases by relaxing the hypothesis in several ways. We first prove the following.

Theorem 1. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients, such that for some real β,*

$$|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots, n$$

and $k \geq 1$,

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ lie in

$$|z + k - 1| \leq \frac{1}{|a_n|}\left\{(k|a_n| - |a_0|)(\sin\alpha + \cos\alpha) + |a_0| + 2\sin\alpha \sum_{j=0}^{n-1} |a_j|\right\}. \tag{3}$$

We may apply Theorem 1 to the polynomial $P(tz)$ to obtain the following:

COROLLARY 1

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients, such that for some real β,

$$|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots, n,$$

$k \geq 1$ and $t > 0$,

$$kt^n|a_n| \geq t^{n-1}|a_{n-1}| \geq \cdots \geq t|a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ lie in

$$|z + kt - t| \leq \frac{t}{|a_n|}\left\{\left(k|a_n| - \frac{|a_0|}{t^n}\right)(\sin\alpha + \cos\alpha) + \frac{|a_0|}{t^n} + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|t^{j-n}\right\}.$$

For $\alpha = \beta = 0$, Corollary 1 reduces to a result of Aziz and Zargar (Corollary 1 of [4]) and for $k = 1$, it gives a result of Aziz and Mohammad [2].

The next result is obtained by taking $k = \left|\frac{a_n}{a_{n-1}}\right| \geq 1$ in Theorem 1.

COROLLARY 2

If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial with complex coefficients, such that for some real β,

$$|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots, n$$

and

$$|a_n| \leq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ lie in

$$\left|z + \left|\frac{a_n}{a_{n-1}}\right| - 1\right| \leq \frac{1}{|a_n|}\left\{(|a_{n-1}| - |a_0|)(\sin\alpha + \cos\alpha) + |a_0| + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|\right\}.$$

It is natural to ask, what happens, if instead of a_j, $j = 0, 1, 2, \ldots, n$ only $\operatorname{Re} a_j$ or $\operatorname{Im} a_j$ satisfy the property (2). In this direction, we prove the following:

Theorem 2. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re} a_j = \alpha_j$ and $\operatorname{Im} a_j = \beta_j$ for $j = 0, 1, 2, \ldots, n$, such that for some $k \geq 1$,*

$$k\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$
$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\left|z + \frac{\alpha_n}{a_n}(k-1)\right| \leq \frac{1}{|a_n|}\{k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}. \tag{4}$$

Or equivalently,

$$\left|z + \frac{\alpha_n}{a_n}(k-1)\right| \leq \sqrt{1+k^2} + \frac{1}{|a_n|}(|\alpha_0| - \alpha_0). \tag{4'}$$

The result of Aziz and Zargar (Theorem 2 of [4]) follows from Theorem 2, if we assume $\beta_j = 0$, and the result of Joyal *et al* (Theorem 3 of [8]) is a special case of this result when $k = 1$ and $\beta_j = 0$, $j = 0, 1, 2, \ldots, n$.

As a generalization of Theorem C, we prove as follows:

Theorem 3. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$, $j = 0, 1, 2, \ldots, n$, *and for some* $k \geq 1$,

$$k\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_\lambda \geq \alpha_{\lambda-1} \geq \cdots \geq \alpha_1 \geq \alpha_0, \quad 0 \leq \lambda \leq n-1$$
$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{1}{|a_n|}\{2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}. \tag{5}$$

We may apply Theorem 3 to the polynomial $P(tz)$ to obtain the following result.

COROLLARY 3

Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$, $j = 0, 1, 2, \ldots, n$, *and for some* $k \geq 1$,

$$kt^n\alpha_n \leq t^{n-1}\alpha_{n-1} \leq \cdots \leq t^\lambda\alpha_\lambda \geq t^{\lambda-1}\alpha_{\lambda-1} \geq \cdots \geq t\alpha_1 \geq \alpha_0, \quad 0 \leq \lambda \leq n-1$$
$$t^n\beta_n \geq t^{n-1}\beta_{n-1} \geq \cdots \geq t\beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{t\alpha_n}{a_n}(k-1) \right| \leq \frac{t}{|a_n|}\left\{2\frac{\alpha_\lambda}{t^{n-\lambda}} - k\alpha_n + \frac{1}{t^n}(|\alpha_0| - \alpha_0) + \beta_n\right\}.$$

A result of Dewan and Bidkham (Theorem 1 of [5]) is a special case of Corollary 3 when $k = 1$and $\beta_j = 0$, $j = 0, 1, 2, \ldots, n$.

As a generalization of Corollary 2, we also prove the following result which considerably improves the bound of Theorem B.

Theorem 4. *Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial with complex coefficients, such that for some real β,*

$$|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots, n$$

and for some λ, $0 \leq \lambda \leq n-1$,

$$|a_n| \leq |a_{n-1}| \leq \cdots \leq |a_\lambda| \geq |a_{\lambda-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|}\left\{2|a_\lambda|\cos\alpha - |a_{n-1}|(\cos\alpha - \sin\alpha) + 2\sin\alpha \right.$$
$$\left. \times \sum_{j=0}^{n-1}|a_j| - |a_0|(\cos\alpha + \sin\alpha - 1)\right\}. \tag{6}$$

For $\alpha = \beta = 0$, this reduces to a result of Aziz and Zargar (Theorem 4 of [4]).

Finally, we turn to the study of the zeros of a class of related analytic functions and here we prove the following generalization of Theorem E.

Theorem 5. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$, *be analytic in* $|z| < t$. *If for some* $k \geq 1$,

$$k|a_0| \geq t|a_1| \geq t^2|a_2| \geq \cdots$$

and for some β,

$$|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots,$$

then $f(z)$ *does not vanish in*

$$\left| z - \frac{(k-1)t}{M^2 - (k-1)^2} \right| < \frac{Mt}{M^2 - (k-1)^2}, \tag{7}$$

where

$$M = k(\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_0|} \sum_{j=1}^{\infty} |a_j| t^j.$$

For $\alpha = \beta = 0$, Theorem 5 reduces to Theorem E. Also Theorem D is a special case of Theorem 5 when $\alpha = \beta = 0$ and $k = 1$.

Theorem 6. *Let* $f(z) = \sum_{j=0}^{\infty} a_j z^j (\not\equiv 0)$, *be analytic in* $|z| < t$. *If for some* $k \geq 1$ *and* $\lambda > 0$,

$$k|a_0| \leq t|a_1| \leq \cdots \leq t^{\lambda}|a_{\lambda}| \geq t^{\lambda+1}|a_{\lambda+1}| \geq \cdots$$

and for some β,

$$|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, \ldots$$

then $f(z)$ *does not vanish in*

$$\left| z - \left(\frac{(k-1)t}{M^{*2} - (k-1)^2} \right) \right| \leq \frac{M^* t}{M^{*2} - (k-1)^2}, \tag{8}$$

where

$$M^* = \left\{ \left(\frac{2|a_{\lambda}|}{|a_0|} t^{\lambda} - k \right) \cos\alpha + k \sin\alpha + 2\frac{\sin\alpha}{|a_0|} \sum_{j=1}^{\infty} |a_j| t^j \right\}.$$

The result (Theorem 6 of [2]) due to Aziz and Mohammad follows from Theorem 6, if we take $k = 1$.

2. Lemma

For the proofs of these theorems, we need the following lemma, which is due to Govil and Rahman [7].

Lemma. If $|\arg\ a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, *and for some* $t > 0$, $|ta_j| \geq |a_{j-1}|$, *then*

$$|ta_j - a_{j-1}| \leq \{(|ta_j| - |a_{j-1}|)\cos\alpha + (|ta_j| + |a_{j-1}|)\sin\alpha\}.$$

3. Proofs of the theorems

Proof of Theorem 1. Consider a polynomial

$$\begin{aligned}F(z) &= (1-z)P(z)\\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0\\ &= -a_n z^n(z+k-1) + (ka_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1}\\ &\quad + \cdots + (a_1 - a_0)z + a_0.\end{aligned}$$

Let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1,\ 0 \le j < n$, and we have

$$\begin{aligned}|F(z)| &\ge |z|^n\Bigg[|a_n||z+k-1|\\ &\quad - \left\{|ka_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\right\}\Bigg]\\ &\ge |z|^n[|a_n||z+k-1| - \{|ka_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + \cdots\\ &\quad + |a_1 - a_0| + |a_0|\}].\end{aligned}$$

Using the Lemma, we get

$$\begin{aligned}|F(z)| &\ge |z|^n|a_n||z+k-1| - \left[\{k|a_n| - |a_0|\}(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|\right]\\ &> 0,\end{aligned}$$

if

$$|z+k-1| > \frac{\{k|a_n| - |a_0|\}(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|}{|a_n|}.$$

This shows that the zeros of $F(z)$ having modulus greater than 1 lie in

$$|z+k-1| \le \frac{1}{|a_n|}\{k|a_n| - |a_0|\}(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|.$$

But the zeros of $F(z)$ of modulus not greater than 1 already satisfy (3), and therefore all the zeros of $F(z)$ lie in the disk

$$|z+k-1| \le \frac{1}{|a_n|}\{k|a_n| - |a_0|\}(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha\sum_{j=0}^{n-1}|a_j|.$$

Since the zeros of $P(z)$ are also the zeros of $F(z)$, Theorem 1 is proved completely.

Proof of Theorem 2. Consider a polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z)\\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0\\
&= \{-\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0\}\\
&\quad + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \cdots + (\beta_1 - \beta_0)z + \beta_0\}\\
&= -\alpha_n z^n (z + k - 1) + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1}\\
&\quad + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n\\
&\quad + \cdots + (\beta_1 - \beta_0)z + \beta_0\}\\
&= -z^n\{(\alpha_n + i\beta_n)z + \alpha_n(k-1)\} + (k\alpha_n - \alpha_{n-1})z^n\\
&\quad + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0\\
&\quad + i\{(\beta_n - \beta_{n-1})z^n + \cdots + (\beta_1 - \beta_0)z + \beta_0\}.
\end{aligned}$$

Let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1, \quad j = 0, 1, 2, \ldots, n-1$. We have

$$\begin{aligned}
|F(z)| \geq |z|^n \Bigg[& |a_n z + \alpha_n(k-1)|\\
& - \left\{ |k\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \cdots + \frac{|\alpha_1 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} \right\}\\
& - \left\{ |\beta_n - \beta_{n-1}| + \frac{|\beta_{n-1} - \beta_{n-2}|}{|z|} + \cdots + \frac{|\beta_1 - \beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} \right\} \Bigg]\\
= |z|^n & \left[|a_n| \left| z + \frac{\alpha_n}{a_n}(k-1) \right| - \{k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\} \right] > 0,
\end{aligned}$$

if

$$\left| z + \frac{\alpha_n}{a_n}(k-1) \right| > \frac{\{k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}}{|a_n|}.$$

This shows that the zeros of $F(z)$ having modulus greater than 1 lie in

$$\left| z + \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{\{k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}}{|a_n|}.$$

But the zeros of $F(z)$ of modulus less than or equal to one already satisfy (4). Since all the zeros of $P(z)$ are also the zeros of $F(z)$, we conclude that all the zeros of $P(z)$ lie in the disk

$$\left| z + \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{\{k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}}{|a_n|}.$$

Using the fact that $k\alpha_n + \beta_n \leq \sqrt{1+k^2}|a_n|$, the equivalent statement also follows.

This completes the proof of Theorem 2.

Proof of Theorem 3. Consider a polynomial

$$
\begin{aligned}
F(z) &= (1-z)P(z) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0 \\
&= \{-\alpha_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0\} \\
&\quad + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n + \cdots + (\beta_1 - \beta_0)z + \beta_0\} \\
&= -\alpha_n z^n(z + k - 1) + (k\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\
&\quad + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{-\beta_n z^{n+1} + (\beta_n - \beta_{n-1})z^n \\
&\quad + \cdots + (\beta_1 - \beta_0)z + \beta_0\} \\
&= -z^n\{(\alpha_n + i\beta_n)z + \alpha_n(k-1)\} + (k\alpha_n - \alpha_{n-1})z^n \\
&\quad + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 \\
&\quad + i\{(\beta_n - \beta_{n-1})z^n + \cdots + (\beta_1 - \beta_0)z + \beta_0\}.
\end{aligned}
$$

Let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1$, $j = 0, 1, 2, \ldots, n-1$. We have

$$
\begin{aligned}
|F(z)| \geq |z|^n \Bigg[|a_n z + \alpha_n(k-1)| - \Bigg\{ |k\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1} - \alpha_{n-2}|}{|z|} + \cdots + \frac{|\alpha_{\lambda+1} - \alpha_\lambda|}{|z|^{n-\lambda-1}} \\
+ \frac{|\alpha_\lambda - \alpha_{\lambda-1}|}{|z|^{n-\lambda}} + \frac{|\alpha_{\lambda-1} - \alpha_{\lambda-2}|}{|z|^{n-\lambda+1}} + \cdots + \frac{|\alpha_1 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} \Bigg\} \\
- \Bigg\{ |\beta_n - \beta_{n-1}| + \frac{|\beta_{n-1} - \beta_{n-2}|}{|z|} + \cdots + \frac{|\beta_1 - \beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} \Bigg\} \Bigg] \\
= |z|^n \left[|a_n| \left| z + \frac{\alpha_n}{a_n}(k-1) \right| - \{2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\} \right] \\
> 0,
\end{aligned}
$$

if

$$
\left| z + \frac{\alpha_n}{a_n}(k-1) \right| > \frac{\{2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}}{|a_n|}.
$$

This shows that the zeros of $F(z)$ having modulus greater than 1 lie in

$$
\left| z + \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{\{2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}}{|a_n|}.
$$

But those zeros of $F(z)$ whose modulus is less than or equal to one already satisfy (5). Hence we conclude that all the zeros of $F(z)$ lie in the disk

$$
\left| z + \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{\{2\alpha_\lambda - k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\}}{|a_n|}.
$$

Since every zero of $P(z)$ is also a zero of $F(z)$, Theorem 3 is proved completely.

Proof of Theorem 4. Consider a polynomial

$$\begin{aligned} F(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0. \end{aligned}$$

Let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1$, $j = 0, 1, 2, \ldots, n-1$. By using the Lemma, we have

$$\begin{aligned} |F(z)| &\geq |a_n z^{n+1} + (a_n - a_{n-1})z^n| + |(a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_1 - a_0)z + a_0| \\ &\geq |z|^n |a_n z + (a_n - a_{n-1})| + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_1 - a_0||z| + |a_0| \\ &\geq |z|^n \Bigg[|a_n z + a_{n-1} - a_n| - \Bigg\{ \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_{\lambda+1} - a_\lambda|}{|z|^{n-\lambda-1}} \\ &\quad + \frac{|a_\lambda - a_{\lambda-1}|}{|z|^{n-\lambda}} + \frac{|a_{\lambda-1} - a_{\lambda-2}|}{|z|^{n-\lambda+1}} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Bigg\} \Bigg] \\ &\geq |z|^n \Bigg\{ \Bigg[|a_n z + a_{n-1} - a_n| - 2|a_\lambda| \cos\alpha - |a_{n-1}|(\cos\alpha - \sin\alpha) \\ &\quad + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0|(\cos\alpha + \sin\alpha - 1) \Bigg] \Bigg\} > 0, \end{aligned}$$

if,

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| > \frac{1}{|a_n|} \Bigg\{ 2|a_\lambda| \cos\alpha - |a_{n-1}|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0|(\cos\alpha + \sin\alpha - 1) \Bigg\}.$$

This shows that the zeros of $F(z)$ having modulus greater than 1 lie in

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \Bigg\{ 2|a_\lambda| \cos\alpha - |a_{n-1}|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0|(\cos\alpha + \sin\alpha - 1) \Bigg\}.$$

But the zeros of $F(z)$ of modulus not greater than 1 already satisfy (6). Hence we conclude that all the zeros of $F(z)$ lie in the disk

$$\left| z + \frac{a_{n-1}}{a_n} - 1 \right| \leq \frac{1}{|a_n|} \Bigg\{ 2|a_\lambda| \cos\alpha - |a_{n-1}|(\cos\alpha - \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| - |a_0|(\cos\alpha + \sin\alpha - 1) \Bigg\}.$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, Theorem 4 is proved completely.

Proof of Theorem 5. Consider

$$\begin{aligned} F(z) &= (z-t)f(z) \\ &= -ta_0 + (a_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots + (a_{n-1} - ta_n)z^n + \cdots \\ &= -a_0\{(k-1)z + t\} + (ka_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots \\ &\quad + (a_k - ta_{k+1})z^{k+1} + \cdots + (a_{n-1} - ta_n)z^n + \cdots \\ &= -a_0\{(k-1)z + t\} + G(z). \end{aligned} \tag{9}$$

Since $|\arg\, a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0, 1, 2, \ldots,$ therefore by the Lemma, for $|z| = t$,

$$\begin{aligned} |G(z)| &= |(ka_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots + (a_{n-1} - ta_n)z^n + \cdots| \\ &\le |ka_0 - ta_1||z| + |a_1 - ta_2||z|^2 + \cdots + |a_{n-1} - ta_n||z|^n + \cdots \\ &\le t[(k|a_0| - |ta_1|)\cos\alpha + (k|a_0| + |ta_1|)\sin\alpha \\ &\quad + (|ta_1| - |t^2 a_2|)\cos\alpha + (|ta_1| + |t^2 a_2|)\sin\alpha + \cdots] \\ &= t|a_0|\left[k(\cos\alpha + \sin\alpha) + \frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j\right] = t|a_0|M, \end{aligned}$$

where

$$M = k(\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j.$$

Since $G(0) = 0$, it follows by Schwarz's lemma that

$$|G(z)| \le t|a_0|M \quad \text{for } |z| \le t.$$

This gives, with the help of (9),

$$\begin{aligned} |F(z)| &\ge |a_0\{(k-1)z + t\}| - |G(z)| \\ &\ge |a_0|\{|(k-1)z + t| - |z|M\} > 0, \end{aligned}$$

if

$$|z|M < |(k-1)z + t|.$$

Since, it is easy to verify that the region defined by

$$\{z\colon |z|M < |(k-1)z + t|\}$$

is precisely the disk

$$\left\{z\colon \left|z - \left(\frac{(k-1)t}{M^2 - (k-1)^2}\right)\right| < \frac{Mt}{M^2 - (k-1)^2}\right\},$$

we conclude that $F(z)$ and hence $f(z)$ does not vanish in the disk defined by (7). This proves Theorem 5 completely.

Proof of Theorem 6. Consider

$$
\begin{aligned}
F(z) &= (z-t)f(z) \\
&= -ta_0 + (a_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots + (a_{n-1} - ta_n)z^n + \cdots \\
&= -a_0\{(k-1)z + t\} + (ka_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots \\
&\quad + (a_k - ta_{k+1})z^{k+1} + \cdots + (a_{n-1} - ta_n)z^n + \cdots \\
&= -a_0\{(k-1)z + t\} + G(z). \qquad (10)
\end{aligned}
$$

Since $|\arg\ a_j - \beta| \le \alpha \le \frac{\pi}{2}$, $j = 0, 1, 2, \ldots$, therefore by the lemma and the hypothesis, for $|z| = t$,

$$
\begin{aligned}
|G(z)| &= |(ka_0 - ta_1)z + (a_1 - ta_2)z^2 + \cdots + (a_\lambda - ta_{\lambda+1})z^{\lambda+1} \\
&\quad + \cdots + (a_{n-1} - ta_n)z^n + \cdots| \\
&\le t[(|ta_1| - k|a_0|)\cos\alpha + (|ta_1| + k|a_0|)\sin\alpha \\
&\quad + (|t^2a_2| - |ta_1|)\cos\alpha + (|t^2a_2| + |ta_1|)\sin\alpha + \cdots \\
&\quad + (|t^\lambda a_\lambda| - |t^{\lambda-1}a_{\lambda-1}|)\cos\alpha + (|t^\lambda a_\lambda| + |t^{\lambda-1}a_{\lambda-1}|)\sin\alpha \\
&\quad + (|t^\lambda a_\lambda| - |t^{\lambda+1}a_{\lambda+1}|)\cos\alpha + (|t^\lambda a_\lambda| + |t^{\lambda+1}a_{\lambda+1}|)\sin\alpha \\
&\quad + \cdots + (|t^{n-1}a_n| - |t^n a_n|)\cos\alpha \\
&\quad + (|t^{n-1}a_n| + |t^n a_n|)\sin\alpha + |a_n t^n| + \cdots] \\
&= t|a_0|\left[\left(2\left|\frac{a_\lambda}{a_0}\right| t^\lambda \cos\alpha\right) + k\sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j\right] \\
&= t|a_0|M^*,
\end{aligned}
$$

where

$$
M^* = \left(2\left|\frac{a_\lambda}{a_0}\right| t^\lambda \cos\alpha\right) + k\sin\alpha + \frac{2\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j.
$$

Since $G(0) = 0$, it follows by Schwarz's lemma that

$$
|G(z)| \le t|a_0|M^* \quad \text{for } |z| \le t.
$$

This gives, with the help of (10),

$$
\begin{aligned}
|F(z)| &\ge |a_0\{(k-1)z + t\}| - |G(z)| \\
&\ge |a_0|\{|(k-1)z + t| - |z|M^*\} \\
&> 0,
\end{aligned}
$$

if

$$
|z|M^* < |(k-1)z + t|.
$$

Since the region defined by

$$\{z\colon |z|M^* < |(k-1)z+t|\}$$

is precisely the disk

$$\left\{z\colon \left|z-\left(\frac{(k-1)t}{M^{*2}-(k-1)^2}\right)\right| < \frac{M^*t}{M^{*2}-(k-1)^2}\right\},$$

we conclude that $F(z)$ and therefore $f(z)$ does not vanish in the disk defined by (8). This proves Theorem 6 completely.

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