The notion of complete reducibility in group theory

Lecture 1

Let Γ be a group. We will discuss *linear representations* of Γ over some fixed field k of characteristic $p \geq 0$. By this we mean a group homomorphism $\Gamma \to \mathrm{GL}(V)$ for some finite dimensional vector space V over k. We will usually refer to V instead as a Γ-module, though of course technically we should say $k[\Gamma]$-module where $k[\Gamma]$ denotes the group algebra of Γ over k. Recall that V is *irreducible* or *simple* if:

(1) $V \neq 0$;
(2) no subspace of V is Γ-stable apart from 0 and V.

One says that V is *completely reducible* or *semisimple* if V is a direct sum of irreducible submodules; equivalently, V is semisimple if V is generated by irreducible submodules.

The category of semisimple Γ-modules is stable under the usual operations of linear algebra. In other words one can take Γ-stable subspaces, quotients, direct sums and duals all within this category. Indeed, all of these statements (apart from dual spaces) are true for modules over an arbitrary ring. But when we consider groups, we can also consider the operations of multilinear algebra. For instance, given two Γ-modules V_1, V_2 we can impose a Γ-module structure upon $V_1 \otimes V_2$ using the diagonal map $\Gamma \to \Gamma \times \Gamma$. From this we can construct exterior powers, symmetric powers, etc....

Around 1950, Chevalley proved the following simple looking result:

Theorem 1. (cf. [C]) *Suppose that k has characteristic 0. If V_1, V_2 are semisimple Γ-modules, then $V_1 \otimes V_2$ is again semisimple.*

An interesting feature of this result is that, although it is stated in elementary terms, the only known proofs involve some algebraic geometry. We sketch the idea. One starts with a series of reductions, reducing to the case that k is algebraically closed and Γ is a subgroup of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Then one replaces Γ by its Zariski closure in $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. So now Γ is an algebraic group. The connected component Γ° of Γ containing the identity is a normal subgroup of Γ of finite index (this is one bonus of using the Zariski topology). In other words, Γ/Γ° is a finite group and since the characteristic is 0, one easily then reduces to the case that $\Gamma = \Gamma^\circ$. So now, Γ is connected. Let $R^u\Gamma$ be the unipotent radical of Γ, i.e. its largest normal unipotent subgroup. In any semisimple representation, $R^u\Gamma$ acts trivially, and the converse is known to be true in characteristic zero. Since V_1 and V_2 are semisimple and the representation of Γ on $V_1 \oplus V_2$ is faithful, we deduce that $R^u\Gamma$ is trivial, and we are done.

Now we ask what happens for $p > 0$. Chevalley's result does not remain true in general. For instance, consider $\Gamma = \mathrm{SL}_2(k) = \mathrm{SL}(V)$ with $\dim V = 2$. Let $\mathrm{Sym}^n(V)$ be n^{th} symmetric power of V. If $n < p$ then $\mathrm{Sym}^n(V)$ is an irreducible representation of Γ. But if $n = p$, the subspace $V^{[1]} \subset \mathrm{Sym}^p(V)$ generated by x^p and y^p, where $\{x, y\}$ is any basis of V, is stable under the action of Γ. This gives a short exact sequence

$$0 \to V^{[1]} \to \mathrm{Sym}^p(V) \to L \otimes \mathrm{Sym}^{p-2}(V) \to 0$$

where $L = \det V$ is one-dimensional. This sequence does not split (unless $p = |k| = 2$). So $\mathrm{Sym}^p(V)$ is not semisimple in general. Hence, $V \otimes \ldots \otimes V$ (p times) is not semisimple either.

Now a general principle is that if a statement is true in characteristic zero then it is also true for "large" p. In keeping with this, we have the following:

Theorem 2. ([S1]) *Let $V_1, \ldots, V_n$ be semisimple Γ-modules. Then*

$$V_1 \otimes \ldots \otimes V_n \textit{ is semisimple if } p > \sum_{i=1}^{n} (\dim V_i - 1).$$

The proof again uses a reduction to algebraic group theory. As above we may assume that k is algebraically closed, the representation $\Gamma \to \mathrm{GL}(V)$ is faithful and Γ is a closed subgroup of $\mathrm{GL}(V)$ in the Zariski topology, where $V = V_1 \oplus \ldots \oplus V_n$. But we can no longer reduce to the case that Γ is connected. Indeed, if Γ is finite of order divisible by p, this assumption will be no help at all. So we need to do more. We need Γ to be *saturated*.

To define this notion (cf. [N],[S1]), suppose that $x \in \mathrm{GL}_n(k)$ has order p. Write $x = 1 + \varepsilon$ for some matrix ε and note that $\varepsilon^p = 0$. For any $t \in k$ define $x^t := 1 + t\varepsilon + \binom{t}{2}\varepsilon^2 + \ldots + \binom{t}{p-1}\varepsilon^{p-1}$. Since $\varepsilon^p = 0$ we have constructed a one parameter subgroup $\{x^t \mid t \in k\}$ of $\mathrm{GL}_n(k)$. By definition, a subgroup $\Gamma \subset \mathrm{GL}_n(k)$ is said to be *saturated* if it is Zariski closed and $x \in \Gamma$ with $x^p = 1$ implies that $x^t \in \Gamma$ for all $t \in k$. One can define the *saturated closure* of a subgroup Γ denoted by Γ^{sat}. It is the smallest saturated subgroup of $\mathrm{GL}_n(k)$ containing Γ.

Here are some examples:

- If $p > 2$ every classical group in its natural representation is saturated.
- If $p > 3$ the group $G_2(k)$, embedded in $\mathrm{GL}_7(k)$, is saturated.

- If $p = 2$ the group $\mathrm{PGL}_2(k)$, embedded in $\mathrm{GL}_3(k)$ by its adjoint representation, is not saturated.
- If $p = 11$ the Janko group J_1, embedded in $\mathrm{GL}_7(k)$, has for saturated closure the group $G_2(k)$.

It can be checked that our problem is stable under replacing Γ by Γ^{sat}. So, we may assume that Γ is saturated. This implies that Γ/Γ° is finite of order prime to p, so we can reduce as before to the case where Γ is a connected reductive algebraic group. Then we resort to the general theory of representations of algebraic groups to complete the proof, which is somewhat technical. (cf. [S1])

One can also ask about various converse theorems (cf. [S2]). For instance:

(1) Does $V_1 \otimes V_2$ semisimple imply V_2 semisimple?
(2) Does $\bigwedge^2 V$ semisimple imply V semisimple?
(3) Does $\mathrm{Sym}^2 V$ semisimple imply V semisimple?

For question (1) the answer in characteristic zero is yes unless $\dim V_1 = 0$. In characteristic $p > 0$, the answer is yes unless $\dim V_1 = 0$ in k, i.e. unless $\dim V_1 \equiv 0 \pmod p$.

For question (2) the answer in characteristic zero is yes unless $\dim V = 2$. In characteristic $p > 0$ the answer is yes unless $\dim V \equiv 2 \pmod p$.

For question (3) the answer is yes in characteristic zero, while in characteristic $p > 0$ the answer is yes unless $\dim V \equiv -2 \pmod p$.

Remarks. These questions make sense more generally in the setting of a "tensor category", cf. [D]. Such a category has tensor products and duals, as well as a distinguished object $\underline{1}$. There is the notion of dimension of an object: consider the composition of the natural maps

$$\underline{1} \to V \otimes V^* \to \underline{1}.$$

This determines an element of $k = \mathrm{End}(\underline{1})$, which is called the dimension of V. In particular it is possible for the dimension to be -2 in k. In this formalism, there is a way of transforming symmetric powers into exterior powers, by changing categories. Deligne noticed that if one proves in this setting one of the two statements:

$$\textstyle\bigwedge^2 V \text{ semisimple} \Rightarrow V \text{ semisimple if } \dim V \neq 2 \text{ in } k$$
$$\mathrm{Sym}^2 V \text{ semisimple} \Rightarrow V \text{ semisimple if } \dim V \neq -2 \text{ in } k$$

then the other is true as well (cf. [S2],§6.2). (Here k is assumed to be of characteristic not equal to 2.)

W. Feit has provided various counterexamples showing that the results are essentially the 'best possible' for questions (1) and (2), (cf. [S2], appendix). The situation is different for question (3). For instance, with $p = 7$ there is no known example in which $\mathrm{Sym}^2 V$ is semisimple but V is not.

We turn now to giving a generalization of the notion of complete reducibility (cf. [T2]). Let k be algebraically closed, G be a connected, reductive algebraic k-group and $\Gamma \subset G(k)$. I shall say that Γ is *G-completely reducible* (G-cr for short) if for every parabolic subgroup P of $G(k)$ containing Γ there exists a Levi subgroup of P, also containing Γ.

The definition of G-cr may be reformulated within the context of Tits buildings (cf. [T1]). The Tits building of G is the simplicial complex X, with simplices corresponding to the parabolic subgroups of $G(k)$ and inclusions being reversed. The group $G(k)$ acts simplicially on X. So if $\Gamma \subset G(k)$, we can consider the complex X^Γ of all Γ-fixed points. One can prove that there are precisely two possibilities:

(1) X^Γ is contractible (homotopy type of a point);
(2) X^Γ has the homotopy type of a bouquet of spheres.

One can show that (2) occurs precisely when Γ is G-cr.

The property of Γ being G-cr relates nicely to the usual property of a Γ-module being semisimple. If we take G to be $\mathrm{GL}(V)$ for some vector space V, it is clear that Γ is G-cr if and only if V is a semisimple Γ-module. More generally, if $p \neq 2$ and G is any symplectic group, orthogonal group, or G_2 then Γ is G-cr if and only if the natural representation of $G(k)$ is a semisimple Γ-module. We would like in a general setting, given $\Gamma \subset G(k)$ and a linear representation V of $G(k)$, to relate the property "Γ is G-cr" to the property that V is a semisimple Γ-module (for p larger that some bound $n(V)$). This will be discussed in the later lectures.

Finally, we give an application of these ideas. The Dynkin diagram of D_4 has a symmetry of order 3 which gives rise to an automorphism τ of Spin_8. Consequently, there are three irreducible modules for Spin_8 of dimension 8, say V_1, V_2, and V_3. Suppose that Γ is a subgroup of Spin_8. Is it true that:

$$V_1 \text{ is } \Gamma\text{-semisimple} \Rightarrow V_2 \text{ and } V_3 \text{ are } \Gamma\text{-semisimple?}$$

The answer is yes if $p > 2$ (and sometimes no if $p = 2$): this follows from the fact that V_i is Γ-semisimple if and only if Γ is Spin_8-cr.

Lecture 2

Fix an algebraically closed field k and let G be a connected, reductive algebraic k-group. We are interested only in the case where $p = \operatorname{char} k > 0$. Recall that a subgroup $\Gamma \subset G(k)$ is called G-cr if for every parabolic subgroup P of $G(k)$ containing Γ, there exists a Levi subgroup of P also containing Γ. We wish to relate this to the usual notion of complete reducibility.

Let T be a maximal torus of G, and B be a Borel subgroup containing T with U its unipotent radical. This determines a root system and a set of positive roots. Let $X(T) = \operatorname{Hom}(T, \mathbb{G}_m)$ be the character group, and $Y(T) = \operatorname{Hom}(\mathbb{G}_m, T) = \operatorname{Hom}(X(T), \mathbb{Z})$ the cocharacter group. We have a natural pairing $\langle ., . \rangle : X(T) \times Y(T) \to \mathbb{Z}$ and for each α in the root system we have the coroot $\alpha^\vee \in Y(T)$.

For each $\lambda \in X(T)$ define

$$n_G(\lambda) = n(\lambda) := \sum_{\alpha > 0} \langle \lambda, \alpha^\vee \rangle.$$

Note we can also write this as $\langle \lambda, \phi \rangle$ where $\phi = \sum_{\alpha > 0} \alpha^\vee$, the *principal homomorphism* of $\mathbb{G}_m$ into T. If V is any finite dimensional G-module, let us put;

$$n_G(V) = n(V) := \sup n(\lambda)$$

where the supremum is taken over all the weights λ of T in V.

As an example, consider $G = \mathrm{GL}_m$, with V the natural m-dimensional representation. Then $n(V) = m - 1 = \dim V - 1$ and $n(\bigwedge^i V) = i(\dim V - i)$. In general, if V_1 and V_2 are any G-modules, $n(V_1 \otimes V_2) = n(V_1) + n(V_2)$.

Note that if V is a *nondegenerate* linear representation of G, i.e. the connected kernel of the representation is a torus, then $n(V) \geq h - 1$, where h is the Coxeter number of G. Indeed, let λ be a highest weight of V. So $n(V) = n(\lambda) = \langle \lambda, \sum_{\alpha > 0} \alpha^\vee \rangle \geq \langle \lambda + \rho, \beta^\vee \rangle - 1$, where ρ is half the sum of positive roots and $\beta^\vee$ is the highest coroot. Since λ is nonzero and dominant we have $\langle \lambda, \beta^\vee \rangle \geq 1$ and $\langle \rho, \beta^\vee \rangle = h - 1$.

Our goal is to prove the following result:

Main Theorem. *Let V be G-module with $p > n(V)$. Let Γ be a subgroup of $G(k)$. Then*

$$\Gamma \textit{ is } G\textit{-cr} \Rightarrow V \textit{ is } \Gamma\textit{-semisimple.}$$

Moreover, the converse is true if V is nondegenerate, i.e. the connected kernel of the representation is a torus.

Some of the results mentioned in Lecture 1 are immediate consequences. For example, let $\{V_1, \dots, V_m\}$ be a collection of semisimple Γ-modules and $p > \sum_i (\dim V_i - 1)$. Then the theorem, applied to $G = \prod \mathrm{GL}(V_i)$ and $V = \bigotimes V_i$, tells us that $V_1 \otimes \cdots \otimes V_m$ is also semisimple. Alternatively, suppose that V is a semisimple Γ-module with $p > i(\dim V - i)$. Then the theorem shows that $\bigwedge^i V$ is Γ semisimple. (This was stated as an open question at the end of [S2]; and the special case where V is irreducible had been proved by McNinch.)

The proof of the main theorem uses the notion of *saturation* with respect to the group G. In order to define it, we need to introduce the "exponential" x^t, for x unipotent in G and $t \in k$. This is possible (for p not too small) thanks to:

Theorem 3. *Assume $p \geq h$ (resp. $p > h$ if G is not simply connected). There exists a unique isomorphism of varieties* $\log : G^u \to \mathfrak{g}_{nilp}$ *with the following properties:*

(i) $\log(\sigma u) = \sigma \log u$ *for all* $\sigma \in \mathrm{Aut}\, G$;
(ii) *the restriction of* $\log$ *to* $U(k)$ *is an isomorphism of algebraic groups* $U(k) \to \mathrm{Lie}\, U$, *whose tangent map is the identity;*
(iii) $\log(x_\alpha(\theta)) = \theta X_\alpha$, *for every root* α *and every* $\theta \in k$.

Here, $\mathfrak{g}_{\mathrm{nilp}}$ is the nilpotent variety of $\mathrm{Lie}\, G$, $x_\alpha : \mathbb{G}_a \to U_\alpha$ denotes some fixed parameterization of the root group U_α of U, and $X_\alpha = \frac{d}{d\theta}(x_\alpha(\theta))|_{\theta=0}$ is the corresponding basis element of $\mathrm{Lie}\, U_\alpha$. We are viewing $\mathrm{Lie}\, U$ as an algebraic group over k via the Campbell-Hausdorff formula: $XY := X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \dots$ (cf. [B], Chap II, §6) which makes sense in characteristic p because of the assumption $p \geq h$ and the fact that the nilpotency class of $\mathrm{Lie}\, U$ is at most h.

For the proof, uniqueness is obvious since the U_α generate U and G^u is the union of conjugates of U. (Moreover, one can show that (iii) is a consequence of (i) and (ii).) However, the existence part is less easy. One possible method is to define first log on U and then extend it to G^u. This approach uses the fact that $\mathfrak{g}_{\mathrm{nilp}}$ is a normal variety (cf. [D], [BR]) when p is good, that G^u is a normal variety (cf. [St]) and draws on work by Springer (cf. [Sp2]).

Given the theorem, let $\exp : \mathfrak{g}_{\mathrm{nilp}} \to G^u$ denote the inverse to log. For $x \in G^u(k)$ and $t \in k$ we define x^t as $\exp(t \log x)$. Note that the exponential map $x, t \mapsto x^t$ may be viewed as a morphism $F : G^u \times \mathbb{A}^1 \to G^u$. Moreover this map is the "reduction mod p" of the corresponding well-known map in characteristic zero, and this gives a convenient way to compute it.

Lecture 3

Continue with the notation of the previous lecture. Recall that we have just defined the map $x \mapsto x^t$ for any unipotent element $x \in G(k)$ and any $t \in k$. We can now at last define the saturation process (assuming $p \geq h$). A subgroup Γ of $G(k)$ is *saturated* if

(1) Γ is Zariski closed;
(2) whenever $x \in \Gamma \cap G^u$, we have $x^t \in \Gamma$ for all $t \in k$.

We wish in the remainder of this lecture to describe some basic properties of saturated subgroups and G-cr subgroups. We will apply these properties in Lecture 4 to prove the Main Theorem.

We begin by mentioning some elementary examples: every parabolic subgroup is saturated; the centralizer of any subgroup of $G(k)$ is saturated; Levi subgroups are saturated, since they may be realized as the centralizer of a torus. We also note that in the case of saturated subgroups lying in U, there are various alternative characterizations giving further 'unipotent' examples:

Property 1. *Let V be a closed subgroup of $U(k)$. The following are equivalent:*

(i) *V is saturated;*
(ii) *$V = \exp(\mathfrak{v})$ for $\mathfrak{v}$ a Lie subalgebra of* $\operatorname{Lie} U$*;*
(iii) *$\log V$ is a vector subspace of* $\operatorname{Lie} U$.

Another basic property is as follows:

Property 2. *Let H be a semisimple subgroup of G with $H(k)$ saturated. If x is any unipotent element of $H(k)$, then the element x^t (defined relative to H) coincides with x^t (defined relative to G).*

Even to state Property 2 correctly, we need first to know that the Coxeter number h_H of H does not exceed the Coxeter number h_G of G. In fact, a stronger result holds:

Theorem 4. *Let p be any prime, and H be a semisimple subgroup of a semisimple group G. Let $d_{i,H}$ and $d_{j,G}$ be the invariant degrees of the Weyl groups of H and G respectively. Then, the polynomial $\prod(1 - T^{d_{i,H}})$ divides $\prod(1 - T^{d_{j,G}})$.*
(For the properties of the invariant degrees, see [B], Chap V, §5.)

As a corollary we see that each $d_{i,H}$ divides some $d_{j,G}$. For, choosing T to be a primitive ${d_{j,H}}^{\text{th}}$ root of unity, the theorem implies that $\prod(1-T^{d_{j,G}})$ vanishes, so $(1-T^{d_{j,G}})$ vanishes for some j. Since the Coxeter number h_H is the largest degree $d_{i,H}$, and similarly for G, we deduce in particular that $h_H \leq h_G$, as required for the statement of Property 2 to make sense.

We sketch the proof of Property 2. Assume that H is a semisimple subgroup of G with $H(k)$ saturated. We may assume that there is a maximal unipotent subgroup U_H of H with $U_H \subset U$. Note that $U_H(k)$ is also a saturated subgroup of G. We need to show that $\log_G(x) = \log_H(x)$ for any unipotent $x \in H(k)$. Conjugating, it suffices to prove this for $x \in U_H(k)$. We have an isomorphism $\log : U(k) \to \operatorname{Lie} U$. Viewing $\operatorname{Lie} U_H$ as a subgroup of $\operatorname{Lie} U$, we conclude that the restriction of $\log_G$ gives a isomorphism $U_H \cong \operatorname{Lie} U_H$ which is compatible with conjugation and whose tangent map is the identity. By the uniqueness in the definition of $\log_H$ we conclude that the restriction of $\log_G$ is equal to $\log_H$, as required.

Property 3. *If $H \subset G$ is saturated then the index $(H : H^\circ)$ is prime to p.*

To prove Property 3, suppose p divides $(H : H^0)$ and take some element x of the finite group H/H^0 of order p. One proves, from general principles, that there exists $\tilde{x} \in H^u(k)$ which maps onto x in the quotient. By saturation, $\{\tilde{x}^t \mid t \in k\}$ is a subgroup of $H(k)$, hence of $H^0(k)$ since it is connected. So $\tilde{x} \in H^0(k)$, a contradiction.

We turn to discussing some basic properties of G-cr subgroups, as defined in Lectures 1 and 2. Recall that given a completely reducible H-module for an algebraic group H, the unipotent radical of H acts trivially. The next property that we will need is similar, but stated intrinsically within the groups.

Property 4. *If Γ is G-cr and V is a normal unipotent subgroup of Γ then $V = 1$. In particular, if in addition Γ is Zariski closed, then Γ^0 is reductive.*

The proof of this depends on the construction of Borel and Tits (cf. [BT]) which associates to the subgroup V a parabolic subgroup P of G with $V \subset R^u(P)$. Now Γ normalizes V, and since P is defined in a canonical fashion, Γ normalizes P. Therefore $\Gamma \subset P$. Now we use the fact that Γ is G-cr to deduce that $\Gamma \cap R^u(P) = 1$, whence $V = 1$.

Property 5. *Let $\Gamma_0 \subset \Gamma$ be a normal subgroup of Γ of index prime to p. Then, Γ_0 is G-cr $\Rightarrow \Gamma$ is G-cr.*

(Before sketching the proof of this, we mention an open problem: if $\Gamma_0 \subset \Gamma$ is normal, is it true that Γ is G-cr $\Rightarrow \Gamma_0$ is G-cr?)

Now for the proof, let P be a parabolic subgroup of G containing Γ, and let L be a Levi subgroup of P which contains Γ_0. Write $P = R^u P \rtimes L$. Let Γ_L be the image of Γ under the projection $P \to L$. The kernel of this projection is $\Gamma \cap R^u P = 1$ so we have an isomorphism $\Gamma \to \Gamma_L$. Then Γ is obtained from Γ_L by a 1-cocycle $a : \Gamma \to R^u P$, with a equal to a coboundary on restriction to Γ_0. This implies that a is induced by a 1-cocycle a' on Γ/Γ_0 with values in $V = R^u P \cap Z(\Gamma_0) = (R^u P)^{\Gamma_0}$. Now, V has a composition series made up of k-vector spaces, and since $|\Gamma/\Gamma_0|$ is prime to p, the cocycle induced by a' on each such composition factor is a coboundary. This implies that a', whence a, is a coboundary, so that we can conjugate Γ to a subgroup of L, as required.

Lecture 4

Now we proceed to prove the Main Theorem. We begin with:

Theorem 5. *Suppose $p \geq h$. Let V be a G-module with associated representation $\rho_V : G \to \mathrm{GL}(V)$. For every unipotent element u of G, let $d_u(V)$ be the degree of the polynomial map $t \mapsto \rho_V(u^t) \in \mathrm{End}(V)$. Then $d_u(V) \leq n(V)$, and there is equality if u is regular.*

The proof is in several steps.

(1) *The case $G = \mathrm{SL}_2$.* In this case we may assume $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $u^t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, and we have to prove $d_u(V) = n(V)$.

(1.1) *One has $d_u(V) \leq n(V)$.* Write $\rho_V(u^t)$ as $1+\sum_{i\geq 1} a_i t^i$, $a_i \in \mathrm{End}(V)$. If $s_\lambda = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ with $\lambda \in k^*$, we have $s_\lambda u^t s_\lambda^{-1} = u^{\lambda^2 t}$, hence

$$\rho^V(s_\lambda).\sum a_i t^i.\rho_V(s_\lambda^{-1}) = \sum a_i \lambda^{2i} t^i,$$

which implies $\rho_V(s_\lambda) a_i \rho_V(s_\lambda)^{-1} = \lambda^{2i} a_i$ for every i. Hence a_i has weight $2i$ in $\mathrm{End}(V) = V \otimes V^*$. By definition of the invariant n this shows that $a_i \neq 0 \Rightarrow 2i \leq n(V \otimes V^*) = n(V) + n(V^*) = 2n(V)$, i.e. $i \leq n(V)$. Hence $d_u(V) \leq n(V)$.

(1.2) *One has $d_u(V) \geq n(V)$.* If V has Jordan-Hölder quotients V_α, it is clear that $n(V) = \sup n(V_\alpha)$, $d_u(V) \geq \sup d_u(V_\alpha)$. Hence we may assume that V is simple. In that case, the equality $n(V) = d_u(V)$ is obvious from the explicit description of V à la Steinberg.

(2) *The case G arbitrary, u regular.* Choose a principal homomorphism $\mathrm{SL}_2 \to G$, (cf. [Te] - see also [S3]). It is known that a nontrivial unipotent

of SL_2 gives a regular unipotent of G. On the other hand, one has $n_G(V) = n_{\mathrm{SL}_2}(V)$, almost by definition. Hence the result follows from (1).

(3) *General case.* For u unipotent of G, write $\rho_V(u^t)$ as above:

$$\rho_V(u^t) = 1 + \sum a_i(u)t^i \in \mathrm{End}(V).$$

The a_i are regular functions of u (viewed as a point of the unipotent variety G^u). If $i < n(V)$ then $a_i(u)$ is 0 when u is regular, by (2). Since the regular unipotents are dense in G^u, this implies $a_i(u) = 0$ for every u.

Corollary 1. *If H is a reductive and saturated subgroup of G, one has $n_H(V) \leq n_G(V)$.*

Choose a regular unipotent element $u \in H$. One gets $n_H(V) = d_u(V) \leq n_G(V)$ by Theorem 5, applied to both H and G.

Corollary 2. *The following are equivalent:*

(i) $p > n(V)$;
(ii) $\rho_V(u^t) = \rho_V(u)^t$ *for every unipotent* u *of* G, *and every* $t \in k$.

Indeed (ii) holds if and only if the degree of $t \mapsto \rho_V(u^t)$ is $< p$, i.e. if and only if $d_u(V) < p$. Since $n(V) = \sup_u d_u(V)$, this shows the equivalence of (i) and (ii). (The same proof shows that (i) and (ii) are equivalent to:

(ii$'$) $\rho_V(u^t) = \rho_V(u)^t$ *for every regular* u, *and every* $t \in k$.)

Theorem 6. *Let G be reductive connected, and let V be a G-module. Assume $p > n(V)$. Let Γ be a subgroup of $G(k)$, which is G-cr. Then V is Γ-semisimple.*

The proof is in several steps.

(1) We may assume that $\rho_V : G \to \mathrm{GL}(V)$ has trivial kernel.

(2) *We have $p \geq h$.* This follows from $p > n(V) \geq h - 1$ (cf. Lecture 2).

(3) *The G-module V is semisimple.* Write G as $T.S_1 \ldots S_m$, where T is the maximal central torus, and $S_1 \ldots S_m$ is the decomposition of (G, G) into quasi-simple factors. To prove (3), it is enough to show that V is S_i-semisimple for every i (this is an easy lemma, cf. [J2] and comments below); since $n_{s_i}(V) \leq n_G(V)$ we are reduced to the case where G is quasi-simple. With the usual notation we have, for every weight λ of V, $\lambda \neq 0$,

$$\langle \lambda + \rho, \beta^\vee \rangle \leq 1 + \sum_{\alpha > 0} \langle \lambda, \alpha^\vee \rangle \leq p,$$

where the inequality on the left is in [S1], p.519. This shows that the simple modules $L(\lambda_i)$ in a Jordan-Hölder series of V are of two types: $\lambda_i = 0$, or $\langle \lambda_i + \rho, \beta^\vee \rangle \leq p$. But it is known (cf. [J1]) that this implies $L(\lambda_i)$ $\mathrm{Ext}^1_G(L(\lambda_i), L(\lambda_i)) = 0$ for every pair λ_i, λ_j (e.g. because these $L(\lambda_i)$ are Weyl modules). Hence V is semisimple.

We pause to discuss a variant of this proof. If λ is a dominant weight with $\sum_{\alpha>0} \langle \lambda, \alpha^\vee \rangle < p$, then $L(\lambda) = V(\lambda)$, where $V(\lambda)$ is the Weyl module. The proof is by reduction to G quasi-simple, and one distinguishes between two cases: $\lambda = 0$, where it is obvious, and $\lambda \neq 0$, where we have $\langle \lambda + \rho, \beta^\vee \rangle \leq p$. Moreover, if λ, μ have the property $L(\lambda) = V(\lambda)$, $L(\mu) = V(\mu)$ then $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = 0$. See [J2] for more details.

(4) *The* Γ^{sat}*-module* V *is semisimple.* (Note that we can define Γ^{sat} since $p \geq h$ by (2).) Let H be the connected component of Γ^{sat}. Since Γ^{sat} is G-cr (because Γ is), H is a reductive group. By Corollary 1 to Theorem 5, we have $n_H(V) \leq n_G(V)$ hence $n_H(V) < p$ and part (3) above (applied to H) shows that V is H-semisimple. Since $(\Gamma^{\mathrm{sat}} : H)$ is prime to p, this implies that V is Γ^{sat}-semisimple (cf. [S1], p.523).

(5) *If a subspace* W *of* V *is* Γ*-stable, it is* Γ^{sat}*-stable.* Let H_W be the stabilizer of W in G. If $u \in H_W$ is unipotent, one has $\rho_V(u^t) = \rho_V(u)^t$ by Corollary 2 to Theorem 5 above. Since $\rho_V(u)W = W$ the same is true for $\rho_V(u)^t$ for every t. This shows that H_W is saturated. Since it contains Γ, it also contains Γ^{sat}.

(6) *End of proof.* By (5), the subspaces of V which are Γ-stable are the same as those which are Γ^{sat}-stable. Since, by (4), V is Γ^{sat}-semisimple, it is Γ-semisimple.

Note that this is the "Main Theorem" announced at the beginning of these lectures. It implies for instance the following (where k is arbitrary of characteristic p):

If V_α *are semisimple* Γ *modules, and* $i_\alpha \geq 0$ *integers with*

$$\sum i_\alpha (\dim V_\alpha - i_\alpha) < p,$$

then $\bigotimes_\alpha \bigwedge^{i_\alpha} V_\alpha$ *is semisimple.*

(Sketch of proof. Apply Theorem 6 to $\prod_\alpha \mathrm{GL}(V_\alpha)$ and $V = \bigotimes_\alpha \bigwedge^{i_\alpha} V_\alpha$, and deduce the statement when k is algebraically closed. Next show that one can assume $i_\alpha \leq (\dim V_\alpha)/2$, and $\dim V_\alpha < p$ for all α; deduce that V_α is absolutely semisimple (i.e. remains semisimple after extension of scalars from k to $\overline{k}$); hence $\bigotimes_\alpha \bigwedge^{i_\alpha} V_\alpha$ is absolutely semisimple.)

Theorem 7 (Eugene). (cf. [J2], [Mc], [LS]) *Let* $H \subset G$ *be connected reductive, and* $p \geq h_G$*. Then* H *is* G*-cr.*

The proof starts by reducing to the case G is quasi-simple. Then there are separate proofs for type A_n (Jantzen), B_n, C_n, D_n (Jantzen-McNinch), and exceptional type (Liebeck-Seitz). There is a little extra work involved in the B_n, C_n, D_n cases when H is of type A_1. (Note that in special cases $p \geq h_G$ can be improved.)

Theorem 8. *Let $\Gamma \subset G$. Assume $p \geq h_G$. The following are equivalent:*

(i) Γ *is G-cr;*
(ii) *the connected component of Γ^{sat} is reductive.*

The direction (i)$\Rightarrow$(ii) is clear since Γ is G-cr $\Rightarrow$ Γ^{sat} is G-cr, and hence $(\Gamma^{\text{sat}})^0$ is reductive.

For (ii)$\Rightarrow$(i) apply Theorem 7 to $H = (\Gamma^{\text{sat}})^0$. One sees that H is G-cr, hence also Γ^{sat}, hence also Γ.

Theorem 9. *Let V be a nondegenerate G-module. Assume $n(V) < p$. If $\Gamma \subset G$, the following are equivalent:*

(i) Γ *is G-cr;*
(ii) V *is Γ-semisimple.*

The direction (i)$\Rightarrow$(ii) is Theorem 6. Conversely, if V is Γ-semisimple, it is also Γ^{sat}-semisimple (cf. argument of Theorem 6), hence $(\Gamma^{\text{sat}})^0$-semisimple and by Theorem 8 this shows that Γ is G-cr.

Note: The implication (ii)$\Rightarrow$(i) proved above under the condition $p > n(V)$ is far from best possible. Example: take $G = \mathrm{GL}(W)$, and $V = \bigwedge^2 W$, which is nondegenerate if $\dim W \neq 2$. One has $n(V) = 2(\dim W - 2)$ and one sees that:

$$\textstyle\bigwedge^2 W \text{ is } \Gamma\text{-semisimple} \;\Rightarrow\; W \text{ is } \Gamma\text{-semisimple}$$

if $p > 2(\dim W - 2)$. However, an elementary argument [S2], shows that this remains true as long as p does not divide $\dim W - 2$.

Example of Theorem 8: If one takes for V the adjoint representation $\operatorname{Lie} G$, which is nondegenerate, one has $n(\operatorname{Lie} G) = 2h - 2$ and Theorem 8 gives:

$$\Gamma \text{ is } G\text{-cr} \iff \operatorname{Lie} G \text{ is } \Gamma\text{-semisimple}$$

provided $p > 2h - 2$. (In fact, for $G = \mathrm{GL}_n$, no condition on p is needed for $\Leftarrow$, cf. [S2], Theorem 3.3.)

A few references

[BR] P. Bardsley and R. W. Richardson, Étale slices for algebraic transformation groups in characteristic p, *Proc. London Math. Soc.* **51** (1985), 295–317.

[Bo1] A. Borel, Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes, *Tôhoku Math. J.*, (2) **13** (1961), 216-240

[BS] A. Borel and J.-P. Serre, Sur certains sous-groupes des groupes de Lie compacts, *Comm. Math. Helv.* **27** (1953), 128–139.

[BT] A. Borel and J. Tits, Éléments unipotents et sous-groupes paraboliques des groupes réductifs, *Invent. math.* **12** (1971), 95-104.

[B] N. Bourbaki, *Groupes et algèbres de Lie*, Chap I-VI, Hermann, Paris (1968).

[C] C. Chevalley, *Théorie des groupes de Lie*, vol. III, Hermann, Paris, (1954).

[D] P. Deligne, Catégories Tannakiennes, *The Grothendieck Festschrift*, Vol II, Birkhäuser, Boston (1990), 111-195.

[De] M. Demazure, Invariants symétriques entiers des groupes de Weyl et torsion, *Invent. math.* **21** (1973), 287-301.

[Dy] E. Dynkin, Semisimple subalgebras of semisimple Lie algebras, *A.M.S. Translations series 2* **6** (1957), 111-245.

[G] R. Griess, Elementary abelian p-subgroups of algebraic groups, *Geom. Dedicata* **39** (1991), 253–305.

[GR] R. Griess and A. Ryba, Finite simple groups which projectively embed in an exceptional Lie group are classified!, preprint, April 14, (1998).

[J1] J. Jantzen, *Representations of Algebraic Groups*, Academic Press, Orlando, Pure and Applied Mathematics **131** (1987).

[J2] J. Jantzen, Low dimensional representations of reductive groups are semisimple, *Algebraic Groups and Lie groups: a volume in honor of R.W. Richardson.* Cambridge (1997).

[K] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, *Amer. J. Math.* **81** (1959), 973-1032.

[LS] M. Liebeck G. Seitz, Reductive subgroups of exceptional algebraic groups, *Memoirs of the A.M.S.* **580** (1996).

[Mc] G. McNinch, Dimensional criteria for semisimplicity of representations, *Proc. London Math. Soc.*(3) **76** (1998), 95–149.

[Min] H. Minkowski, *Gesammelte Abhandlungen*, New York, Chelsea Pub. Co. (1967).

[N] M. V. Nori, On subgroups of $\mathrm{GL}_n(\mathbb{F}_q)$, *Invent. math.* **88** (1987), 257–275.

[S1] J.-P. Serre, Sur la semi-simplicité des produits tensoriels de représentations de groupes, *Invent. math.* **116** (1994), 513–530.

[S2] J.-P. Serre, Semisimplicity and tensor products of group representations: converse theorems, *J. Algebra* **194** (1997), 496–520 (with an appendix by W. Feit).

[S3] J.-P. Serre, Exemples de plongements des groupes $\mathrm{PSL}_2(\mathbb{F}_p)$ dans des groupes de Lie simples, *Invent. math.* **124** (1996), 525–562.

[S4] J.-P. Serre, *Galois Cohomology*, Springer-Verlag (1997).

[Sp1] T. Springer, Some arithmetical results on semi-simple Lie algbras, *Publ. Math I.H.E.S.* **30**, (1966),115-141.

[Sp2] T. Springer, Regular elements of finite reflection groups, *Invent. math.* **25** (1974), 159–198.

[SpSt] T. Springer, R. Steinberg, Seminar on algebraic groups and related finite groups: conjugacy classes, algebraic groups, *Lect. Notes in Math* **131**, Springer-Verlag (1970).

[St] R. Steinberg, Regular elements of semisimple algebraic groups, *Publ. Math. IHES* **25** (1965), 49–80 (= Coll. Papers no. 20).

[Te] D. Testerman, The construction of the maximal A_1's in the exceptional algebraic groups. *Proc. A.M.S* **161**, 635-644 (1993).

[T1] J. Tits, Buildings of spherical type and finite BN-pairs, *Lect. Notes in Math.* **386**, Springer-Verlag (1974).

[T2] J. Tits, Résumé de cours au Collège de France, (1997-1998), 93-98.