Least-Squared Ordered Weighted Averaging Operator Weights

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The ordered weighted averaging (OWA) operator by Yager (*IEEE Trans Syst Man Cybern* 1988; 18; 183–190) has received much more attention since its appearance. One key point in the OWA operator is to determine its associated weights. Among numerous methods that have appeared in the literature, we notice the maximum entropy OWA (MEOWA) weights that are determined by taking into account two appealing measures characterizing the OWA weights. Instead of maximizing the entropy in the formulation for determining the MEOWA weights, a new method in the paper tries to obtain the OWA weights that are evenly spread out around equal weights as much as possible while strictly satisfying the orness value provided in the program. This consideration leads to the least-squared OWA (LSOWA) weighting method in which the program is to obtain the weights that minimize the sum of deviations from the equal weights since entropy is maximized when all the weights are equal. Above all, the LSOWA method allocates the positive and negative portions to the equal weights that are identical but opposite in sign from the middle point in the number of criteria. Furthermore, interval LSOWA weights can be constructed when a decision maker specifies his or her orness value in uncertain numerical bounds and we present a method, with those uncertain interval LSOWA weights, for prioritizing alternatives that are evaluated by multiple criteria. © 2008 Wiley Periodicals, Inc.

1. INTRODUCTION

Usually, decision problems are categorized as follows. If a decision maker knows in advance which state of nature will occur, he or she is in a position to choose the alternative that gives the best payoff for this state of nature and the decision is said to take place under certainty. If the decision maker does not know which state of nature will occur but does know its probability of occurring, the decision is said to take place under risk. At the other extreme, if the decision maker has no information at all concerning the relative likelihood of the various states of nature, the decision is said to take place under uncertainty. In almost all decision-making problems, there are multiple criteria for judging alternatives. A multiple criteria decision making (MCDM) method under certainty largely consists

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of two phases: (1) construction of a decision problem and information specification and (2) aggregation and exploitation.[1,2] Among others, synthesizing judgments is an important part of MCDM methods. Yager[3] introduced the ordered weighted averaging (OWA) operator to provide a method for aggregating multiple inputs that lie between the max and min operators. As the term "ordered" implies, the OWA operator pursues a nonlinear aggregation of objects considered, so it is clearly different from the existent multicriteria aggregation methods such as, for example, multiattribute utility theory (MAUT), simple weighted sum, the analytic hierarchical process (AHP), and so on. The OWA operator is generally composed of the following three steps:[4]

(1) Reorder the input arguments in descending order,
(2) Determine the weights associated with the OWA operator by using a proper method, and
(3) Utilize the OWA weights to aggregate these reordered arguments.

In the short time since its first appearance, the OWA operators have been used in an astonishingly wide range of applications in the fields including neural networks,[5,6] database systems,[7] fuzzy logic controllers,[8,9] group decision making problems with linguistic assessments,[10,11] data mining,[12] location-based service (LBS)[13] or more generally geographical information system (GIS)[14,15] and so on. The main reason for this is their great flexibility to model a wide variety of aggregators, as their nature is defined by a weighting vector and not by a single parameter.[16] By appropriately selecting the weighting vector, it is possible to model different kinds of relations among the criteria aggregated. Xu and Da[17] presented a survey of the main aggregation operators that encompass a broad range of existing operators (more than 20 aggregators). Clearly the actual type of aggregation performed by an OWA operator heavily depends upon the weighting vector, which plays a key role in the aggregation process. Filev and Yager[18] presented a way of obtaining weights associated with the OWA aggregation in the situation, where we have observed data on the arguments and the aggregated value.

An appealing point in the OWA operator was the introduction of the concept of *orness* and the definition of an *orness* measure that could establish how "orlike" a certain operator is, based on the values of its weighting function. Thus, the measure can be interpreted as the mode of decision-making circumstances by conferring the semantic meaning to the weights used in the aggregation process. If an aggregated value is close to the maximum of the ordered objects, the aggregation pursues the "orlike" aggregation. If an aggregated value is close to the minimum of the ordered objects, on the other hand, the aggregation pursues the "andlike" aggregation. This concept perfectly coincides with the traditional decision-making theory in which *max* decision principle denotes the optimistic decision context and *min* decision principle denotes the pessimistic decision context.

On the other hand, Yager,[3] based on a measure of entropy, proposed a measure of dispersion which gauges the degree of utilization of information in the sense that each of weighting vectors considered can be different to each other by degree of dispersion though they have the same value of *orness*. Recently, Xu[4] reviewed existing main methods for determining the weights associated with the OWA operator. One of the approaches, suggested by O'Hagan,[19] determines a special class of OWA operators having maximal entropy of the OWA weights for a given value of *orness*, algorithmically based on the solution of a constrained optimization problem. The resulting weights are called maximum entropy OWA (MEOWA) weights for a given degree of *orness* and analytic forms and property for these weights are further investigated by several researchers.[20,21] Instead of maximizing the degree of dispersion, Fuller and Majlender[22] presented a method of deriving the minimal variability weighting vector for any level of *orness*, using Kuhn–Tucker second-order sufficiency conditions for optimality. Ahn[23] presented analytic forms of OWA operator-weighting functions, each of which has properties of rank-based weights and a constant value of orness, irrespective of the number of objectives considered. These analytic forms provide significant advantages for generating OWA weights over previously reported methods. Instead of maximizing the entropy in the formulation for determining the MEOWA weights, a new method in this paper tries to obtain the OWA weights that are evenly spread out around the equal weights as much as possible while strictly satisfying the orness value provided in the program. This consideration leads to the least-squared OWA (LSOWA) weighting method in which the program is to obtain the weights that minimize the sum of deviations from the equal weights since entropy is maximized when all the weights are equal. Several properties of the LSOWA weights are investigated in detail. Interval LSOWA weights are constructed when a decision maker specifies his or her orness value in uncertain numerical bounds, and furthermore we present a method, with those uncertain interval LSOWA weights, for prioritizing alternatives that are evaluated by multiple criteria.

The paper is organized as follows: in Section 2, we will show the method for determining the LSOWA weights. The analytic form of the weights is derived, and then their properties and limitations are presented in detail. In Section 3, a method for constructing interval LSOWA weights is introduced and then the OWA aggregation under interval LSOWA weights is presented. Finally, concluding remarks follow in Section 4.

2. DETERMINING THE LSOWA WEIGHTS

An OWA operator[3] of dimension n is a mapping $f\colon R^n \to R$ that has an associated weighting n vector $W = [w_1, w_2, \ldots, w_n]$, such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Furthermore,

$$f(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i b_i$$

where b_i is the ith largest element of the collection of the n aggregated objects $a_1, a_2, \ldots, a_n$, thus satisfying the relation $\min_i[a_i] \leq f(a_1, a_2, \ldots, a_n) \leq \max_i[a_i]$. Central to this operator is the reordering of the arguments, based on their values, in particular an argument a_i is not associated with a particular weight w_i but rather a weight w_i is associated with a particular ordered position i of the arguments. The OWA aggregation is a nonlinear aggregation because of the ordering process used.

Yager[3] introduced two characterizing measures associated with the weighting vector W of an OWA operator. The first one, the measure of orness of the aggregation, is defined as

$$\text{orness}(W) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i,$$

and it characterizes the degree to which the aggregation is like an *or* operation. If we consider the special cases of OWA operators,

$W^* = [1, 0, 0, \ldots, 0]$ (maximum operator),
$W_* = [0, 0, 0, \ldots, 1]$ (minimum operator),
$W_{\text{Ave}} = [1/n, 1/n, 1/n, \ldots, 1/n]$ (average operator),

then it can easily be shown that

(1) $\text{orness}(W^*) = 1$,
(2) $\text{orness}(W_*) = 0$, and
(3) $\text{orness}(W_{\text{Ave}}) = 0.5$.

A measure of *andness* for an OWA operator with the weights W is also defined as

$$\text{andness}(W) = 1 - \text{orness}(W) = 1 - \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i.$$

The OWA operators with many of the weights near the top will be an "orlike" operator ($\text{orness}(W) \geq 0.5$), whereas those operators with most of the weights at the bottom will be "andlike" operators ($\text{orness}(W) \leq 0.5$). As to the semantics of the OWA's measure of orness, Yager suggests that, based on Hurwicz's model, the measure of orness can be interpreted as a measure of optimistic decision making context, whereas the measure of andness is a measure of pessimism.

Yager[3] used the entropy of the OWA operator weights for the purpose of gauging the degree of utilization of information.

$$\text{disp}(W) = -\sum_{i=1}^{n} w_i \ln w_i.$$

This measure can be used to gauge the degree to which information about the individual aggregates is used in the aggregation process. We note that, since this dispersion is really a measure of entropy, the following properties are valid:

(1) If $w_i = 1$ for some i, then the dispersion is minimum and $\text{disp}(W) = 0$.
(2) If $w_i = 1/n$, $i = 1, \ldots, n$, then the dispersion is maximum, and $disp(W) = \ln n$.

O'Hagan[19] determined a special class of OWA operators having a maximal entropy of the OWA weights for some prescribed value of orness. This approach is based on the solution of the following mathematical programming problem:

$$\text{Maximize} \quad -\sum_{i=1}^{n} w_i \ln w_i \tag{1a}$$

$$\text{subject to} \quad \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \Omega, \quad 0 \leq \Omega \leq 1 \tag{1b}$$

$$\sum_{i=1}^{n} w_i = 1, \quad 0 \leq w_i \leq 1, \quad i = 1, \ldots, n. \tag{1c}$$

Filev and Yager[20] provided an analytic solution to the above constrained optimization problem with an aim to use the MEOWA weights among others in dynamic environments, in which a prescribed orness value Ω changes, without having to solve a new constrained optimization problem.

In the LSOWA method, the program is to obtain the OWA weights that minimize the sum of deviations from the equal weights instead of maximizing entropy itself since it is known that the entropy is maximized when all the weights are equal. This consideration can be set forth by the following constrained mathematical program:

$$\text{Mimimize} \quad \sum_{i=1}^{n}\left(w_i - \frac{1}{n}\right)^2 \tag{2a}$$

$$\text{subject to} \quad \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i = \Omega, \quad 0 \leq \Omega \leq 1 \tag{2b}$$

$$\sum_{i=1}^{n} w_i = 1, \quad 0 \leq w_i \leq 1, \quad i = 1, \ldots, n \tag{2c}$$

The program 2a–2c is a quadratic mathematical program, thus well-known nonlinear software package such as, for example, the *Lindo* solver suite can be used to obtain the LSOWA weights. If we omit the nonnegative constraints on w_i in the formulation, we can find a nice analytic solution for determining the LSOWA weights, It will help us to deeply understand the LSOWA weights and simplify the process used for generating the LSOWA weights. In doing so, a composite function can be built such as

$$L(W, \alpha, \beta) = \sum_{i=1}^{n}\left(w_i - \frac{1}{n}\right)^2 + \alpha\left(\sum_{i=1}^{n} w_i - 1\right) + \beta\left(\frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i - \Omega\right)$$

which denotes the Lagrange function of the constrained optimization problem 2a–2c, where α and β are real numbers. Then, the partial derivatives of L are computed as

$$\frac{\partial L}{\partial w_j} = 2w_j - \frac{2}{n} + \alpha + \beta\frac{n-j}{n-1} = 0, \quad \forall j \tag{3a}$$

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} w_i - 1 = 0, \tag{3b}$$

$$\frac{\partial L}{\partial \beta} = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)w_i - \Omega = 0. \tag{3c}$$

From Equations 3a and 3b, we obtain, for the w_is

$$w_i = \frac{1}{n} - \frac{1}{2}\alpha - \frac{n-i}{2(n-1)}\beta, \quad \forall i \tag{4a}$$

$$\sum_{i=1}^{n} w_i = \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{1}{2}\alpha - \frac{n-i}{2(n-1)}\beta\right) = 1 \tag{4b}$$

From Equation 4b, we obtain the following relationship between α and β:

$$\alpha = -\frac{1}{2}\beta \tag{5}$$

and substituting this equation into Equation 4a, we get

$$w_i = \frac{1}{n} + \frac{1}{4}\beta - \frac{n-i}{2(n-1)}\beta. \tag{6}$$

Finally, by substituting Equation 6 for w_i Equation 3c, we obtain an equation relating the specified degree of orness and the Lagrange parameter β:

$$\beta = \frac{24(n-1)(0.5-\Omega)}{n(n+1)}. \tag{7}$$

Furthermore, Equation 6 can be simplified by using Equation 7

$$w_i = \frac{1}{n} + \frac{6(2i-n-1)}{n(n+1)}(0.5-\Omega), \quad i = 1, \ldots, n. \tag{8}$$

The analytic solution shown in Equation 8 takes a simple and closed form as compared to the analytic solution for determining the MEOWA weights. Thus, the LSOWA weights can be easily determined once the value of orness and the number of objects to be aggregated are specified in advance. As was intended by the mathematical formulation in 2a–2c, the LSOWA weights are determined possibly around the equal weights (i.e., $1/n$) while satisfying the prescribed value of orness. Table I shows the LSOWA weights and the MEOWA weights when the value of orness is set at 0.7. The numerical data in Table I indicate that there is not much difference in entropy between the two weighting methods. Rather, there exist some peculiar characteristics in the LSOWA weights that are not in the MEOWA weights. We shall show them in several theorems and corollaries.

THEOREM 1. *If a specified value of orness is 0.5, then the LSOWA weights are* $W_{ave} = [1/n, 1/n, \ldots, 1/n]$.

Proof. It is obvious from Equation 8. ■

THEOREM 2. *For orness* $0.5 < \Omega \leq 1$, *the LSOWA weights form a decreasing sequence,* $w_i > w_j$ *for* $i < j$. *For orness* $0 \leq \Omega < 0.5$, *on the contrary, the LSOWA weights form an increasing sequence,* $w_i < w_j$ *for* $i < j$.

Proof.

$$w_i - w_j = \frac{1}{n} + \frac{6(2i - n - 1)}{n(n+1)}(0.5 - \Omega) - \frac{1}{n} - \frac{6(2j - n - 1)}{n(n+1)}(0.5 - \Omega)$$

$$= \frac{12(0.5 - \Omega)}{n(n+1)}(i - j)$$

Therefore, for $0.5 < \Omega \leq 1$ and $i < j$, $w_i > w_j$ and for $0 \leq \Omega < 0.5$ and $i < j$, $w_i < w_j$. ■

THEOREM 3. *The LSOWA weight at median is* $1/n$. *That is,* $w_{\frac{n+1}{2}} = \frac{1}{n}$, *when n is odd and* $(w_{\frac{n}{2}} + w_{\frac{n}{2}+1})/2 = \frac{1}{n}$ *when n is even.*

Proof. When n is odd,

$$w_{\frac{n+1}{2}} = \frac{1}{n} + \frac{6(2 \times (n+1)/2 - n - 1)}{n(n+1)}(0.5 - \Omega) = \frac{1}{n}.$$

Table I. The LSOWA and the MEOWA weights when a value of orness is set at 0.70.

	LSOWA weights											MEOWA weights										
n	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E^a	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	E
2	.700	.300									0.611	.700	.300									0.611
3	.533	.333	.133								0.970	.554	.292	.154								0.975
4	.430	.310	.190	.070							1.228	.461	.276	.165	.098							1.237
5	.360	.280	.200	.120	.040						1.429	.396	.257	.167	.109	.071						1.443
6	.310	.252	.195	.138	.081	.024					1.595	.347	.240	.165	.114	.079	.054					1.614
7	.271	.229	.186	.143	.100	.057	.014				1.736	.310	.224	.161	.117	.084	.061	.044				1.759
8	.242	.208	.175	.142	.108	.075	.042	.008			1.895	.279	.209	.156	.117	.087	.065	.049	.037			1.885
9	.218	.191	.164	.138	.111	.084	.058	.031	.004		1.968	.254	.196	.151	.116	.089	.069	.053	.041	.031		1.997
10	.198	.176	.155	.133	.111	.089	.067	.045	.024	.002	2.046	.234	.184	.145	.114	.090	.071	.056	.044	.035	.027	2.098

[a] E denotes entropy.

When n is even,

$$w_{\frac{n}{2}} + w_{\frac{n}{2}+1} = \frac{1}{n} + \frac{6(2 \times n/2 - n - 1)}{n(n+1)}(0.5 - \Omega) + \frac{1}{n} + \frac{6(2 \times (n/2+1) - n - 1)}{n(n+1)}(0.5 - \Omega) = \frac{2}{n}.$$

Thus, $(w_{\frac{n}{2}} + w_{\frac{n}{2}+1})/2 = \frac{1}{n}$. ■

COROLLARY 1. *Let us denote* $\Delta_i = w_i - \frac{1}{n}$, *that is,* $\Delta_i = \frac{6(2i-n-1)}{n(n+1)}(0.5 - \Omega)$. *Then,* $|\Delta_i| = |\Delta_{n-i+1}|$, *i=1, . . . , n.*

Proof.

$$\Delta_{n-i+1} = \frac{6(2 \times (n - i + 1) - n - 1)}{n(n+1)}(0.5 - \Omega) = -\frac{6(2i - n - 1)}{n(n+1)}(0.5 - \Omega).$$

■

The LSOWA weights are rank-based weights and allocate some portion of weights symmetrically on the basis of median. More specifically, if the value of orness is greater than 0.5, positive portions of weights (i.e., Δ_i) are added to the left-sided weights at median and the same portions of weights are subtracted to the right-sided weights at median. On the contrary, if the value of orness is less than 0.5, positive portions of weights are added to the right-sided weights at median and the same portions of weights are subtracted to the left-sided weights at median. According to Corollary 1, it is obvious that $\sum_{i=1}^{\frac{n+1}{2}} |\Delta_i| = \sum_{i=\frac{n+1}{2}}^{n} |\Delta_i|$, when n is odd and $\sum_{i=1}^{\frac{n}{2}} |\Delta_i| = \sum_{i=\frac{n}{2}+1}^{n} |\Delta_i|$, when n is even.

Example. For $n = 5$ and $\Omega = 0.7$, the LSOWA weights from Table I are given

$$W(0.7) = (0.36, 0.28, 0.2, 0.12, 0.04).$$

It is evident that $w_3 = 0.2$ from Theorem 3. Furthermore, it holds that $|\Delta_i| = |\Delta_{n-i+1}|$, $i = 1,2,3$, according to Corollary 1 because

$$\Delta_1 = \frac{6(2 \times 1 - 5 - 1)}{5 \times 6}(0.5 - 0.7) = 0.16,$$

$$\Delta_2 = \frac{6(2 \times 2 - 5 - 1)}{5 \times 6}(0.5 - 0.7) = 0.08, \quad \Delta_3 = 0,$$

$$\Delta_4 = \frac{6(2 \times 4 - 5 - 1)}{5 \times 6}(0.5 - 0.7) = -0.08,$$

$$\Delta_5 = \frac{6(2 \times 5 - 5 - 1)}{5 \times 6}(0.5 - 0.7) = -0.16.$$

Thus, the LSOWA weights can be rewritten as

$$W(0.7) = (0.2 + 0.16, 0.2 + 0.08, 0.2, 0.2 - 0.08, 0.2 - 0.16).$$

It is well known that if a weighting vector W is optimal under some predefined value of orness Ω, then its reverse, denoted by W^R and defined as

$$w_i^R = w_{n-i+1}$$

is also optimal under degree of orness $(1 - \Omega)$. Indeed, as was shown by Yager,[3] we find that

$$\text{disp}(W^R) = \text{disp}(W) \text{ and}$$

$$\text{orness}(W^R) = 1 - \text{orness}(W).$$

It should be noted that Equation 8 for determining the LSOWA weights is derived by omitting the nonnegative conditions on the weights w_is and thus has some drawbacks. In other words, if the specific LSOWA weights are determined by using the analytic solution and they are nonnegative, the solution is fine, but if the solution results in negative weights, we cannot use the LSOWA weights as they are. Instead, we have to solve the nonlinear program by the use of software package. The index of weights (i.e., the number of criteria or objects) and the prescribed value of orness in the analytic solution are two parameters that determine the usefulness of the LSOWA weights. Figure 1 shows the valid combinations concerning the two parameters when we use Equation 8 to obtain the LSOWA weights.

As can be seen in Figure 1, the lower area of the curve indicates the feasible combinations of prescribed values of onress and indexes of weights when we use Equation 8 to determine the LSOWA weights. On the contrary, the upper area of the curve indicates the combinations of prescribed values of orness and indexes of weights that may result in negative weights. The feasible index of weights by the analytic solution rapidly decreases as the value of orness increases from 0.7, and thus Equation 8 can be used without much concern when the value of orness belongs to the one ranging approximately from 0.7 to 0.3.

If the number of objects to be aggregated is small, then the LSOWA weights that are generated by using the analytic solution at the value of orness larger than 0.7 are also acceptable. At any rate, all of these discussions cannot restrict the usefulness of the LSOWA weighting method because it is always possible to obtain

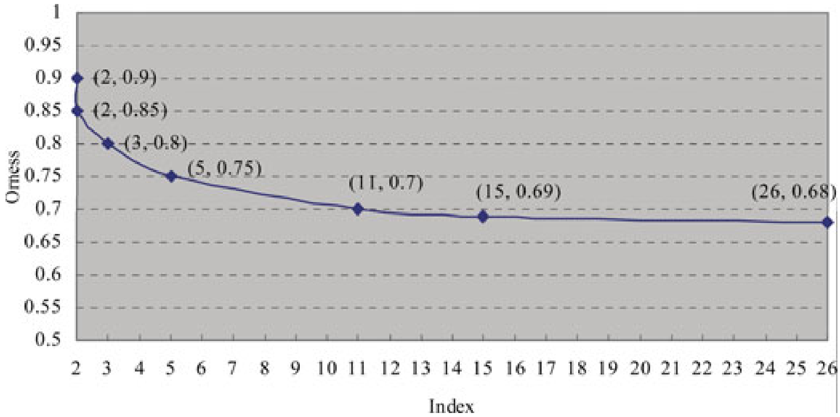

Figure 1. The feasible combinations of orness and index of weights.

all the positive LSOWA weights only if we resort to the nonlinear software readily available to obtain the LSOWA weights while the prescribed value of orness is satisfied.

3. AGGREGATION OF MULTIPLE OBJECTS WITH INTERVAL LSOWA WEIGHTS

In this section, we deal with a situation in which the LSOWA weights are specified not in the form of exact numerical values but in the form of uncertain forms. This is because when we work with vague or imprecise knowledge, it is difficult to estimate the weights with precision. Then, a more realistic approach may be to use imprecise assessments instead of exact numerical values, that is, by assuming that the parameters which are allowed in the problem are assessed by means of interval, weak ordinal, or set inclusion. Specifically, it seems difficult for a decision maker to specify an exact orness value and thus it seems reasonable to assume that the value of orness is specified in interval numbers when we formulate a mathematical program for determining the LSOWA weights. Such interval orness renders the LSOWA weights intervally specified as well. We consider a simple method for deriving interval LSOWA weights. This approach is appropriate to many problems, because it allows for the representation of information in a more direct and adequate form if we are unable to express it with precision.

THEOREM 4. *For given two values of orness Ω_1 and Ω_2, if $\Omega_1 > \Omega_2$, then when n is odd,*

$$w_i(\Omega_1) > w_i(\Omega_2) \quad \text{for} \quad i = 1, \ldots, \frac{n+1}{2} - 1$$

$$w_i(\Omega_1) < w_i(\Omega_2) \quad \text{for} \quad i = \frac{n+1}{2} + 1, \ldots, n$$

and when n is even,

$$w_i(\Omega_1) > w_i(\Omega_2) \quad \text{for} \quad i = 1, \dots, \frac{n}{2}$$

$$w_i(\Omega_1) < w_i(\Omega_2) \quad \text{for} \quad i = \frac{n}{2} + 1, \dots, n$$

where $w_i(\Omega_j)$*,* $j = 1, 2$ *denote the ith LSOWA weights at the value of orness* Ω_j*.*

Proof.

$$\begin{aligned} w_i(\Omega_1) - w_i(\Omega_2) &= \frac{1}{n} + \frac{6(2i - n - 1)}{n(n+1)}(0.5 - \Omega_1) \\ &\quad - \frac{1}{n} - \frac{6(2i - n - 1)}{n(n+1)}(0.5 - \Omega_2) \\ &= \frac{6(2i - n - 1)}{n(n+1)}(\Omega_2 - \Omega_1). \end{aligned}$$

For $\Omega_1 > \Omega_2$, the sign of $w_i(\Omega_1) - w_i(\Omega_2)$ is determined depending on the index number i as in the statements. ■

Let us denote $Q_k(\Omega)$ as a cumulative LSOWA weight from $i = 1$ to k, when a value of orness is given as Ω. That is,

$$Q_k(\Omega) = \sum_{i=1}^{k} w_i(\Omega).$$

COROLLARY 2. *For given two values of orness* Ω_1 *and* Ω_2*, if* $\Omega_1 > \Omega_2$*, then* $Q_k(\Omega_1) \geq Q_k(\Omega_2)$ *for* $k = 2, \dots, n$ *where* $Q_k(\Omega_1) = \sum_{i=1}^{k} w_i(\Omega_1)$ *and* $Q_k(\Omega_2) = \sum_{i=1}^{k} w_i(\Omega_2)$*.*

Proof.

$$Q_k(\Omega_1) - Q_k(\Omega_2) = \sum_{i=1}^{k} [w_i(\Omega_1) - w_i(\Omega_2)] = \sum_{i=1}^{k} \frac{6(2i - n - 1)}{n(n+1)}(\Omega_2 - \Omega_1)$$

$$= \frac{6(\Omega_2 - \Omega_1)}{n(n+1)} \sum_{i=1}^{k} (2i - n - 1) = \frac{6(\Omega_2 - \Omega_1)}{n(n+1)} k(k - n) \geq 0$$

since $\Omega_1 > \Omega_2$ and $0 < k \leq n$ (equality holds when $k = n$). ■

As was previously mentioned, let us consider a situation in which a decision maker specifies his or her optimistic value for the aggregation in uncertain ways. If the value of orness is specified in the form of interval, then for $\Omega_2 < \Omega_1$, the constraint in Equation 1b should be replaced by Inequality 9.

$$\Omega_2 \leq \frac{1}{n-1} \sum_{i=1}^{n} (n-i) w_i \leq \Omega_1 \tag{9}$$

If we solve the constrained optimization problem with an interval orness constraint, the optimal LSOWA weights will be determined whereas the value of orness is set at not Ω_1 but Ω_2 because the optimal objective value is minimized at less value of orness. Thus, rather than solving the mathematical program directly, it seems reasonable to think that uncertain orness indicates uncertain LSOWA weights ranging from the weights designated by Ω_1 to the weights designated by Ω_2. This consideration leads to the following Corollary 3.

COROLLARY 3. *When the value of orness is specified in the form of interval* ($\Omega_2 < \Omega_1$), *interval LSOWA weights can be constructed in such a way that when n is odd,* $[w_1(\Omega_2), w_1(\Omega_1)], \ldots, [w_{\frac{n+1}{2}-1}(\Omega_2), w_{\frac{n+1}{2}-1}(\Omega_1)]$, $[1/n, 1/n]$, $[w_{\frac{n+1}{2}+1}(\Omega_1), w_{\frac{n+1}{2}+1}(\Omega_2)], \ldots, [w_n(\Omega_1), w_n(\Omega_2)]$ *and when n is even,* $[w_1(\Omega_2), w_1(\Omega_1)], \ldots, [w_{\frac{n}{2}}(\Omega_2), w_{\frac{n}{2}}(\Omega_1)]$, $[w_{\frac{n}{2}+1}(\Omega_1), w_{\frac{n}{2}+1}(\Omega_2)], \ldots, [w_n(\Omega_1), w_n(\Omega_2)]$.

Proof. It directly follows from the results in Theorem 4. ■

It can be easily shown that the sum of lower bounds in interval LSOWA weights is less than 1 and the sum of upper bounds in interval LSOWA weights is greater than 1. Furthermore, for any orness value $\Omega \in [\Omega_2, \Omega_1]$ and odd number of objects, it holds that $w_i(\Omega) \in [w_i(\Omega_2), w_i(\Omega_1)]$, $i = 1, \ldots, \frac{n+1}{2} - 1$ and $w_i(\Omega) \in [w_i(\Omega_1), w_i(\Omega_2)]$, $i = \frac{n+1}{2} + 1, \ldots, n$. Similarly, for any orness value $\Omega \in [\Omega_2, \Omega_1]$ and even number of objects, $w_i(\Omega) \in [w_i(\Omega_2), w_i(\Omega_1)]$, $i = 1, \ldots, \frac{n}{2}$ and $w_i(\Omega) \in [w_i(\Omega_1), w_i(\Omega_2)]$, $i = \frac{n}{2} + 1, \ldots, n$.

Example. Suppose that a decision maker specifies his or her orness (i.e., degree of optimism) in [0.6, 0.7], then the LSOWA weights can also be specified in interval ones. For $n = 5$ (odd case), interval LSOWA weights are determined by combining these two weights,
$[w_1(0.6), w_2(0.6), w_3(0.6), w_4(0.6), w_5(0.6)] = [0.28, 0.24, 0.2, 0.16, 0.12]$
and
$[w_1(0.7), w_2(0.7), w_3(0.7), w_4(0.7), w_5(0.7)] = [0.36, 0.28, 0.2, 0.12, 0.04]$.
For $n = 6$ (even case), interval LSOWA weights are determined by combining these two weights,

Table II. Interval LSOWA weights with orness ranging between 0.6 and 0.7.

n	w_1(0.6–0.7)	w_2(0.6–0.7)	w_3(0.6–0.7)	w_4(0.6–0.7)	w_5(0.6–0.7)	w_6(0.6–0.7)
$n = 5$	[0.28, 0.36]	[0.24, 0.28]	[0.2, 0.2]	[0.12, 0.16]	[0.04, 0.12]	
$n = 6$	[0.238, 0.310]	[0.210, 0.252]	[0.181, 0.195]	[0.138, 0.152]	[0.081, 0.124]	[0.024, 0.095]

$[w_1(0.6), w_2(0.6), w_3(0.6), w_4(0.6), w_5(0.6), w_6(0.6)] = [0.238, 0.210, 0.181, 0.152, 0.124, 0.095]$
and
$[w_1(0.7), w_2(0.7), w_3(0.7), w_4(0.7), w_5(0.7), w_6(0.7)] = [0.310, 0.252, 0.195, 0.138, 0.081, 0.024]$.

Interval LSOWA weights for $n = 5$ and $n = 6$, respectively, are shown in Table II.

Now, we shall introduce an MCDM method under certainty when the LSOWA weights are specified in interval numbers. Assume, we have the following payoff matrix

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array}.$$

In the decision matrix, a set $A = \{A_1, A_2, \dots, A_m\}$ represents available alternatives and a set $C = \{C_1, C_2, \dots, C_n\}$ represents all relevant criteria that characterize the alternatives. In the above, a_{ij} $i = 1, \dots, m$, $j = 1, \dots, n$ indicate the payoff for selecting alternative A_i when the criterion is C_j.

When we have to determine the superiority between two alternatives $A_i = (a_{i1}, a_{i2}, \dots, a_{in})$ and $A_j = (a_{j1}, a_{j2}, \dots, a_{jn})$, it is said that A_i is superior to A_j if and only if Min $f(a_{i1}, a_{i2}, \dots, a_{in}) >$ Max $f(a_{j1}, a_{j2}, \dots, a_{jn})$ subject to the weights constraints given as interval descriptions. For specific, we have to solve linear programs that can be formulated as in Problem 1 for identifying the superiority between the two alternatives, A_i and A_j.[2,24]

Problem 1. Minimize (Maximize) $f(A_i) = \sum_{k=1}^{n} \tilde{a}_{ik} w_k \left(f(A_j) = \sum_{k=1}^{n} \tilde{a}_{jk} w_k \right)$

subject to

$$w_k \in [l_k, u_k], \text{ where } l_k, u_k \in [0, 1]$$

$$\left(\sum_{k=1}^{n} l_k < 1 \quad \text{and} \quad \sum_{k=1}^{n} u_k > 1 \right)$$

$$\sum_{k=1}^{n} w_k = 1, w_k \geq 0 \quad \text{for} \quad k = 1, \dots, n.$$

where $\tilde{a}_{ik}$ and $\tilde{a}_{jk}$ are the kth largest element in $(a_{i1}, a_{i2}, \ldots, a_{in})$ and $(a_{j1}, a_{j2}, \ldots, a_{jn})$, respectively.

In case the arguments, $\tilde{a}_{ik}$ (and $\tilde{a}_{jk}$) are specified in numerical values, the linear program can be solved by inspection since Problem 1 is a simple knapsack problem. The procedure for obtaining minimized OWA value of alternative A_i is described below in steps of an algorithm and illustrated with a numerical example.

Step 1. Assign the l_k weight to all w_k, $k = 1$ to n.
Step 2. Successively allocate the remaining available weight $(1 - \sum_{k=1}^{n} l_k)$ to the $\tilde{a}_{ik}$ from $k = n$ to 1 until the constraints become binding. Note that in the final solution at least $(n-1) w_k$s will take values l_k or u_k; only one w_k may take some value between l_k and u_k.

Remark. The procedure for obtaining minimized OWA value is equally applicable to a maximization problem, only except starting to allocate the remaining available weight $(1 - \sum_{k=1}^{n} l_k)$ to the $\tilde{a}_{ik}$ from $k = 1$ to n instead of $k = n$ to 1 until the constraints become binding.

COROLLARY 4. *When the value of orness is specified in the form of interval $\Omega \in [\Omega_2, \Omega_1]$ and interval LSOWA weights are constructed as in Corollary 3, the solution of Problem 1 can be found in such a way that, for an even number of objects,*

$$\min f(A_i) = \sum_{k=1}^{\frac{n}{2}} \tilde{a}_{ik} w_k(\Omega_2) + \sum_{k=\frac{n}{2}+1}^{n} \tilde{a}_{ik} w_k(\Omega_2)$$

$$\max f(A_i) = \sum_{k=1}^{\frac{n}{2}} \tilde{a}_{ik} w_k(\Omega_1) + \sum_{k=\frac{n}{2}+1}^{n} \tilde{a}_{ik} w_k(\Omega_1).$$

Proof. The sum of lower bound weights that should be assigned to all n objects is $\sum_{k=1}^{n} l_k = \sum_{k=1}^{\frac{n}{2}} w_k(\Omega_2) + \sum_{k=\frac{n}{2}+1}^{n} w_k(\Omega_1) = 1 + \frac{3n(\Omega_2-\Omega_1)}{2(n+1)}$. Therefore, the remaining weight for further allocations is $1 - \sum_{k=1}^{n} l_k = \frac{3n(\Omega_1-\Omega_2)}{2(n+1)}$, which is the very weights that are totally utilized to be allocated to the upper half of the reordered objects for a maximum problem (or lower half for a minimum problem) since $\sum_{k=1}^{\frac{n}{2}} [w_k(\Omega_1) - w_k(\Omega_2)] = \sum_{k=\frac{n}{2}+1}^{n} [w_k(\Omega_2) - w_k(\Omega_1)]$ by the symmetry property of the LSOWA weights and $\sum_{k=1}^{\frac{n}{2}} [w_k(\Omega_1) - w_k(\Omega_2)] = \frac{3n(\Omega_1-\Omega_2)}{2(n+1)}$. For an odd number of objects, we only have to replace a term in the $\frac{n+1}{2}$ place simply by $\tilde{a}_{i,\frac{n+1}{2}} \cdot \frac{1}{n}$. ■

Example. The interval LSOWA weights for $0.6 \le \Omega \le 0.7$ and $n = 4$ are specified as $w_1 = [0.34, 0.43]$, $w_2 = [0.28, 0.31]$, $w_3 = [0.19, 0.22]$, $w_4 = [0.07, 0.16]$, and two alternatives A_i and A_j are given as $A_i = (0.5, 0.8, 0.2, 0.3)$ and $A_j = (0.7, 0.4, 0.6, 0.5)$. Then, this information is sufficient to conclude that A_j is superior to A_i.

To show the computations, reorder the alternatives according to the magnitude of elements such as $\tilde{A}_i = (0.8, 0.5, 0.3, 0.2)$ and $\tilde{A}_j = (0.7, 0.6, 0.5, 0.4)$ in which $\tilde{a}_{ik}$ and $\tilde{a}_{jk}$, $k = 1, \ldots, 4$ are kth largest elements in A_i and A_j, respectively. Then, we assign the l_k weight to all w_k in the objective function $0.8w_1 + 0.5w_2 + 0.3w_3 + 0.2w_4$ for A_i to obtain a total of weight usage 0.88. Now, from the available weight 0.12, assign 0.09 to w_4 ($w_4 \leq 0.16$ constraint becomes binding) and 0.03 to w_3 to obtain a minimized OWA value 0.51. Then, the final optimal weights that were assigned to each of reordered arguments are $(w_1^*, w_2^*, w_3^*, w_4^*) = (0.34, 0.28, 0.22, 0.16)$ for a minimization problem. On the other hand, the optimal weights, which make an alternative A_i attain a maximum value, are $(w_1^*, w_2^*, w_3^*, w_4^*) = (0.43, 0.31, 0.19, 0.07)$ and the resulting maximized OWA value is

$$0.8 \times 0.43 + 0.5 \times 0.31 + 0.3 \times 0.19 + 0.2 \times 0.07 = 0.57.$$

Therefore, a range for A_i becomes $[f_{\min}(A_i), f_{\max}(A_i)] = [0.51, 0.57]$. Analogously, we can obtain the end points in which an alternative A_j lies: $[f_{\min}(A_j), f_{\max}(A_j)] = [0.58, 0.61]$. From the results, it can be concluded that A_j is superior to A_i since $f_{\min}(A_j) = 0.58 > 0.57 = f_{\max}(A_i)$.

Remark. If a superior alternative is not identified, and if the decision maker does not wish to further reduce the uncertainty by tightening the bounds on w_ks, one of the several secondary criteria can be adopted.[24]

4. CONCLUDING REMARKS

In this paper, we present an alternative weighting method, the LSOWA for determining the OWA weights. The method is basically in line with the MEOWA weighting method in that we intend to obtain the OWA weights minimizing the variations from the equal weights while satisfying the prescribed value of orness. In other words, the LSOWA weights minimizing the variations from the equal weights indicate the weights that are evenly spread out around the equal weights as much as possible and that hence possibly maximize entropy.

When a decision maker specifies an uncertain interval value of orness, we can construct interval LSOWA weights. Furthermore, a method for prioritizing multiple alternatives characterized by multiple criteria is presented under those interval LSOWA weights.

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