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# A nonterminating ${}_8\phi_7$ summation for the root system $C_r$

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## Abstract

Using multiple  $q$ -integrals and a determinant evaluation, we establish a nonterminating  ${}_8\phi_7$  summation for the root system  $C_r$ . We also give some important specializations explicitly.

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## 1. Introduction

Bailey's [6, Eq. (3.3)] nonterminating very-well-poised  ${}_8\phi_7$  summation,

$$\begin{aligned}
& {}_8\phi_7 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, e, f \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e, aq/f \end{matrix}; q, q \right] \\
& + \frac{(aq, c, d, e, f, b/a, bq/c, bq/d, bq/e, bq/f; q)_\infty}{(a/b, aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a, b^2q/a; q)_\infty} \\
& \times {}_8\phi_7 \left[ \begin{matrix} b^2/a, qb/\sqrt{a}, -qb/\sqrt{a}, b, bc/a, bd/a, be/a, bf/a \\ b/\sqrt{a}, -b/\sqrt{a}, bq/a, bq/c, bq/d, bq/e, bq/f \end{matrix}; q, q \right] \\
& = \frac{(aq, b/a, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef; q)_\infty}{(aq/c, aq/d, aq/e, aq/f, bc/a, bd/a, be/a, bf/a; q)_\infty}, \tag{1.1}
\end{aligned}$$

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where  $a^2q = bcdef$  (cf. [9, Eq. (2.11.7)]), is one of the deepest results in the classical theory of basic hypergeometric series. It contains many important identities as special cases (such as the nonterminating  ${}_3\phi_2$  summation, the terminating  ${}_8\phi_7$  summation, and all their specializations including the  $q$ -binomial theorem). One way to derive (1.1) is to start with a particular rational function identity, namely Bailey's [5] very-well-poised  ${}_{10}\phi_9$  transformation, and apply a nontrivial limit procedure, see the exposition in [9, Sections 2.10 and 2.11].

Basic hypergeometric series (and, more generally,  $q$ -series) have various applications in combinatorics, number theory, representation theory, statistics, and physics, see [1,2]. For a general account of the importance of basic hypergeometric series in the theory of special functions see [4].

There are different types of multivariable series. The one we are concerned with are so-called *multiple basic hypergeometric series associated to root systems* (or, equivalently, to *Lie algebras*). This is mainly just a classification of certain multiple series according to the type of specific factors (such as a Vandermonde determinant) appearing in the summand. We omit giving a precise definition here, but instead refer to papers of Bhatnagar [7] or Milne [15, Section 5].

The significance of the nonterminating  ${}_8\phi_7$  summation (1.1) lies in the fact that it can be used for deriving other nonterminating transformation formulae; see [9, Sections 2.12 and 3.8, 18]. Thus, it is apparently desirable to find (various) multivariable generalizations of Bailey's nonterminating  ${}_8\phi_7$  summation.

In this paper, we give a multivariable nonterminating  ${}_8\phi_7$  summation for the root system  $C_r$  (or, equivalently, the symplectic group  $Sp(r)$ ); see Corollary 5.1. We deduce this result from an equivalent multiple  $q$ -integral evaluation (Theorem 4.1). In our proof of the latter we utilize a simple determinant method, essentially the same which was introduced in [11] and which we further exploited in [17] to derive a number of identities for multiple basic hypergeometric series. The difference here is that now we apply the method to *integrals* and  $q$ -*integrals* whereas in [17] we had only applied it to sums. Our new  $C_r$  nonterminating  ${}_8\phi_7$  summation is not the first multivariable nonterminating  ${}_8\phi_7$  sum that has been found. In fact, Degenhardt and Milne [8] already derived such a result for the root system  $A_n$  (or, equivalently, the unitary group  $U(n)$ ), a result we consider to be deeper than ours. While Corollary 5.1 is derived by elementary means, by combining known one-variable results with the argument of interchanging the order of summations, or of summation and ( $q$ -)integration (this is what the determinant method in this article really does), Degenhardt and Milne deduce their multivariable summation formula from Milne's and Newcomb's [16] extension of Gasper and Rahman's [9, Section 2.10] analysis to higher dimensions which appears to be fairly nontrivial. However, supported by the combinatorial applications (in [14,10]) of identities of type strikingly similar to the one being investigated in this paper, we believe that the identities derived here will have future applications and deserve being written out in detail.

Our paper is organized as follows: In Section 2, we review some basics in the theory of basic hypergeometric series. Further, we also note a determinant lemma which we need as an ingredient in proving our results in Sections 3 and 4. We demonstrate the method of proof in Section 3 by deriving a simple multidimensional beta integral evaluation. In Section 4, we derive an (attractive) multiple  $q$ -integral evaluation, Theorem 4.1, which in Section 5 is used to explicitly write out a nonterminating  ${}_8\phi_7$  summation for the root system  $C_r$ ; see Corollary 5.1. Finally, in Section 6 we explicitly list several interesting specializations of Theorem 4.1 (and of the equivalent Corollary 5.1).

## 2. Basic hypergeometric series and a determinant lemma

Here we recall some standard notation for  $q$ -series, and basic hypergeometric series (cf. [9]).

Let  $q$  be a complex number such that  $0 < |q| < 1$ . We define the  $q$ -shifted factorial for all integers  $k$  by

$$(a; q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j) \quad \text{and} \quad (a; q)_k := \frac{(a; q)_\infty}{(aq^k; q)_\infty}.$$

For brevity, we employ the condensed notation

$$(a_1, \dots, a_m; q)_k \equiv (a_1; q)_k \cdots (a_m; q)_k,$$

where  $k$  is an integer or infinity. Further, we utilize

$${}_s\phi_{s-1} \left[ \begin{matrix} a_1, a_2, \dots, a_s \\ b_1, b_2, \dots, b_{s-1} \end{matrix}; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_s; q)_k}{(q, b_1, \dots, b_{s-1}; q)_k} z^k \tag{2.1}$$

to denote the *basic hypergeometric  ${}_s\phi_{s-1}$  series*. In (2.1),  $a_1, \dots, a_s$  are called the *upper parameters*,  $b_1, \dots, b_{s-1}$  the *lower parameters*,  $z$  is the *argument*, and  $q$  the *base* of the series. The series in (2.1) terminates if one of the upper parameters, say  $a_s$ , equals  $q^{-n}$ , where  $n$  is a nonnegative integer. If the series does not terminate, we need  $|z| < 1$  for convergence.

The classical theory of basic hypergeometric series contains numerous summation and transformation formulae involving  ${}_s\phi_{s-1}$  series. Many of these summation theorems require that the parameters satisfy the condition of being either balanced and/or very-well-poised. An  ${}_s\phi_{s-1}$  basic hypergeometric series is called *balanced* if  $b_1 \cdots b_{s-1} = a_1 \cdots a_s q$  and  $z = q$ . An  ${}_s\phi_{s-1}$  series is *well-poised* if  $a_1 q = a_2 b_1 = \cdots = a_s b_{s-1}$  and *very-well-poised* if it is well-poised and if  $a_2 = -a_3 = q\sqrt{a_1}$ . Note that the factor

$$\frac{1 - a_1 q^{2k}}{1 - a_1}$$

appears in a very-well-poised series. The parameter  $a_1$  is usually referred to as the *special parameter* of such a series.

One of the most important summation theorems in the theory of basic hypergeometric series is Jackson’s [13] terminating very-well-poised balanced  ${}_8\phi_7$  summation (cf. [9, Eq. (2.6.2)]):

$$\begin{aligned} & {}_8\phi_7 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, d, a^2 q^{1+n}/bcd, q^{-n} \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, bcdq^{-n}/a, aq^{1+n} \end{matrix}; q, q \right] \\ &= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}. \end{aligned} \tag{2.2}$$

Clearly, (2.2) is the special case  $f \rightarrow q^{-n}$  of (1.1).

For studying nonterminating basic hypergeometric series it is often convenient to utilize Jackson's [12]  $q$ -integral notation, defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad (2.3)$$

where

$$\int_0^a f(t) d_q t = a(1-q) \sum_{k=0}^{\infty} f(aq^k) q^k. \quad (2.4)$$

If  $f$  is continuous on  $[0, a]$ , then it is easily seen that

$$\lim_{q \rightarrow 1^-} \int_0^a f(t) d_q t = \int_0^a f(t) dt,$$

see [9, Eq. (1.11.6)].

Using the above  $q$ -integral notation, the nonterminating  ${}_8\phi_7$  summation (1.1) can be conveniently expressed as

$$\begin{aligned} & \int_a^b \frac{(qt/a, qt/b, t/\sqrt{a}, -t/\sqrt{a}, qt/c, qt/d, qt/e, qt/f; q)_{\infty}}{(t, bt/a, qt/\sqrt{a}, -qt/\sqrt{a}, ct/a, dt/a, et/a, ft/a; q)_{\infty}} d_q t \\ &= \frac{b(1-q)(q, a/b, bq/a, aq/cd, aq/ce, aq/cf, aq/de, aq/df, aq/ef; q)_{\infty}}{(b, c, d, e, f, bc/a, bd/a, be/a, bf/a; q)_{\infty}}, \end{aligned} \quad (2.5)$$

where  $a^2q = bcdef$  (cf. [9, Eq. (2.11.8)]).

A standard reference for basic hypergeometric series is Gasper and Rahman's text [9]. In our computations in the subsequent sections we frequently use some elementary identities of  $q$ -shifted factorials, listed in [9, Appendix I].

The following determinant evaluation was given as Lemma A.1 in [17] where it was derived from a determinant lemma of Krattenthaler [14, Lemma 34].

**Lemma 2.1.** *Let  $X_1, \dots, X_r, A, B,$  and  $C$  be indeterminate. Then there holds*

$$\begin{aligned} \det_{1 \leq i, j \leq r} \left( \frac{(AX_i, AC/X_i; q)_{r-j}}{(BX_i, BC/X_i; q)_{r-j}} \right) &= \prod_{1 \leq i < j \leq r} (X_j - X_i)(1 - C/X_i X_j) \\ &\times A \binom{r}{2} q \binom{r}{3} \prod_{i=1}^r \frac{(B/A, ABCq^{2r-2i}; q)_{i-1}}{(BX_i, BC/X_i; q)_{r-1}}. \end{aligned} \quad (2.6)$$

The above determinant evaluation was generalized to the elliptic case (more precisely, to an evaluation involving Jacobi theta functions) in [20, Corollary 5.4].

### 3. A multidimensional beta integral evaluation

Here, we present a simple multivariable extension of Euler’s beta integral evaluation. The proof serves as an illustration of the determinant method which we use in Section 4 to derive a multivariable extension of (2.5).

**Proposition 3.1.** *Let  $a, b$ , and  $x_1, \dots, x_r$  be indeterminate. Then*

$$\int_0^1 \cdots \int_0^1 \prod_{1 \leq i < j \leq r} (u_i - u_j) \prod_{i=1}^r u_i^{a-1+x_i} (1 - u_i)^{b-1} du_r \cdots du_1$$

$$= \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \frac{\Gamma(a + x_i) \Gamma(b + i - 1)}{\Gamma(a + b + x_i + r - 1)}, \tag{3.1}$$

provided  $\Re(a + x_i), \Re(b) > 0$ , for  $i = 1, \dots, r$ .

**Remark 3.2.** We note the differences between Proposition 3.1 and Selberg’s [19] integral,

$$\int_0^1 \cdots \int_0^1 \prod_{1 \leq i < j \leq r} |u_i - u_j|^{2c} \prod_{i=1}^r u_i^{a-1} (1 - u_i)^{b-1} du_r \cdots du_1$$

$$= \prod_{i=1}^r \frac{\Gamma(a + (i - 1)c) \Gamma(b + (i - 1)c) \Gamma(ic + 1)}{\Gamma(a + b + (r + i - 2)c) \Gamma(c + 1)}, \tag{3.2}$$

where  $\Re(a), \Re(b) > 0$ , and  $\Re(c) > \max(-1/r, -\Re(a)/(r - 1), -\Re(b)/(r - 1))$ . In (3.1) we have additional parameters  $x_1, \dots, x_r$ , while in (3.2) the absolute value of the discriminant  $\prod_{1 \leq i < j \leq r} (u_i - u_j)$  in the integrand is taken to an arbitrary power  $2c$ , which makes the computation considerably more difficult.

**Proof of Proposition 3.1.** First, we note that if in Lemma 2.1 we let  $C \rightarrow 0$ , replace  $A, B$ , and  $X_i$  by  $q^a, q^{a+b}$ , and  $q^{x_i}$ , for  $i = 1, \dots, r$ , respectively, and then let  $q \rightarrow 1$ , we have

$$\det_{1 \leq i, j \leq r} \left( \frac{(a + x_i)_{r-j}}{(a + b + x_i)_{r-j}} \right) = \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \frac{(b)_{i-1}}{(a + b + x_i)_{r-1}}, \tag{3.3}$$

where

$$(\alpha)_k := \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} \tag{3.4}$$

is the *shifted factorial*. Thus,

$$\int_0^1 \cdots \int_0^1 \prod_{1 \leq i < j \leq r} (u_i - u_j) \prod_{i=1}^r u_i^{a-1+x_i} (1 - u_i)^{b-1} du_r \cdots du_1$$

$$= \det_{1 \leq i, j \leq r} \left( \int_0^1 u_i^{a-1+x_i+r-j} (1 - u_i)^{b-1} du_i \right)$$

$$\begin{aligned}
 &= \det_{1 \leq i, j \leq r} \left( \frac{\Gamma(a + x_i + r - j) \Gamma(b)}{\Gamma(a + b + x_i + r - j)} \right) = \det_{1 \leq i, j \leq r} \left( \frac{\Gamma(a + x_i) \Gamma(b)}{\Gamma(a + b + x_i)} \frac{(a + x_i)_{r-j}}{(a + b + x_i)_{r-j}} \right) \\
 &= \prod_{i=1}^r \frac{\Gamma(a + x_i) \Gamma(b)}{\Gamma(a + b + x_i)} \cdot \det_{1 \leq i, j \leq r} \left( \frac{(a + x_i)_{r-j}}{(a + b + x_i)_{r-j}} \right) \\
 &= \prod_{i=1}^r \frac{\Gamma(a + x_i) \Gamma(b)}{\Gamma(a + b + x_i)} \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \frac{(b)_{i-1}}{(a + b + x_i)_{r-1}} \\
 &= \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \frac{\Gamma(a + x_i) \Gamma(b + i - 1)}{\Gamma(a + b + x_i + r - 1)},
 \end{aligned}$$

where we have used linearity of the determinant with respect to rows, the Vandermonde determinant evaluation  $\det_{1 \leq i, j \leq r} (u_i^{r-j}) = \prod_{1 \leq i < j \leq r} (u_i - u_j)$ , Euler’s beta integral evaluation  $\int_0^1 u^{a-1} (1 - u)^{b-1} dt = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ , for  $\Re(a), \Re(b) > 0$ , the definition of the shifted factorial (3.4), and the determinant evaluation (3.3).  $\square$

#### 4. A multiple $q$ -integral evaluation

By iteration, the extension of (2.4) to multiple  $q$ -integrals is straightforward:

$$\begin{aligned}
 &\int_0^{a_1} \cdots \int_0^{a_r} f(t_1, \dots, t_r) d_q t_r \cdots d_q t_1 \\
 &= a_1 \cdots a_r (1 - q)^r \sum_{k_1, \dots, k_r=0}^{\infty} f(a_1 q^{k_1}, \dots, a_r q^{k_r}) q^{k_1 + \dots + k_r}.
 \end{aligned} \tag{4.1}$$

Similarly, the extension of (2.3) is

$$\begin{aligned}
 &\int_{a_1}^{b_1} \cdots \int_{a_r}^{b_r} f(t_1, \dots, t_r) d_q t_r \cdots d_q t_1 \\
 &= \sum_{S \subseteq \{1, 2, \dots, r\}} \left( \prod_{i \in S} (-a_i) \right) \left( \prod_{i \notin S} b_i \right) (1 - q)^r \\
 &\quad \times \sum_{k_1, \dots, k_r=0}^{\infty} f(c_1(S) q^{k_1}, \dots, c_r(S) q^{k_r}) q^{k_1 + \dots + k_r},
 \end{aligned} \tag{4.2}$$

where the outer sum runs over all  $2^r$  subsets  $S$  of  $\{1, 2, \dots, r\}$ , and, where  $c_i(S) = a_i$  if  $i \in S$  and  $c_i(S) = b_i$  if  $i \notin S$ , for  $i = 1, \dots, r$ .

We give our main result, a  $C_r$  extension of (2.5):

**Theorem 4.1.** *Let  $a^2q^{2-r} = bcdef$ . Then there holds*

$$\begin{aligned}
 & \int_{ax_1}^b \cdots \int_{ax_r}^b \prod_{1 \leq i < j \leq r} (t_i - t_j)(1 - t_it_j/a) \prod_{i=1}^r (1 - t_i^2/a) \\
 & \times \prod_{i=1}^r \frac{(qt_i/ax_i, qt_i/b, qt_i/c, qt_i/d, qt_i/e, qt_ix_i/f; q)_\infty}{(t_ix_i, bt_i/a, ct_i/a, dt_i/a, et_i/a, ft_i/ax_i; q)_\infty} d_q t_r \cdots d_q t_1 \\
 & = a^{\binom{r}{2}} b^r (1 - q)^r \prod_{1 \leq i < j \leq r} (x_i - x_j)(1 - ax_ix_j/f) \\
 & \times \prod_{i=1}^r \frac{(q, ax_i/b, bq/ax_i, aq^{2-i}/cd, aq^{2-i}/ce; q)_\infty}{(bx_i, cx_i, dx_i, ex_i, f; q)_\infty} \\
 & \times \prod_{i=1}^r \frac{(ax_iq/cf, aq^{2-i}/de, ax_iq/df, ax_iq/ef; q)_\infty}{(bcq^{i-1}/a, bdq^{i-1}/a, beq^{i-1}/a, bf/ax_i; q)_\infty}. \tag{4.3}
 \end{aligned}$$

**Proof.** We have

$$\begin{aligned}
 \prod_{1 \leq i < j \leq r} (t_i - t_j)(1 - t_it_j/a) &= \prod_{i=1}^r \frac{(q^{2-r}t_i/d, aq^{2-r}/dt_i; q)_{r-1}}{(aq^{2-r}/cd, cq^{2+r-2i}/d; q)_{i-1}} t_i^{r-1} \\
 & \times c^{-\binom{r}{2}} q^{-\binom{r}{3}} \det_{1 \leq i, j \leq r} \left( \frac{(ct_i/a, c/t_i; q)_{r-j}}{(q^{2-r}t_i/d, aq^{2-r}/dt_i; q)_{r-j}} \right)
 \end{aligned}$$

due to the  $X_i \mapsto t_i$ ,  $A \mapsto c/a$ ,  $B \mapsto q^{2-r}/d$ , and  $C \mapsto a$  case of Lemma 2.1. Hence, using some elementary identities from [9, Appendix I], we may write the left-hand side of (4.3) as

$$\begin{aligned}
 & \left(\frac{a}{d}\right)^{\binom{r}{2}} q^{-\binom{r}{3}} \prod_{i=1}^r (aq^{2-r}/cd, cq^{2+r-2i}/d; q)_{i-1}^{-1} \\
 & \times \det_{1 \leq i, j \leq r} \left( \int_{ax_i}^b \frac{(1 - t_i^2/a)(t_iq/ax_i, t_iq/b; q)_\infty}{(t_ix_i, bt_i/a; q)_\infty} \right. \\
 & \left. \times \frac{(t_iq^{1-r+j}/c, t_iq^{2-j}/d, t_iq/e, t_ix_iq/f; q)_\infty}{(ct_iq^{r-j}/a, dt_iq^{j-1}/a, et_i/a, ft_i/ax_i; q)_\infty} d_q t_i \right).
 \end{aligned}$$

Now, to the integral inside the determinant we apply the  $q$ -integral evaluation (2.5), with the substitution  $t \mapsto t_ix_i$ , and the replacements  $a \mapsto ax_i^2$ ,  $b \mapsto bx_i$ ,  $c \mapsto cq^{r-j}x_i$ ,  $d \mapsto dq^{j-1}x_i$ , and  $e \mapsto ex_i$ .

Thus we obtain

$$\left(\frac{a}{d}\right)^{\binom{r}{2}} q^{-\binom{r}{3}} \prod_{i=1}^r (aq^{2-r}/cd, cq^{2+r-2i}/d; q)_{i-1}^{-1} \\ \times \det_{1 \leq i, j \leq r} \left( \frac{b(1-q)(q, ax_i/b, bq/ax_i, aq^{2-r}/cd, aq^{1-r+j}/ce; q)_{\infty}}{(bx_i, cx_i q^{r-j}, dx_i q^{j-1}, ex_i, f; q)_{\infty}} \right. \\ \left. \times \frac{(ax_i q^{1-r+j}/cf, aq^{2-j}/de, ax_i q^{2-j}/df, ax_i q/ef; q)_{\infty}}{(bcq^{r-j}/a, bdq^{j-1}/a, be/a, bf/ax_i; q)_{\infty}} \right).$$

Now, by using linearity of the determinant with respect to rows and columns, we take some factors out of the determinant and obtain

$$\left(\frac{a}{d}\right)^{\binom{r}{2}} q^{-\binom{r}{3}} b^r (1-q)^r \prod_{i=1}^r \frac{(q, ax_i/b, bq/ax_i; q)_{\infty}}{(aq^{2-r}/cd, cq^{2+r-2i}/d; q)_{i-1}} \\ \times \prod_{i=1}^r \frac{(aq^{2-r}/cd, aq^{1-r+i}/ce, ax_i q/cf, aq^{2-i}/de, ax_i q^{2-r}/df, ax_i q/ef; q)_{\infty}}{(bx_i, cx_i, dx_i q^{r-1}, ex_i, f, bcq^{r-i}/a, bdq^{i-1}/a, be/a, bf/ax_i; q)_{\infty}} \\ \times \left(\frac{a}{cdf}\right)^{\binom{r}{2}} q^{-3\binom{r}{3}} \det_{1 \leq i, j \leq r} \left( \frac{(cx_i, cf/ax_i; q)_{r-j}}{(ax_i q^{2-r}/df, q^{2-r}/dx_i; q)_{r-j}} \right).$$

The determinant can be evaluated by means of Lemma 2.1 with  $X_i \mapsto x_i$ ,  $A \mapsto c$ ,  $B \mapsto aq^{2-r}/df$ , and  $C \mapsto f/a$ ; specifically

$$\det_{1 \leq i, j \leq r} \left( \frac{(cx_i, cf/ax_i; q)_{r-j}}{(ax_i q^{2-r}/df, q^{2-r}/dx_i; q)_{r-j}} \right) \\ = c^{\binom{r}{2}} q^{\binom{r}{3}} \prod_{1 \leq i < j \leq r} (x_j - x_i)(1 - f/ax_i x_j) \prod_{i=1}^r \frac{(aq^{2-r}/cdf, cq^{2+r-2i}/d; q)_{i-1}}{(ax_i q^{2-r}/df, q^{2-r}/dx_i; q)_{r-1}}.$$

Substituting our calculations and performing further elementary manipulations we arrive at the right-hand side of (4.3). □

### 5. A multivariable nonterminating ${}_8\phi_7$ summation

Note that if the integrand  $f(t_1, \dots, t_r)$  of the multiple integral in (4.3) were an antisymmetric function in  $t_1, \dots, t_r$ , the multiple sum in (4.2) would simplify considerably. In fact, if  $t_i = bq^{k_i}$  and  $t_j = bq^{k_j}$ , for a pair  $i < j$ , we would then have

$$\sum_{k_i, k_j=0}^{\infty} f(\dots, bq^{k_i}, \dots, bq^{k_j}, \dots) = 0.$$



(A double sum of any function antisymmetric in its two summation indices vanishes.) Thus, the multiple  $q$ -integral  $\int_{ax_1}^b \cdots \int_{ax_r}^b f(t_1, \dots, t_r) d_q t_r \cdots d_q t_1$ , being a sum of  $2^r$  sums according to (4.2), would reduce to a sum of  $r + 1$  nonzero sums. In particular, we would have

$$\begin{aligned} & \int_{ax_1}^b \cdots \int_{ax_r}^b f(t_1, \dots, t_r) d_q t_r \cdots d_q t_1 \\ &= (-1)^r a^r x_1 \cdots x_r (1 - q)^r \sum_{k_1, \dots, k_r=0}^{\infty} f(ax_1 q^{k_1}, \dots, ax_r q^{k_r}) q^{\sum_{i=1}^r k_i} \\ & \quad + (-1)^{r-1} a^{r-1} b x_1 \cdots x_r (1 - q)^r \sum_{l=1}^r x_l^{-1} \\ & \quad \times \sum_{k_1, \dots, k_r=0}^{\infty} f(ax_1 q^{k_1}, \dots, ax_{l-1} q^{k_{l-1}}, b q^{k_l}, ax_{l+1} q^{k_{l+1}}, \dots, ax_r q^{k_r}) q^{\sum_{i=1}^r k_i}. \end{aligned} \tag{5.1}$$

A very similar situation occurs in the  $U(n)$  (or  $A_r$ ) nonterminating  ${}_8\phi_7$  summation by [8] (however, their argument is reversed, i.e., they first derive a nonterminating summation and then deduce the multiple  $q$ -integral evaluation). Unfortunately, in our case the multiple integrand in (4.3) is *not* antisymmetric in  $t_1, \dots, t_r$ , whence we have all  $2^r$  sums on the right-hand side of (4.2).

We write out (4.2) explicitly for our integral in (4.3):

$$\begin{aligned} & \int_{ax_1}^b \cdots \int_{ax_r}^b \prod_{1 \leq i < j \leq r} (t_i - t_j)(1 - t_i t_j / a) \prod_{i=1}^r (1 - t_i^2 / a) \\ & \quad \times \prod_{i=1}^r \frac{(qt_i / ax_i, qt_i / b, qt_i / c, qt_i / d, qt_i / e, qt_i x_i / f; q)_{\infty}}{(t_i x_i, bt_i / a, ct_i / a, dt_i / a, et_i / a, f t_i / ax_i; q)_{\infty}} d_q t_r \cdots d_q t_1 \\ &= \sum_{S \subseteq \{1, 2, \dots, r\}} (-1)^{|S|} a^{|S|} b^{r-|S|} (1 - q)^r a^{\binom{|S|}{2}} b^{\binom{r-|S|}{2}} \prod_{i \in S} x_i \\ & \quad \times \sum_{k_1, \dots, k_r=0}^{\infty} \prod_{\substack{1 \leq i < j \leq r \\ i, j \in S}} (x_i q^{k_i} - x_j q^{k_j})(1 - ax_i x_j q^{k_i+k_j}) \prod_{i \in S} (1 - ax_i^2 q^{2k_i}) \\ & \quad \times \prod_{\substack{1 \leq i < j \leq r \\ i, j \notin S}} (q^{k_i} - q^{k_j})(1 - b^2 q^{k_i+k_j} / a) \prod_{i \notin S} (1 - b^2 q^{2k_i} / a) \\ & \quad \times \prod_{i \in S, j \notin S} (ax_i q^{k_i} - b q^{k_j})(1 - bx_i q^{k_i+k_j}) (-1)^{\chi(i > j)} \end{aligned}$$

$$\begin{aligned} &\times \prod_{i \in S} \frac{(q^{1+k_i}, ax_i q^{1+k_i}/b, ax_i q^{1+k_i}/c, ax_i q^{1+k_i}/d, ax_i q^{1+k_i}/e, ax_i^2 q^{1+k_i}/f; q)_\infty}{(ax_i^2 q^{k_i}, bx_i q^{k_i}, cx_i q^{k_i}, dx_i q^{k_i}, ex_i q^{k_i}, f q^{k_i}; q)_\infty} \\ &\times \prod_{i \notin S} \frac{(bq^{1+k_i}/ax_i, q^{1+k_i}, bq^{1+k_i}/c, bq^{1+k_i}/d, bq^{1+k_i}/e, bx_i q^{1+k_i}/f; q)_\infty}{(bx_i q^{k_i}, b^2 q^{k_i}/a, bcq^{k_i}/a, bdq^{k_i}/a, beq^{k_i}/a, bfq^{k_i}/ax_i; q)_\infty} q^{\sum_{i=1}^r k_i}, \end{aligned}$$

where  $|S|$  denotes the number of elements of  $S$ , and  $\chi$  is the truth function (which evaluates to one if the argument is true and evaluates to zero otherwise). Now, if we set the obtained sum of  $2^r$  sums equal to the right-hand side of (4.3) and divide both sides by

$$\begin{aligned} &(-1)^r a^{\binom{r+1}{2}} x_1 \cdots x_r (1-q)^r \prod_{1 \leq i < j \leq r} (x_i - x_j)(1 - ax_i x_j) \\ &\times \prod_{i=1}^r \frac{(q, ax_i q/b, ax_i q/c, ax_i q/d, ax_i q/e, ax_i^2 q/f; q)_\infty}{(ax_i^2 q, bx_i, cx_i, dx_i, ex_i, f; q)_\infty}, \end{aligned}$$

and simplify, we obtain the following result which reduces to (1.1) when  $r = 1$ .

**Corollary 5.1** (A  $C_r$  nonterminating  ${}_8\phi_7$  summation). *Let  $a^2 q^{2-r} = bcdef$ . Then there holds*

$$\begin{aligned} &\sum_{S \subseteq \{1, 2, \dots, r\}} \left(\frac{b}{a}\right)^{\binom{r-|S|}{2}} \prod_{i \notin S} \frac{(ax_i^2 q, cx_i, dx_i, ex_i, f; q)_\infty}{(ax_i/b, ax_i q/c, ax_i q/d, ax_i q/e, ax_i^2 q/f; q)_\infty} \\ &\times \prod_{i \notin S} \frac{(b/ax_i, bq/c, bq/d, bq/e, bx_i q/f; q)_\infty}{(b^2 q/a, bc/a, bd/a, be/a, bf/ax_i; q)_\infty} \\ &\times \sum_{k_1, \dots, k_r=0}^\infty \prod_{\substack{1 \leq i < j \leq r \\ i, j \in S}} \frac{(x_i q^{k_i} - x_j q^{k_j})(1 - ax_i x_j q^{k_i+k_j})}{(x_i - x_j)(1 - ax_i x_j)} \prod_{i \in S} \frac{(1 - ax_i^2 q^{2k_i})}{(1 - ax_i^2)} \\ &\times \prod_{\substack{1 \leq i < j \leq r \\ i, j \notin S}} \frac{(q^{k_i} - q^{k_j})(1 - b^2 q^{k_i+k_j}/a)}{(x_i - x_j)(1 - ax_i x_j)} \prod_{i \notin S} \frac{(1 - b^2 q^{2k_i}/a)}{(1 - b^2/a)} \\ &\times \prod_{i \in S, j \notin S} \frac{(x_i q^{k_i} - bq^{k_j}/a)(1 - bx_i q^{k_i+k_j})}{(x_i - x_j)(1 - ax_i x_j)} \\ &\times \prod_{i \in S} \frac{(ax_i^2, bx_i, cx_i, dx_i, ex_i, f; q)_{k_i}}{(q, ax_i q/b, ax_i q/c, ax_i q/d, ax_i q/e, ax_i^2 q/f; q)_{k_i}} \\ &\times \prod_{i \notin S} \frac{(b^2/a, bx_i, bc/a, bd/a, be/a, bf/ax_i; q)_{k_i}}{(q, bq/ax_i, bq/c, bq/d, bq/e, bx_i q/f; q)_{k_i}} q^{\sum_{i=1}^r k_i} \end{aligned}$$

$$\begin{aligned}
 &= \prod_{1 \leq i < j \leq r} \frac{(1 - ax_i x_j / f)}{(1 - ax_i x_j)} \prod_{i=1}^r \frac{(ax_i^2 q, b / ax_i, aq^{2-i} / cd, aq^{2-i} / ce; q)_\infty}{(ax_i q / c, ax_i q / d, ax_i q / e, ax_i^2 q / f; q)_\infty} \\
 &\quad \times \prod_{i=1}^r \frac{(ax_i q / cf, aq^{2-i} / de, ax_i q / df, ax_i q / ef; q)_\infty}{(bcq^{i-1} / a, bdq^{i-1} / a, beq^{i-1} / a, bf / ax_i; q)_\infty}. \tag{5.2}
 \end{aligned}$$

**6. Specializations**

It is clear that in Corollary 5.1, if we replace  $e$  by  $a^2 q^{2-r} / bcd f$  and then let  $f \rightarrow q^{-N}$ , we obtain a  $C_r$  extension of Jackson’s  ${}_8\phi_7$  summation (2.2). This result

$$\begin{aligned}
 &\sum_{k_1, \dots, k_r=0}^N \prod_{1 \leq i < j \leq r} \frac{(x_i q^{k_i} - x_j q^{k_j})(1 - ax_i x_j q^{k_i+k_j})}{(x_i - x_j)(1 - ax_i x_j)} \prod_{i=1}^r \frac{(1 - ax_i^2 q^{2k_i})}{(1 - ax_i^2)} \\
 &\quad \times \prod_{i=1}^r \frac{(ax_i^2, bx_i, cx_i, dx_i, a^2 x_i q^{2-r+N} / bcd, q^{-N}; q)_{k_i}}{(q, ax_i q / b, ax_i q / c, ax_i q / d, bcd x_i q^{r-1-N} / a, ax_i^2 q^{1+N}; q)_{k_i}} q^{\sum_{i=1}^r k_i} \\
 &= \prod_{1 \leq i < j \leq r} \frac{(1 - ax_i x_j q^N)}{(1 - ax_i x_j)} \prod_{i=1}^r \frac{(ax_i^2 q, aq^{2-i} / bc, aq^{2-i} / bd, aq^{2-i} / cd; q)_N}{(aq^{2-r} / bcd x_i, ax_i q / d, ax_i q / c, ax_i q / b; q)_N}, \tag{6.1}
 \end{aligned}$$

was given as Theorem 4.3 in [17]. An extension of (6.1) to elliptic hypergeometric series was found in [20, Theorem 5.1].

Next, if in (4.3), we first replace  $e$  by  $a^2 q^{2-r} / bcd f$ , then  $b$  by  $aq/b$ , do the substitution  $t_i \mapsto at_i$ ; divide both sides by  $a^{\binom{r+1}{2}}$ , let  $a \rightarrow 0$ , and afterwards do the simultaneous substitutions  $x_i \mapsto x_i \sqrt{a}$ ,  $b \mapsto q\sqrt{a}/b$ ,  $c \mapsto cq^{1-r} / bdf \sqrt{a}$ ,  $d \mapsto d\sqrt{a}$ ,  $f \mapsto fa$ , then multiply both sides by  $\sqrt{a}^{\binom{r+1}{2}}$ , and perform the substitution  $t_i \mapsto t_i / \sqrt{a}$ , we obtain the following multiple  $q$ -integral evaluation which is an  $r$ -dimensional extension of Eq. (2.10.18) in [9].

**Theorem 6.1.** *Let  $c = abdefq^{r-1}$ . Then there holds*

$$\begin{aligned}
 &\int_{ax_1}^b \cdots \int_{ax_r}^b \prod_{1 \leq i < j \leq r} (t_i - t_j) \prod_{i=1}^r \frac{(qt_i / ax_i, qt_i / b, ct_i; q)_\infty}{(dt_i, et_i, ft_i / x_i; q)_\infty} d_q t_r \cdots d_q t_1 \\
 &= a^{\binom{r}{2}} b^r (1 - q)^r \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \frac{(q, ax_i / b, bq / ax_i, cq^{1-i} / d, cq^{1-i} / e, cx_i / f; q)_\infty}{(adx_i, aex_i, af, bdq^{i-1}, beq^{i-1}, bf / x_i; q)_\infty}. \tag{6.2}
 \end{aligned}$$

Similarly, we can specialize Corollary 5.1 to a multivariable nonterminating  $q$ -Pfaff–Saalschütz summation by first replacing  $b$  and  $e$  by  $aq/b$  and  $abq^{1-r} / cd f$ , respectively, then letting  $a \rightarrow 0$ , and finally performing the simultaneous substitutions  $b \mapsto e$ ,  $c \mapsto a$ ,  $d \mapsto b$  and  $f \mapsto c$ . We obtain the following multivariable extension of Eq. (II.24) in [9].

**Corollary 6.2** (An  $A_r$  nonterminating  ${}_3\phi_2$  summation). *Let  $ef = abcq^r$ . Then there holds*

$$\begin{aligned} & \sum_{S \subseteq \{1, 2, \dots, r\}} \left(\frac{q}{e}\right)^{\binom{r}{2} - \binom{|S|}{2}} \prod_{i \notin S} \frac{(ax_i, bx_i, c, q/ex_i, fq/e; q)_\infty}{(aq/e, bq/e, cq/ex_i, ex_i/q, fx_i; q)_\infty} \\ & \times \sum_{k_1, \dots, k_r=0}^\infty \prod_{\substack{1 \leq i < j \leq r \\ i, j \in S}} \frac{(x_i q^{k_i} - x_j q^{k_j})}{(x_i - x_j)} \prod_{\substack{1 \leq i < j \leq r \\ i, j \notin S}} \frac{(q^{k_i} - q^{k_j})}{(x_i - x_j)} \prod_{i \in S, j \notin S} \frac{(ex_i q^{1+k_i} - q^{k_j})}{(x_i - x_j)} \\ & \times \prod_{i \in S} \frac{(ax_i, bx_i, c; q)_{k_i}}{(q, ex_i, fx_i; q)_{k_i}} \prod_{i \notin S} \frac{(aq/e, bq/e, cq/ex_i; q)_{k_i}}{(q, q^2/ex_i, fq/e; q)_{k_i}} q^{\sum_{i=1}^r k_i} \\ & = \prod_{i=1}^r \frac{(q/ex_i, fq^{1-i}/a, fq^{1-i}/b, fx_i/c; q)_\infty}{(aq^i/e, bq^i/e, cq/ex_i, fx_i; q)_\infty}. \end{aligned} \tag{6.3}$$

It is again clear that Corollary 6.2 above can be specialized to an  $A_r$  terminating  $q$ -Pfaff–Saalschütz summation, Namely, by first replacing  $f$  by  $abcq^r/e$  and then letting  $c \rightarrow q^{-N}$ , we obtain

$$\begin{aligned} & \sum_{k_1, \dots, k_r=0}^N \prod_{1 \leq i < j \leq r} \frac{(x_i q^{k_i} - x_j q^{k_j})}{(x_i - x_j)} \prod_{i=1}^r \frac{(ax_i, bx_i, cx_i, q^{-N}; q)_{k_i}}{(q, cx_i, abx_i q^{r-N}/c; q)_{k_i}} q^{\sum_{i=1}^r k_i} \\ & = \prod_{i=1}^r \frac{(cq^{1-i}/a, cq^{1-i}/b; q)_N}{(cx_i, cq^{1-r}/abx_i; q)_N} \end{aligned} \tag{6.4}$$

which is Theorem 5.1 in [17].

Finally, we specialize Theorem 6.1 further, for possible future reference. We first replace  $f$  by  $cq^{1-r}/abde$  and then let  $c \rightarrow 0$  and replace  $a, d,$  and  $e$  by  $-a, -c/a,$  and  $d/b,$  respectively. The result is the following.

**Corollary 6.3.** *There holds*

$$\begin{aligned} & \int_{-ax_1}^b \cdots \int_{-ax_r}^b \prod_{1 \leq i < j \leq r} (t_i - t_j) \prod_{i=1}^r \frac{(-qt_i/ax_i, qt_i/b; q)_\infty}{(-ct_i/a, dt_i/b; q)_\infty} d_q t_r \cdots d_q t_1 \\ & = (-a)^{\binom{r}{2}} b^r (1 - q)^r \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \frac{(q, -ax_i/b, -bq/ax_i, cdq^{r-1}x_i; q)_\infty}{(cx_i, -adx_i/b, -bcq^{i-1}/a, dq^{i-1}; q)_\infty}. \end{aligned} \tag{6.5}$$

Corollary 6.3 is a multivariable extension of a  $q$ -integral derived in [3]. Replacing  $a, b, c, d,$  and  $x_i$  by  $c, d, q^a, q^b,$  and  $q^{x_i}$ , for  $i = 1, \dots, r,$  and letting  $q \rightarrow 1,$  we obtain

$$\begin{aligned} & \int_{-c}^d \cdots \int_{-c}^d \prod_{1 \leq i < j \leq r} (t_i - t_j) \prod_{i=1}^r (1 + t_i/c)^{a-1+x_i} (1 - t_i/d)^{b-1} dt_r \cdots dt_1 \\ &= \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{i=1}^r \frac{\Gamma(a + x_i) \Gamma(b + i - 1)}{\Gamma(a + b + r - a + x_i)} \\ & \quad \times c^{r(1-a) - \sum x_i} d^{r(1-b)} (c + d)^{r(a+b-1) + \binom{r}{2} + \sum x_i}, \end{aligned} \tag{6.6}$$

which follows from the multiple beta integral evaluation in Proposition 3.1 by the substitutions

$$u_i \mapsto \frac{c + t_i}{c + d}, \quad i = 1, \dots, r. \tag{6.7}$$

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