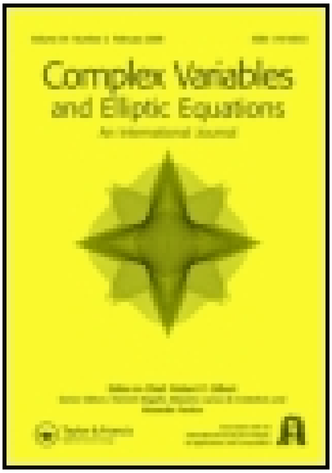

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Subordination of planar harmonic functions

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Subordination of Planar Harmonic Functions*

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In this paper, we initiate a systematic study of subordination of harmonic functions. We establish results about coefficient relationships, integral means, and majorization of the Jacobian. In connection with the last result, we use affine and linear invariant families of harmonic functions to obtain the surprising result that the radius of majorization of the Jacobian for harmonic functions coincides with the same quantity for analytic functions.

Keywords: Subordination; majorization; harmonic; invariant family

AMS Subject Classification: 30C80

1 INTRODUCTION

The notion of subordination is an important concept in the theory of analytic functions. In this paper, we initiate a systematic study of subordination of harmonic functions. We establish results about coefficient relationships, integral means, and majorization of the Jacobian. The concept of a linear invariant family is important in connection with the last result.

Any harmonic function in the open unit disk $\mathbb{D}$ can be written as a sum of an analytic and anti-analytic function, that is

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$f(z) = h(z) + \overline{g(z)}$. For convenience, we will only examine *sense-preserving* functions, that is, functions for which $J_f(z) \geq 0$. If f has $J_f(z) \leq 0$, then $\bar{f}$ is sense-preserving. The *analytic dilatation* of a harmonic function is the quantity $\omega(z) = g'(z)/h'(z)$. We will define subordination of harmonic functions as follows:

Definition 1.1 *We say that f is subordinate to F, denoted $f \prec F$ if $f(z) = F(\psi(z))$, where ψ is analytic, $|\psi(z)| < 1$ in $\mathbb{D}$, and $\psi(0) = 0$.*

Notice that subordination only makes sense if $\psi(z)$ is analytic, because if we allow ψ to be harmonic, we are not guaranteed the harmonicity of the subordinate function f. We require $\psi(0) = 0$, since we will only consider those harmonic functions for which $f(0) = 0$. Also note that though analytic functions have the property that if $f(\mathbb{D}) \subseteq F(\mathbb{D})$ then $f(z) \prec F(z)$, the same is not true for harmonic functions.

Many results about subordination of analytic functions combine subordination with majorization. A function f is said to be majorized by F in a certain region if $|f(z)| \leq |F(z)|$ there. For analytic functions, f is majorized by F if and only if $f(z) = \phi(z)F(z)$, where $\phi(z)$ is a Schwarz function. Majorization of harmonic functions cannot be linked with multiplication by an analytic function, for if ϕ is analytic and F is harmonic, their product is not necessarily harmonic. Robertson [8] combined the ideas of majorization and subordination of analytic functions with the idea of quasi-subordination. Unfortunately, quasi-subordination does not make sense for harmonic functions, again because multiplication will not preserve harmonicity.

This paper includes a result which will link subordination of harmonic functions with majorization of their Jacobians. Campbell [2] studied subordination when the analytic superordinate function belongs to a linear invariant family of order α. In particular, he showed that if $f \prec F$, then $f'(z)$ is majorized by $F'(z)$ in the disk $|z| \leq \alpha + 1 - \sqrt{\alpha^2 + 2\alpha}$. We obtain a similar result for subordinate harmonic functions in Theorem 3.4.

2 COEFFICIENT INEQUALITIES AND INTEGRAL MEANS

If $f = h + \bar{g} \prec F = H + \bar{G}$, with $f(z) = F(\psi(z))$, it is easy to see that $h(z) = H(\psi(z))$, $g(z) = G(\psi(z))$, and $\omega(z) = \Omega(\psi(z))$, where ω and Ω are the analytic dilatations of f and F, respectively. From this observation we get several initial results that were proved true for analytic subordination. For proofs of the analytic results and references to the original sources, see Duren [4].

THEOREM 2.1 *Suppose $f \prec F$, where*

$$f(z) = \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad \text{and} \quad F(z) = \sum_{k=1}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k}. \quad (1)$$

Then the following inequalities hold:

$$\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |A_k|^2 \quad \text{and} \quad \sum_{k=1}^{n} |b_k|^2 \leq \sum_{k=1}^{n} |B_k|^2 \qquad \text{for } n = 1, 2, \ldots$$

THEOREM 2.2 *Let $f \prec F$, where f and F are defined by* (1). *If $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ and $\beta_1 \geq \beta_2 \geq \cdots \geq 0$ then*

$$\sum_{k=1}^{\infty} \alpha_k |a_k|^2 \leq \sum_{k=1}^{\infty} \alpha_k |A_k|^2 \quad \text{and} \quad \sum_{k=1}^{\infty} \beta_k |b_k|^2 \leq \sum_{k=1}^{\infty} \beta_k |B_k|^2.$$

A result of Littlewood [5] that is also true for harmonic functions is as follows:

THEOREM 2.3 *If $f \prec F$, where f and F are defined by* (1), *and*

(a) *If $A_1, A_2, \ldots, A_n$ are nonnegative, nonincreasing, and convex, then $|a_n| \leq A_1$.*
(b) *If $B_1, B_2, \ldots, B_n$ are nonnegative, nonincreasing, and convex, then $|b_n| \leq B_1$.*
(c) *If $A_1, A_2, \ldots, A_n$ are nonnegative, nondecreasing, and convex, then $|a_n| \leq A_n$.*
(d) *If $B_1, B_2, \ldots, B_n$ are nonnegative, nondecreasing, and convex, then $|b_n| \leq B_n$.*

In particular, if $k(z)$ is the harmonic Koebe function

$$\frac{z - (1/2)z^2 + (1/6)z^3}{(1-z)^3} + \overline{\left(\frac{(1/2)z^2 + (1/6)z^3}{(1-z)^3}\right)},$$

and $f(z) \prec k(z)$, then $|a_n| \leq (1/6)(2n+1)(n+1)$ and $|b_n| \leq (1/6)(2n-1) \times (n-1)$. This is of interest because these are the conjectured sharp bounds on coefficients for the class S_H^0 of all harmonic functions univalent in $\mathbb{D}$ with $f(0)=0$, $h'(0)=1$, and $g'(0)=0$.

THEOREM 2.4 *If $f \prec F$, then $M_p(r,f) \leq M_p(r,F)$ for $p \geq 1$, where $M_p(r,f)$ is the integral mean $((1/2\pi)\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta)^{1/p}$. Equality occurs only when F is constant or when $\psi(z)$ is a rotation ($\psi(z) = e^{i\theta}z$).*

Proof It is known (see, for example, [7, p. 22]) that for $f(z)$ harmonic and $p \geq 1$, $|f(z)|^p$ is subharmonic. For the convenience of the reader, we provide a proof. Since $f(z)$ is harmonic, we know that for $|z_0| < 1$ that

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{i\theta}) \, d\theta,$$

so that

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta,$$

thus $|f(z)|$ is subharmonic. If $p \geq 1$, then

$$|f(z_0)|^p \leq \left(\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})| \, d\theta\right)^p,$$

but by Jensen's inequality, the right hand side is less than or equal to $(1/2\pi)\int_0^{2\pi} |f(z_0 + \rho e^{i\theta})|^p \, d\theta$, since x^p is a convex function. Thus $|f(z)|^p$ is subharmonic, because it has the local sub-mean-value property. The following proof, modeled after a proof in Duren's book [4], shows that if u is subharmonic in $\mathbb{D}$ and $v(z) = u(\psi(z))$, then $\int_0^{2\pi} v(re^{i\theta}) \, d\theta \leq \int_0^{2\pi} u(re^{i\theta}) \, d\theta$. Applying this to $|f(z)|^p = |F(\psi(z))|^p$ will yield the desired result. Fix r with $0 < r < 1$, and let $U(z)$ be the function that is harmonic on $|z| < r$ so that $U(z) = u(z)$ on $|z| = r$. Then $u(z) \leq U(z)$ in $|z| \leq r$ and $v(z) \leq V(z) = U(\psi(z))$ on $|z| = r$. We then have the following equalities:

$$\frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta}) \, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} V(re^{i\theta}) \, d\theta = V(0) = U(0)$$
$$= \frac{1}{2\pi}\int_0^{2\pi} U(re^{i\theta}) \, d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta}) \, d\theta.$$

If equality holds for some r, then $v(z) = V(z)$ on the entire circle $|z| = r$. This can happen if $\psi(z) = e^{i\theta}z$ or if $u(z) = U(z)$ at some interior points of $|z| \leq r$, and so the two functions must agree in the entire disk, by the maximum principle. If the latter possibility happens, that means that equality can occur in the theorem only if $|F(z)|^p$ is harmonic in $|z| = r$, but this would imply that F is constant.

3 INVARIANT FAMILIES

It is best to consider questions relating subordination to majorization of the Jacobian in the context of invariant families, where the questions may be cast in their most natural and general form. Our definition of an affine and linear invariant family of locally univalent harmonic functions is similar to Sheil-Small's definition for families of univalent harmonic maps of the unit disk [9]. We let Aut($\mathbb{D}$) denote the set of analytic automorphisms of $\mathbb{D}$, and we shall employ the notation

$$T_\varphi(f(z)) = \frac{f(\varphi(z)) - f(\varphi(0))}{\varphi'(0)h'(\varphi(0))}, \quad \varphi \in \mathrm{Aut}(\mathbb{D}),$$

$$A_\varepsilon(f(z)) = \frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon g'(0)}, \qquad |\varepsilon| < 1,$$

for the Koebe transforms and the affine transforms of a locally univalent harmonic function $f = h + \bar{g}$.

DEFINITION 3.1 *Let L be a family of functions $f = h + \bar{g}$ that are harmonic and locally univalent on $\mathbb{D}$ and normalized by $f(0) = 0$, $h'(0) = 1$. We say that L is an affine and linear invariant family (ALIF) if for each $f \in L$ the functions $T_\varphi(f)$ and $A_\varepsilon(f)$ belong to L for all $\varphi \in$ Aut($\mathbb{D}$) and $|\varepsilon| < 1$. The order of the family is defined as* $\sup_{f\in L}|a_2(f)|$. *L^0 is the subset of the ALIF L defined by $f \in L^0$ iff $f \in L$ and $g'(0) = 0$.*

The classes C_H of all normalized close-to-convex univalent harmonic functions and K_H of all normalized convex univalent harmonic functions are affine and linear invariant. It is not known whether the class S_H is the closure of an affine and linear invariant family.

In the following theorem, inequality (2) is essentially due to Sheil-Small [9], but the sharpness result is new.

THEOREM 3.2 *If $f \in L^0$, where L is an ALIF of order α, then*

$$J_f(z) \le \frac{(1+r)^{2\alpha-2}}{(1-r)^{2\alpha+2}} = \left(\frac{1+r}{1-r}\right)^{2\alpha} \frac{1}{(1-r^2)^2}, \tag{2}$$

where $|z| = r$. Equality occurs only if f is analytic and is a rotation of the function

$$f(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^{\alpha} - 1\right]. \tag{3}$$

Proof First, consider $f = h + \bar{g}$ in L. If $|z_0| < 1$, then the function

$$T_\varphi(f(z)) = \frac{f((z_0+z)/(1+\bar{z}_0 z)) - f(z_0)}{(1-|z_0|^2)h'(z_0)} = H + \bar{G} \in L.$$

For $T_\varphi(f(z))$,

$$a_2(T) = \frac{1}{2}H''(0) = \frac{(1-|z_0|^2)h''(z_0)}{2h'(z_0)} - \bar{z}_0,$$

and $|a_2(T)| \le \alpha$ by hypothesis. Thus if $|z_0| = r$, we have

$$\frac{-2\alpha + 2r}{1-r^2} \le \frac{1}{r}\mathrm{Re}\left\{\frac{z_0 h''(z_0)}{h'(z_0)}\right\} \le \frac{2\alpha + 2r}{1-r^2}.$$

Integrating the above equation with respect to ρ along the ray $\zeta = \rho e^{i\theta}$ from 0 to r and recognizing that $\mathrm{Re}\{z_0 h''(z_0)/h'(z_0)\} = \rho(\partial/\partial\rho)\log|h'(\zeta)|$, we obtain

$$\frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \le |h'(z_0)| \le \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}. \tag{4}$$

Now suppose that $f = h + \bar{g} \in L^0$. Then for each ε with $|\varepsilon| < 1$, the function $f + \varepsilon\bar{f} = h + \varepsilon g + \overline{(g + \bar{\varepsilon}h)}$ is in L. Thus for $|z| = r$, we have

$$\frac{(1-r)^{\alpha-1}}{(1+r)^{\alpha+1}} \le |h'(z) + \varepsilon g'(z)| \le \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}.$$

By picking ε wisely and letting $|\varepsilon| \to 1$, we can deduce that

$$|h'(z)| - |g'(z)| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}} \tag{5}$$

and

$$|h'(z)| + |g'(z)| \leq \frac{(1+r)^{\alpha-1}}{(1-r)^{\alpha+1}}. \tag{6}$$

Multiplying the two inequalities proves (2).

For the proof of equality, suppose that there is a $z_1 = Re^{i\theta}$, $R \neq 0$, for which $J_f(z_1) = (1+R)^{2\alpha-2}/(1-R)^{2\alpha+2}$. Then equality occurs in both (5) and (6). By subtracting the two, we see that $|g'(z_1)| = 0$, so $|h'(z_1)| = (1+R)^{\alpha-1}/(1-R)^{\alpha+1}$. Now by a method similar to Campbell's [1], we will show that $h(z)$ must be of the form (3), so $g(z)$ must be identically 0. We can assume $z = R > 0$ by considering $e^{i\theta}f(e^{-i\theta}z)$, so we have that $|h'(R)| = (1+R)^{\alpha-1}/(1-R)^{\alpha+1}$. If

$$u(x) = \log|h'(x)| - (\alpha-1)\log(1+x) + (\alpha+1)\log(1-x)$$

for $0 \leq x < 1$, then $u'(x) = \operatorname{Re}\{h''(x)/h'(x)\} - (2\alpha+2x)/(1-x^2)$ is non-positive for $0 < x < 1$. Indeed, if there were some x_0 in $(0,1)$ so that $u'(x_0)$ were positive, we would have

$$\begin{aligned}
\alpha &\geq \sup_{z\in\mathbb{D}} \left| -\bar{z} + \frac{1-|z|^2}{2}\frac{h''(z)}{h'(z)} \right| \\
&\geq \left| -x_0 + \frac{1-x_0^2}{2}\frac{h''(x_0)}{h'(x_0)} \right| \\
&= \left| \alpha + \frac{1-x_0^2}{2}u'(x_0) \right| > \alpha.
\end{aligned}$$

Since $u(0) = u(R) = 0$, and $u'(x) \leq 0$, it must be that $u(x) \equiv 0$ for $0 \leq x \leq R$. Let

$$\Phi(z) = \log h'(z) - (\alpha-1)\log(1+z) + (\alpha+1)\log(1-z) = u(z) + iv(z).$$

Note that

$$\Phi'(z) = \frac{h''(z)}{h'(z)} - \frac{2\alpha + 2z}{1 - z^2}.$$

For $0 \leq x \leq R$, and $f(z) = h(z) + \overline{g(z)}$ in L^0, we have

$$\begin{aligned}\alpha &\geq \left| -x + \frac{1 - x^2}{2} \frac{h''(x)}{h'(x)} \right| \\ &= \left| -x + \frac{1 - x^2}{2} \left(\Phi'(x) + \frac{2\alpha + 2x}{1 - x^2} \right) \right| \\ &= \left| \alpha + \frac{1 - x^2}{2} \mathrm{i} v'(x) \right|,\end{aligned}$$

since $u'(x) \equiv 0$ on $[0, R]$. Thus, we must have that $v'(x) \equiv 0$ on $[0, R]$ to avoid a contradiction. Since $\Phi'(z)$ is identically 0 on $[0, R]$, we have that it is 0 for all $z \in \mathbb{D}$. The solution of the differential equation yields (3).

By using an alternate proof, we can obtain the same result for an ALIF L.

THEOREM 3.3 *If f is in an ALIF L of order α, then*

$$\frac{(1 - r)^{2\alpha - 2}}{(1 + r)^{2\alpha + 2}} \leq J_f(z) \leq \frac{(1 + r)^{2\alpha - 2}}{(1 - r)^{2\alpha + 2}} = \left(\frac{1 + r}{1 - r} \right)^{2\alpha} \frac{1}{(1 - r^2)^2}, \tag{7}$$

where $|z| = r$.

The theorem will be proven by using the transformation

$$S_\varphi(f(z)) = \frac{\overline{h'(z_0)}(f(\varphi(z)) - f(z_0)) - \overline{g'(z_0)}(\overline{f(\varphi(z))} - \overline{f(z_0)})}{\varphi'(0) J_f(z_0)}.$$

Notice that when $f \in L$, $S_\varphi(f(z)) \in L^0$. The function $S_\varphi(f(z)) = A_{\varepsilon_0} \circ T_\varphi(f(z))$, where $\varepsilon_0 = -\overline{g'(\varphi(0))}/h'(\varphi(0))$.

Proof If $f \in L$, clearly $S_\varphi(f(z)) \in L$. Let $H(z, z_0)$ be the analytic part of $S_\varphi(f(z))$, where $\varphi(0) = z_0$. By examining the formula for S_φ, it is clear that

$$H(z,z_0)=\frac{\overline{h'(z_0)}(h(\varphi(z))-h(z_0))-\overline{g'(z_0)}(g(\varphi(z))-g(z_0))}{(1-|z_0|^2)J_f(z_0)}.$$

A straightforward calculation shows that

$$2A_2=H''(0,z_0)=(1-|z_0|^2)\frac{\overline{h'(z_0)}h''(z_0)-\overline{g'(z_0)}g''(z_0)}{J_f(z_0)}-2\overline{z_0}, \qquad (8)$$

where A_2 is the coefficient of z^2 in the power series expansion of $S_\varphi(f(z))$.

We begin by making some observations that will be useful. Note that

$$J_f(z)=\begin{vmatrix} h'(z) & \overline{g'(z)} \\ g'(z) & \overline{h'(z)} \end{vmatrix}$$

and

$$\begin{aligned}\frac{\partial}{\partial r}J_f(z)&=\frac{\partial}{\partial z}J_f(z)\frac{\mathrm{d}z}{\mathrm{d}r}+\frac{\partial}{\partial \bar{z}}J_f(z)\frac{\mathrm{d}\bar{z}}{\mathrm{d}r}\\ &=\begin{vmatrix} h''(z) & \overline{g'(z)} \\ g''(z) & \overline{h'(z)} \end{vmatrix}\mathrm{e}^{\mathrm{i}\theta}+\begin{vmatrix} h'(z) & \overline{g''(z)} \\ g'(z) & \overline{h''(z)} \end{vmatrix}\mathrm{e}^{-\mathrm{i}\theta}\\ &=2\mathrm{Re}\left\{\mathrm{e}^{\mathrm{i}\theta}\frac{\partial}{\partial z}J_f(z)\right\}.\end{aligned}$$

Letting $z_0=r\mathrm{e}^{\mathrm{i}\theta}$ in (8), we have

$$\frac{2A_2}{1-r^2}=\frac{\partial}{\partial z}\log J_f(r\mathrm{e}^{\mathrm{i}\theta})-\frac{2r\mathrm{e}^{-\mathrm{i}\theta}}{1-r^2}.$$

Multiplying by $\mathrm{e}^{\mathrm{i}\theta}$ and taking real parts, we have

$$\mathrm{Re}\left\{A_2\mathrm{e}^{\mathrm{i}\theta}\frac{\mathrm{d}}{\mathrm{d}r}\log\left(\frac{1+r}{1-r}\right)\right\}=\frac{1}{2}\frac{\partial}{\partial r}\log J_f(r\mathrm{e}^{\mathrm{i}\theta})+\frac{\mathrm{d}}{\mathrm{d}r}\log(1-r^2).$$

Multiplying by 2 and using the fact that $|A_2| \leq \alpha$ yields the inequality

$$-2\alpha \frac{\mathrm{d}}{\mathrm{d}r} \log\left(\frac{1+r}{1-r}\right) \leq \frac{\partial}{\partial r} \log J_f(r\mathrm{e}^{\mathrm{i}\theta}) + 2\frac{\mathrm{d}}{\mathrm{d}r}\log(1-r^2)$$
$$\leq 2\alpha \frac{\mathrm{d}}{\mathrm{d}r} \log\left(\frac{1+r}{1-r}\right).$$

Integrating, we have

$$\log\left(\frac{1+r}{1-r}\right)^{-2\alpha} \leq \log J_f(r\mathrm{e}^{\mathrm{i}\theta})(1-r^2)^2 \leq \log\left(\frac{1+r}{1-r}\right)^{2\alpha},$$

and direct calculation gives us (7).

We now prove the following theorem:

THEOREM 3.4 *Let F be in an ALIF L of order α or in L^0, $1.65 \leq \alpha < \infty$. If $f \prec F$, then $J_f(z) \leq J_F(z)$ for $|z| \leq r = \alpha + 1 - \sqrt{\alpha^2 + 2\alpha}$.*

This result is surprising, because the radius of majorization of the Jacobian of a harmonic function coincides with the radius of majorization of the derivative of an analytic function, instead of being smaller.

The proof of the theorem will make use of the following lemma:

LEMMA 3.5 *If f is in an ALIF L of order α, then*

$$\frac{J_f(x)}{J_f(z_0)} \leq \left(\frac{1+|(x-z_0)/(1-\bar{z}_0 x)|}{1-|(x-z_0)/(1-\bar{z}_0 x)|}\right)^{2\alpha}\left(\frac{1-|z_0|^2}{1-|x|^2}\right)^2.$$

The same inequality holds when $f \in L^0$, with equality only if f is analytic, and is of the form (3).

Proof We know that $J_{S(f)}(z) \leq ((1+r)/(1-r))^{2\alpha}\, 1/(1-r^2)^2$. But by directly calculating, we get that

$$J_{S(f)}(z) = \frac{|\varphi'(z)|^2}{(1-|z_0|^2)^2} \cdot \frac{J_f(\varphi(z))}{J_f(z_0)},$$

where $\varphi(z) = (z_0 + z)/(1 + \bar{z}_0 z) \in \mathrm{Aut}(\mathbb{D})$. Combining the two yields

$$\frac{J_f(\varphi(z))}{J_f(z_0)} \leq \left(\frac{1+|z|}{1-|z|}\right)^{2\alpha} \left(\frac{1-|z_0|^2}{1-|x|^2}\right)^2 \frac{1}{|\varphi'(z)|^2}.$$

Now, if we let $z = ((x - z_0)/(1 - \bar{z}_0 x))$, then $\varphi(z) = x$ and $\varphi'(z) = (1 - \bar{z}_0 x)^2/(1 - |z_0|^2)$. Computation proves the lemma. Equality occurs for $f \in L^0$ only if there is equality in Theorem 3.3.

Proof of Theorem 3.4 Since $f \prec F$, we know that $f(z) = F(\psi(z))$, and $J_f(z) = |\psi'(z)|^2 J_F(\psi(z))$. By Lemma 3.5, we know that

$$\frac{J_F(\psi(z_0))}{J_F(z_0)} \leq \left(\frac{1 + |(\psi(z_0) - z_0)/(1 - \bar{z}_0\psi(z_0))|}{1 - |(\psi(z_0) - z_0)/(1 - \bar{z}_0\psi(z_0))|}\right)^{2\alpha} \left(\frac{1-|z_0|^2}{1-|\psi(z_0)|^2}\right)^2$$

so that

$$\frac{J_f(z_0)}{J_F(z_0)} \leq |\psi'(z_0)|^2 \left(\frac{1-|z_0|^2}{1-|\psi(z_0)|^2}\right)^2 \left(\frac{|1 - \bar{z}_0\psi(z_0)| + |\psi(z_0) - z_0|}{|1 - \bar{z}_0\psi(z_0)| - |\psi(z_0) - z_0|}\right)^{2\alpha}.$$

The above quantity is precisely the square of the quantity that Campbell [2] proved is not greater than 1 for $|z| \leq r = \alpha + 1 - \sqrt{\alpha^2 + 2\alpha}$. Campbell [2] showed that this is the best possible bound by letting

$$F(z) = \frac{1}{2\alpha}\left\{1 - \left(\frac{1-z}{1+z}\right)^{\alpha}\right\}$$

and letting $\psi(z) = z \cdot (a + z)/(1 + az)$, with $0 \leq a \leq 1$. There is no harmonic function in L^0 which makes this bound sharp, because equality can only occur when there is equality in Lemma 3.5.

COROLLARY 3.6 *If $f \prec F$, for F in K_H, the family of convex univalent harmonic mappings, $J_f(z) \leq J_F(z)$ for $|z| \leq 3 - \sqrt{8}$. For F in C_H, the family of close-to-convex univalent harmonic mappings, $J_f(z) \leq J_F(z)$ for $|z| \leq 4 - \sqrt{15}$.*

Proof Clunie and Sheil-Small [3] proved that $\alpha(K_H) = 2$ and $\alpha(C_H) = 3$.

4 MAJORIZATION OF THE JACOBIAN FOR FAMILIES OF ORDER $\alpha+\beta$

When we examine families of functions with the order of the family defined as the supremum of $|a_2|$, it is possible that we are losing information about the function by neglecting its anti-analytic part. In this section we examine ALIF's, but change the definition of the order of the family to

$$\alpha+\beta,$$

where

$$\alpha=\sup\{|a_2|: f\in L\} \quad \text{and} \quad \beta=\sup\{|b_2|: f\in L\}.$$

We prove a theorem about the bound of the Jacobian of f, where the bound depends on α and β.

THEOREM 4.1 *For f in an ALIF L of order $\alpha+\beta$, the Jacobian of f is bounded by*

$$\frac{(1-r)^{2\alpha+2\beta-1}}{(1+r)^{2\alpha+2\beta+3}} \le J_f(z) \le \frac{(1+r)^{2\alpha+2\beta-1}}{(1-r)^{2\alpha+2\beta+3}}, \tag{9}$$

where $|z|=r$.

Proof As in the proof of Theorem 3.3, we shall use the transformation $S_\varphi(f(z))$. If $A_2=a_2(S_\varphi)$ and $B_2=b_2(S_\varphi)$, we have that

$$2A_2+2B_2 = -2\bar{z}_0+(1-|z_0|^2)\left(\frac{\overline{h'(z_0)}h''(z_0)-\overline{g'(z_0)}g''(z_0)}{J_f(z_0)} + \frac{h'(z_0)g''(z_0)-g'(z_0)h''(z_0)}{J_f(z_0)}\right),$$

where $\varphi(0)=z_0$. We note that $J_f(z_0)=|h'(z_0)|^2(1-|\omega(z_0)|^2)$, and $\omega(z_0)=g'(z_0)/h'(z_0)$, thus

$$\omega'(z_0)=\frac{h'(z_0)g''(z_0)-g'(z_0)h''(z_0)}{(h'(z_0))^2}.$$

We also recall that

$$\frac{\overline{h'(z_0)}h''(z_0) - \overline{g'(z_0)}g''(z_0)}{J_f(z_0)} = \frac{(\partial/\partial z)J_f(re^{i\theta})}{J_f(re^{i\theta})}.$$

Using these observations, letting $z_0 = re^{i\theta}$, and dividing by $(1-r^2)$, we have

$$\frac{2A_2 + 2B_2}{1-r^2} = \frac{-2re^{-i\theta}}{1-r^2} + \frac{(\partial/\partial z)J_f(re^{i\theta})}{J_f(re^{i\theta})} + \frac{h'(re^{i\theta})\omega'(re^{i\theta})}{\overline{h'(re^{i\theta})}(1-|\omega(re^{i\theta})|^2)}.$$

Multipying by $e^{i\theta}$ and taking real parts, we have

$$\begin{aligned}&\operatorname{Re}\{e^{i\theta}(A_2+B_2)\}\frac{d}{dr}\log\left(\frac{1+r}{1-r}\right)\\ &= \frac{d}{dr}\log(1-r^2) + \frac{1}{2}\frac{\partial}{\partial r}\log J_f(re^{i\theta}) + \operatorname{Re}\left\{\frac{e^{i\theta}h'(re^{i\theta})\omega'(re^{i\theta})}{\overline{h'(re^{i\theta})}(1-|\omega(re^{i\theta})|^2)}\right\}.\end{aligned}$$

Multiplying by 2 and using bounds on $|A_2|$ and $|B_2|$, we have the following two inequalities:

$$\begin{aligned}&-(2\alpha+2\beta)\frac{d}{dr}\log\left(\frac{1+r}{1-r}\right)\\ &\leq \frac{d}{dr}\log(1-r^2) + \frac{1}{2}\frac{\partial}{\partial r}\log J_f(re^{i\theta}) + 2\left|\frac{\omega'(re^{i\theta})}{(1-|\omega(re^{i\theta})|^2)}\right|\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dr}\log(1-r^2) + \frac{1}{2}\frac{\partial}{\partial r}\log J_f(re^{i\theta}) - 2\left|\frac{\omega'(re^{i\theta})}{(1-|\omega(re^{i\theta})|^2)}\right|\\ &\leq (2\alpha+2\beta)\frac{d}{dr}\log\left(\frac{1+r}{1-r}\right).\end{aligned}$$

We now note that $\omega(z)$ is a Schwarz function, and hence

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}$$

(see, for example [6, p. 168]). Applying this to the above inequalities, integrating, exponentiating, and simplifying yields inequality (9).

We now can see how the radius of majorization depends on α and β.

THEOREM 4.2 *If F is in an ALIF L of order $\alpha+\beta$, $1.15 \leq \alpha+\beta < \infty$, $f \prec F$, and $f_z(0) \geq 0$, then*

$$J_f(z) \leq J_F(z) \quad \text{for}$$
$$|z| \leq r = \alpha+\beta+\tfrac{3}{2} - \sqrt{\left(\alpha+\beta+\tfrac{1}{2}\right)^2 + 2\alpha+2\beta+1}.$$

As with the proof of Theorem 3.4, we shall use the following technical lemma.

LEMMA 4.3 *If f is in an ALIF of order $\alpha+\beta$, then*

$$\frac{J_f(x)}{J_f(z_0)} \leq \left(\frac{1+|(x-z_0)/(1-\bar{z}_0x)|}{1-|(x-z_0)/(1-\bar{z}_0x)|}\right)^{2\alpha+2\beta+1}\left(\frac{1-|z_0|^2}{1-|x|^2}\right)^2.$$

Proof We know that

$$J_{S(f)}(z) \leq \left(\frac{1+|z|}{1-|z|}\right)^{2\alpha+2\beta+1}\frac{1}{(1-|z|^2)^2}.$$

But by directly calculating, we have

$$J_{S(f)}(z) = \frac{|\varphi'(z)|^2}{(1-|z_0|^2)^2}\cdot\frac{J_f(\varphi(z))}{J_f(z_0)}.$$

Combining the two yields

$$\frac{J_f(\varphi(z))}{J_f(z_0)} \leq \left(\frac{1+|z|}{1-|z|}\right)^{2\alpha+2\beta+1}\left(\frac{1-|z_0|^2}{1-|x|^2}\right)^2\frac{1}{|\varphi'(z)|^2}.$$

Let $z=((x-z_0)/(1-\bar{z}_0x))$. Then $\varphi(z)=x$ and $\varphi'(z)=(1-\bar{z}_0x)^2/(1-|z_0|^2)$. Computation proves the lemma.

Proof of Theorem 4.2 Since $f \prec F$, we know that $f(z) = F(\psi(z))$, and $J_f(z) = |\psi'(z)|^2 J_F(\psi(z))$. By Lemma 4.3, we know that

$$\frac{J_F(\psi(z_0))}{J_F(z_0)} \le \left(\frac{1 + |(\psi(z_0) - z_0)/(1 - \bar{z}_0\psi(z_0))|}{1 - |(\psi(z_0) - z_0)/(1 - \bar{z}_0\psi(z_0))|}\right)^{2\alpha+2\beta+1} \left(\frac{1 - |z_0|^2}{1 - |\psi(z_0)|^2}\right)^2,$$

hence

$$\frac{J_f(z_0)}{J_F(z_0)} \le |\psi'(z_0)|^2 \left(\frac{1 - |z_0|^2}{1 - |\psi(z_0)|^2}\right)^2 \times \left(\frac{|1 - \bar{z}_0\psi(z_0)| + |\psi(z_0) - z_0|}{|1 - \bar{z}_0\psi(z_0)| - |\psi(z_0) - z_0|}\right)^{2(\alpha+\beta+(1/2))}.$$

The above quantity is precisely the square of the quantity that Campbell [2] proved is not greater than 1 for

$$|z| \le r = \left(\alpha + \beta + \tfrac{1}{2}\right) + 1 - \sqrt{\left(\alpha + \beta + \tfrac{1}{2}\right)^2 + 2\left(\alpha + \beta + \tfrac{1}{2}\right)}.$$

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References

[1] D.M. Campbell, Applications and proof of a uniqueness theorem for linear invariant families of finite order, *Rocky Mountain J. Math.* **4** (1974), 621–634.
[2] D.M. Campbell, Majorization–subordination theorems for locally univalent functions, III, *Trans. Amer. Math. Soc.* **198** (1974), 297–306.
[3] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn.* **9** (1984), 3–25.
[4] P. Duren, *Univalent Functions*, Springer-Verlag, New York, 1980.
[5] J.E. Littlewood, *Lectures on the Theory of Functions*, Oxford University Press, London, 1944.

[6] Z. Nehari, *Conformal Mappings*, McGraw Hill, New York, 1952, Dover reprint, New York, 1975.
[7] T. Radó, *Subharmonic Functions*, Chelsea, New York, 1949.
[8] M.S. Robertson, Quasi-subordination and coefficient conjectures, *AMS Bull. Ser. 2* **76** (1970), 1–9.
[9] T. Sheil-Small, Constants for planar harmonic mappings, *J. London Math. Soc.* **42** (1990), 237–248.