Linear Algebra and its Applications 448 (2014) 1–10

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

New proofs of Bapat and Sivasubramanian's theorems

Iwao Sato [1]

Oyama National College of Technology, Oyama, Tochigi 323-0806, Japan

ARTICLE INFO

Article history:
Received 26 October 2013
Accepted 23 January 2014
Available online 25 February 2014
Submitted by R. Brualdi

MSC:
05C50
15A15

Keywords:
Tree
Distance matrix
Zeta function

ABSTRACT

Recently, Bapat and Sivasubramanian [1] presented formulas for the determinant and the inverse of the product distance matrix of a tree. Furthermore, they defined a bivariant Ihara–Selberg zeta function of a graph and gave its determinant expression. We present new proofs for three results of Bapat and Sivasubramanian.

1. Introduction

Graphs and digraphs treated here are finite. Let G be a connected graph. Then the symmetric digraph D_G corresponding to G is the digraph with vertex set $V(G)$ and arc set $D(G) = \{(u,v),(v,u) \mid uv \in E(G)\}$. For $e = (u,v) \in D(G)$, set $u = o(e)$ and $v = t(e)$. Furthermore, let $e^{-1} = (v,u)$ be the *inverse* of $e = (u,v)$. A *path* P *of length* n in G is a sequence $P = (e_1, \dots, e_n)$ of n arcs such that $e_i \in D(G)$, $t(e_i) = o(e_{i+1})$ $(1 \leqslant i \leqslant n-1)$. If $e_i = (v_{i-1}, v_i)$ for $i = 1, \dots, n$, then we write $P = (v_0, v_1, \dots, v_{n-1}, v_n)$. Set $|P| = n$,

[1] This research was partially supported by Grant-in-Aid for Science Research (C).

http://dx.doi.org/10.1016/j.laa.2014.01.021

$o(P) = o(e_1)$ and $t(P) = t(e_n)$. Also, P is called an $(o(P), t(P))$-*path*. A path P is called *simple* if all vertices of P are distinct. For two vertices $u, v \in V(G)$, let the *distance* $d(u, v)$ between u and v be the minimum length of simple (u, v)-paths in G.

Bapat and Sivasubramanian [1] presented formulas for the determinant and the inverse of the product distance matrix of a tree.

Let $T = (V(T), E(T))$ be a tree with $n = |V(T)|$ vertices, and $D(T) = \{e_1, \ldots, e_{n-1}, e_1^{-1}, \ldots, e_{n-1}^{-1}\}$. Furthermore, we consider a weight function $w : D(T) \longrightarrow \mathbf{C}$ such that $w(e_j) = q_{\bar{e}_j}$ and $w(e_j^{-1}) = t_{\bar{e}_j}$ for each $j = 1, \ldots, n-1$, where $\mathbf{C}$ is the set of complex numbers, and $\bar{e}_j$ is the edge corresponding to e_j and e_j^{-1} for each $j = 1, \ldots, n-1$. For each vertex $u \neq v$ of T, let $P_{u,v}$ be the unique shortest simple path from u to v in T. Then define

$$d_{u,v} = \prod_{e \in P_{u,v}} w(e).$$

When $u = v$, define $d_{u,v} = 1$. The *product distance matrix* $\mathbf{M}_T$ is defined as follows:

$$\mathbf{M}_T = (d_{u,v})_{u,v \in V(T)}.$$

Then we define an $n \times n$ matrix $\mathbf{B} = (b_{xy})$ as follows:

$$b_{xy} = \begin{cases} w(x,y)/(1 - q_{xy}t_{xy}) & \text{if } (x,y) \in D(T), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, an $n \times n$ matrix $\mathbf{D} = (d_{xy})$ is the diagonal matrix defined by

$$d_{xx} = \sum_{o(e)=x} \frac{q_{\bar{e}} t_{\bar{e}}}{1 - q_{\bar{e}} t_{\bar{e}}}.$$

The determinant and the inverse of $\mathbf{M}_T$ were given by Bapat and Sivasubramanian [1].

Theorem 1 *(Bapat and Sivasubramanian). Let T be a tree with n vertices, and $w : D(T) \longrightarrow \mathbf{C}$ a weight function. Then the following two results hold:*

1. $\det(\mathbf{M}_T) = \prod_{i=1}^{n-1}(1 - q_{\bar{e}_i} t_{\bar{e}_i})$.
2. $\mathbf{M}_T^{-1} = \mathbf{I}_n - \mathbf{B} + \mathbf{D}$.

Next, we state the Ihara–Selberg zeta function of a graph. We say that a path $P = (e_1, \ldots, e_n)$ has a *backtracking* if $e_{i+1}^{-1} = e_i$ for some i $(1 \leqslant i \leqslant n-1)$. A (v, w)-path is called a v-*cycle* (or v-*closed path*) if $v = w$. We introduce an equivalence relation between cycles. Two cycles $C_1 = (e_1, \ldots, e_m)$ and $C_2 = (f_1, \ldots, f_m)$ are called *equivalent* if $f_j = e_{j+k}$ for all j. The inverse cycle of C is in general not equivalent to C. Let $[C]$ be the equivalence class which contains a cycle C. Let B^r be the cycle obtained by going

r times around a cycle B. Such a cycle is called a *multiple* of B. A cycle C is *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is *prime* if it is not a multiple of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G.

The *Ihara(–Selberg) zeta function* of G is defined by

$$\mathbf{Z}(G,t) = \prod_{[C]} \bigl(1 - t^{|C|}\bigr)^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Ihara [5] defined Ihara zeta functions of graphs, and showed that the reciprocals of Ihara zeta functions of regular graphs are explicit polynomials. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [7,8]. Hashimoto [4] generalized Ihara's result on the zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial by a determinant containing the edge matrix. Bass [2] presented another determinant expression for the Ihara zeta function of an irregular graph by using its adjacency matrix.

Let G be a connected graph with n vertices $v_1, \dots, v_n$ and m edges. Then the *adjacency matrix* $\mathbf{A}(G) = (a_{ij})$ is the square matrix such that $a_{ij} = 1$ if v_i and v_j are adjacent, and $a_{ij} = 0$ otherwise. Let $\mathbf{D}_v = (d_{ij})$ be the diagonal matrix with $d_{ii} = \deg_G v_i$.

Theorem 2 *(Bass). Let G be a connected graph with n vertices and m edges. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G,t)^{-1} = \bigl(1 - t^2\bigr)^{m-n} \det\bigl(\mathbf{I}_n - t\mathbf{A}(G) + t^2(\mathbf{D}_v - \mathbf{I}_n)\bigr).$$

Bass [2] proved by using a linear algebraic method.

Bapat and Sivasubramanian [1] defined a bivariate Ihara–Selberg zeta function of a graph. Let G be a connected graph with m edges, and $D(G) = \{e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ ($e_{m+i} = e_i^{-1}$ ($1 \leqslant i \leqslant m$)). Furthermore, we consider a weight function $w : D(G) \longrightarrow \mathbf{C}$ such that $w(e_j) = q$ and $w(e_j^{-1}) = t$ for each $j = 1, \dots, m$. For a cycle C, let $a(C)$ and $b(C)$ be the number of arcs with weight q and t, respectively. Note that $a(C) + b(C) = |C|$. Then the *bivariate Ihara–Selberg zeta function* $\eta_G(q,t)$ of G is defined by

$$\eta_G(q,t) = \prod_{[C]} \bigl(1 - q^{a(C)} t^{b(C)}\bigr)^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Bapat and Sivasubramanian [1] presented a determinant expression for the bivariate Ihara–Selberg zeta function of a graph.

Theorem 3 *(Bapat and Sivasubramanian). Let G be a connected graph with n vertices and m edges. Then*

$$\eta_G(q,t)^{-1} = (1-qt)^{m-n} \det(\mathbf{I}_n - \mathbf{W} + \mathbf{K}),$$

where two matrices $\mathbf{W} = (\mathbf{W}_{u,v})_{u,v \in V(G)}$ and $\mathbf{K} = (\mathbf{K}_{u,v})_{u,v \in V(G)}$ are given by

$$\mathbf{W}_{u,v} = \begin{cases} w(u,v) & \text{if } (u,v) \in D(G), \\ 0 & \text{otherwise}, \end{cases} \qquad \mathbf{K}_{u,v} = \begin{cases} (\deg u - 1)qt & \text{if } u = v, \\ 0 & \text{otherwise}. \end{cases}$$

In Section 2, we present a new proof for the second formula of Theorem 1 by a combinatorial method. In Section 3, we give a short review on edge zeta function of a graph, and present a new proof for Theorem 3 by using Watanabe and Fukumizu's Theorem on the edge zeta function of a graph. In Section 4, we give a new proof for the first formula of Theorem 1 by using Watanabe and Fukumizu's Theorem, again.

2. A new proof for the second formula of Theorem 1

Let T be a tree with n vertices, and $w : D(T) \longrightarrow \mathbf{C}$ a weight function of T. Then, let

$$\mathbf{N}_T = -(\mathbf{M}_T - \mathbf{I}_n).$$

Thus, we have

$$\mathbf{M}_T = \mathbf{I}_n - \mathbf{N}_T.$$

Moreover, the (u,v)-entry of $\mathbf{N}_T$ is given by

$$(\mathbf{N}_T)_{uv} = \begin{cases} -d_{u,v} & \text{if } u \neq v, \\ 0 & \text{otherwise}. \end{cases}$$

Furthermore, we have

$$\mathbf{M}_T^{-1} = (\mathbf{I}_n - \mathbf{N}_T)^{-1} = \mathbf{I}_n + \mathbf{N}_T + \mathbf{N}_T^2 + \cdots.$$

Then we consider the (u,v)-entry of $\mathbf{I}_n + \mathbf{N}_T + \mathbf{N}_T^2 + \cdots$ for $u, v \in V(T)$.

Let $u, v \in V(T)$. Furthermore, let P be a (u,v)-path in T. For $k \in \mathbf{N}$, we define a *k-simple path partition* $\Phi = (P_1, \ldots, P_k)$ of P as follows: each P_i $(1 \leqslant i \leqslant k)$ is a simple subpath of P, and

$$P = P_1 \cup \cdots \cup P_k, \qquad P_i \cap P_j = \phi \quad (j \neq i-1, i, i+1) \quad \text{and} \quad P_i \cap P_{i+1} = \{o(P_{i+1})\}.$$

In general, a k-simple path partition of P is called a *simple path partition* of P. The *weight* $w(\Phi)$ of Φ is given by

$$w(\Phi) = (-1)^k w(P_1) \cdots w(P_k).$$

Moreover, let $\Phi(P, k)$ be the set of k-simple path partitions of P, and let

$$f(P, k) = \sum_{\Phi \in \Phi(P,k)} w(\Phi).$$

Note that $f(P, k)$ contributes to the (u, v)-entry of the matrix $\mathbf{N}_T^k$. Furthermore, let

$$f(P) = \sum_{k=1}^{|P|} f(P, k).$$

Then the (u, v)-entry of $\mathbf{M}_T^{-1} = \mathbf{I}_n + \mathbf{N}_T + \mathbf{N}_T^2 + \cdots$ is equal to

$$\sum_P f(P), \tag{$*$}$$

where P runs over all (u, v)-paths in T.

Let $u, v \in V(T)$, $d(u, v) = d \geqslant 2$ and $P_{u,v}$ be a unique simple (u, v)-path in T. For $k = 1, 2, \ldots, d$, let λ be the number of k-simple path partitions of $P_{u,v}$. Then we have

$$f(P_{u,v}, k) = (-1)^k \lambda d_{u,v}.$$

Furthermore, λ is the number of positive integer solutions $(x_1, \ldots, x_k)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_k = d, \quad x_1 \geqslant 1, \ldots, x_k \geqslant 1,$$

and so,

$$\lambda = \binom{d-1}{k-1}.$$

Thus, we have

$$f(P_{u,v}, k) = (-1)^k \binom{d-1}{k-1} d_{u,v}.$$

Therefore, it follows that

$$
\begin{aligned}
f(P_{u,v}) &= f(P_{u,v},1)+\cdots+f(P_{u,v},d) \\
&= -d_{u,v}\left\{1+(-1)\binom{d-1}{1}+(-1)^2\binom{d-1}{2}+\cdots+(-1)^{d-1}\binom{d-1}{d-1}\right\} \\
&= -d_{u,v}\cdot 0 = 0.
\end{aligned}
$$

In the third equality, we use the binomial theorem. Hence, if $d(u,v)\geqslant 2$, then

$$f(P_{u,v})=0. \tag{1}$$

Let P be a (u,v)-path of T. Suppose that P has a simple subpath Q of length $\geqslant 2$. Then, by (1), we have $f(Q)=0$, and so

$$f(P)=cf(Q)=0, \tag{2}$$

where c is a constant. If $u,v\in V(T)$ and $d(u,v)\geqslant 2$, then each (u,v)-path P in T has a simple subpath Q of length $\geqslant 2$. By (2), we have $f(P)=0$. Thus, if $d(u,v)\geqslant 2$, then the (u,v)-entry of $\mathbf{M}_T^{-1}=\mathbf{I}_n+\mathbf{N}_T+\mathbf{N}_T^2+\cdots$ is 0.

Now, we consider the (u,v)-entry of $\mathbf{M}_T^{-1}$ for two adjacent vertices $u,v\in V(T)$. If a (u,v)-path P has a simple subpath of length $\geqslant 2$, then, by (2), $f(P)=0$. Thus, assume that P has no simple subpath of length $\geqslant 2$. Then P must be of form

$$P=(e_1,e_2,\ldots,e_{2m-1}) \quad \text{for } e_{2j-1}=e \ (1\leqslant j\leqslant m) \text{ and } e_{2k}=e^{-1} \ (1\leqslant k\leqslant m-1),$$

where $e=(u,v)$. Therefore, it follows that

$$f(P)=(-1)^{2m-1}w(e)^m w\left(e^{-1}\right)^{m-1}=-w(e)^m w\left(e^{-1}\right)^{m-1}.$$

Note that $f(P)$ contributes to the (u,v)-entry of the matrix $\mathbf{N}_T^{2m-1}$. Hence, the (u,v)-entry of $\mathbf{M}_T^{-1}=\mathbf{I}_n+\mathbf{N}_T+\mathbf{N}_T^2+\cdots$ is equal to

$$-\sum_{m=1}^{\infty} w(e)^m w\left(e^{-1}\right)^{m-1}=\frac{-w(e)}{1-w(e)w(e^{-1})}=\frac{-w(e)}{1-q_{\bar{e}}t_{\bar{e}}}.$$

Now, we consider the (u,u)-entry of $\mathbf{M}_T^{-1}$ for each $u\in V(T)$. Let $u\in V(T)$ and P be a (u,u)-path, i.e., a u-cycle in T. If $f(P)\neq 0$, then, by (2), P must be of form

$$P=(e_1,e_2,\ldots,e_{2m}), \quad e_{2j-1}=e,\ e_{2j}=e^{-1} \ (1\leqslant j\leqslant m),$$

where $e=(u,v)$ for some $v\in V(T)$. Thus, we have

$$f(P)=(-1)^{2m}w(e)^m w\left(e^{-1}\right)^m=w(e)^m w\left(e^{-1}\right)^m.$$

Note that $f(P)$ contributes to the (u,u)-entry of the matrix $\mathbf{N}_T^{2m}$. Therefore, by $(*)$, it follows that the (u,u)-entry of $\mathbf{M}_T^{-1} = \mathbf{I}_n + \mathbf{N}_T + \mathbf{N}_T^2 + \cdots$ is equal to

$$1 + \sum_{o(e)=u} \sum_{m=1}^{\infty} w(e)^m w\left(e^{-1}\right)^m = 1 + \sum_{o(e)=u} \frac{w(e)w(e^{-1})}{1 - w(e)w(e^{-1})} = 1 + \sum_{o(e)=u} \frac{q_{\bar{e}} t_{\bar{e}}}{1 - q_{\bar{e}} t_{\bar{e}}}.$$

Hence,

$$\mathbf{M}_T^{-1} = \mathbf{I}_n - \mathbf{B} + \mathbf{D}.$$

3. A new proof for Theorem 3

Let G be a connected graph with n vertices and m edges, and $D(G) = \{e_1, \ldots, e_m, e_{m+1}, \ldots, e_{2m}\}$ ($e_{m+i} = e_i^{-1}$ $(1 \leqslant i \leqslant m)$). A bivariate zeta function of Bapat and Sivasubramanian [1] is a new generalization of the Ihara zeta function of G. The matrix of the right side in Theorem 3 is the inverse matrix of $\mathbf{M}_T$ for the weight w such that $w(e_j) = q$ and $w(e_j^{-1}) = t$ $(1 \leqslant j \leqslant m)$. They proved Theorem 3 by using Theorem 1.1 and Proposition 8.1 of Foata and Zeilberger [3].

We consider the edge zeta function of G, i.e., the zeta function with respect to a general weight treated in Theorem 1, and give another proof of Theorem 3 by using Watanabe and Fukumizu's Theorem which presents a determinant expression for the edge zeta function by $n \times n$ matrices. A bivariate zeta function of Bapat and Sivasubramanian [1] is the edge zeta function in the bivariate q, t case. Furthermore, it can be remarked that the theorem of Watanabe and Fukumizu in the bivariate q, t case gives the same result as Proposition 8.1 of Foata and Zeilberger [3].

Stark and Terras [6] defined the edge zeta function of a graph G with m edges. We introduce $2m$ variables $u_1, \ldots, u_{2m}$, and set $\mathbf{u} = (u_1, \ldots, u_{2m})$. Furthermore, set $g(C) = u_{i_1} \cdots u_{i_k}$ for each cycle $C = (e_{i_1}, \ldots, e_{i_k})$ of G. Then the *edge zeta function* $\zeta_G(u)$ of G is defined by

$$\zeta_G(\mathbf{u}) = \prod_{[C]} \left(1 - g(C)\right)^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G.

Theorem 4 *(Stark and Terras). Let G be a connected graph with m edges. Then*

$$\zeta_G(\mathbf{u})^{-1} = \det(\mathbf{I}_{2m} - \mathbf{T}\mathbf{U}) = \det(\mathbf{I}_{2m} - \mathbf{U}\mathbf{T}),$$

where the matrix $\mathbf{T} = (\mathbf{T}_{e,f})_{e,f \in D(G)}$ is given by

$$\mathbf{T}_{e,f} = \begin{cases} 1 & \text{if } t(e) = o(f) \text{ and } f \neq e^{-1}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbf{U} = \operatorname{diag}(u_1, \dots, u_m, u_{m+1}, \dots, u_{2m}).$$

Watanabe and Fukumizu [9] presented a determinant expression for the edge zeta function of a graph G with n vertices by means of $n \times n$ matrices. Then we define an $n \times n$ matrix $\widehat{\mathbf{A}} = \widehat{\mathbf{A}}(G) = (a_{xy})$ as follows:

$$a_{xy} = \begin{cases} u_{(x,y)}/(1 - u_{(x,y)}u_{(y,x)}) & \text{if } (x,y) \in D(G), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, an $n \times n$ matrix $\widehat{\mathbf{D}} = \widehat{\mathbf{D}}(G) = (\hat{d}_{xy})$ is the diagonal matrix defined by

$$\hat{d}_{xx} = \sum_{o(e)=x} \frac{u_e u_{e^{-1}}}{1 - u_e u_{e^{-1}}}.$$

Theorem 5 *(Watanabe and Fukumizu). Let G be a graph G with m edges, and $\mathbf{u} = (u_1, \dots, u_{2m})$. Then*

$$\zeta_G(\mathbf{u})^{-1} = \det(\mathbf{I}_n + \widehat{\mathbf{D}} - \widehat{\mathbf{A}}) \prod_{i=1}^{m} (1 - u_{e_i} u_{e_i^{-1}}).$$

A new proof for Theorem 3:

Let G be a connected graph with n vertices and m edges, and $D(G) = \{e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ ($e_{m+i} = e_i^{-1}$ $(1 \leqslant i \leqslant m)$). In Theorem 5, we set

$$u_{e_j} = u_j = q, \qquad u_{e_j^{-1}} = u_{m+j} = t \quad (1 \leqslant j \leqslant m).$$

Then two matrices $\widehat{\mathbf{A}} = (a_{xy})$ and $\widehat{\mathbf{D}} = (\hat{d}_{xy})$ are given as follows:

$$a_{xy} = \begin{cases} u_{(x,y)}/(1 - qt) & \text{if } (x,y) \in D(G), \\ 0 & \text{otherwise,} \end{cases} \qquad \hat{d}_{xx} = qt \deg x/(1 - qt), \quad x \in V(G).$$

Thus, we have

$$\widehat{\mathbf{A}} = \frac{1}{1 - qt}\mathbf{W} \quad \text{and} \quad \widehat{\mathbf{D}} = \frac{qt}{1 - qt}\mathbf{D}_v.$$

Again, by Theorem 5, we have

$$\begin{aligned} \eta_G(q,t)^{-1} = \zeta_G(\mathbf{u})^{-1} &= (1 - qt)^m \det\left(\mathbf{I}_n - \frac{1}{1 - qt}\mathbf{W} + \frac{qt}{1 - qt}\mathbf{D}_v\right) \\ &= (1 - qt)^{m-n} \det\bigl((1 - qt)\mathbf{I}_n - \mathbf{W} + qt\mathbf{D}_v\bigr) \\ &= (1 - qt)^{m-n} \det(\mathbf{I}_n - \mathbf{W} + qt\mathbf{D}_v - qt\mathbf{I}_n) \\ &= (1 - qt)^{m-n} \det(\mathbf{I}_n - \mathbf{W} + \mathbf{K}). \end{aligned}$$

4. A new proof for the first formula of Theorem 1

Bapat and Sivasubramanian [1] proved the first formula of Theorem 1 by introduction on the number of vertices. We present another proof of it by using the fact that the edge zeta function of a tree is "1" (the empty product).

Let T be a tree with n vertices, and $D(T)=\{e_1,\dots,e_{n-1},e_n,\dots,e_{2n-2}\}$ ($e_{n-1+i}=e_i^{-1}$ $(1\leqslant i\leqslant n-1)$). In Theorem 5, we set

$$u_{e_j}=u_j=q_{\bar{e}_j},\qquad u_{e_j^{-1}}=u_{m+j}=t_{\bar{e}_j}\quad (1\leqslant j\leqslant n-1).$$

Then we have

$$\widehat{\mathbf{A}}=\mathbf{B}\quad\text{and}\quad\widehat{\mathbf{D}}=\mathbf{D}.$$

By Theorem 5, we have

$$\begin{aligned}\zeta_G(\mathbf{u})^{-1}&=\det(\mathbf{I}_n-\widehat{\mathbf{A}}+\widehat{\mathbf{D}})\prod_{j=1}^{n-1}(1-q_{\bar{e}_j}t_{\bar{e}_j})\\&=\det(\mathbf{I}_n-\mathbf{B}+\mathbf{D})\prod_{j=1}^{n-1}(1-q_{\bar{e}_j}t_{\bar{e}_j})\\&=\det(\mathbf{M}_T)^{-1}\prod_{j=1}^{n-1}(1-q_{\bar{e}_j}t_{\bar{e}_j}).\end{aligned}$$

Since T is a tree, we have

$$\zeta_G(\mathbf{u})^{-1}=1.$$

Therefore, it follows that

$$\det(\mathbf{M}_T)=\prod_{j=1}^{n-1}(1-q_{\bar{e}_j}t_{\bar{e}_j}).$$

Acknowledgement

We would like to thank the referee for many valuable comments and many helpful suggestions.

References

[1] R.B. Bapat, Sivaramakrishnan Sivasubramanian, The product distance matrix of a tree and a bivariate zeta function of a graph, Electron. J. Linear Algebra 23 (2012) 275–286.

[2] H. Bass, The Ihara–Selberg zeta function of a tree lattice, Internat. J. Math. 3 (1992) 717–797.
[3] D. Foata, D. Zeilberger, A combinatorial proof of Bass's evaluations of the Ihara–Selberg zeta function for graphs, Trans. Amer. Math. Soc. 351 (1999) 2257–2274.
[4] K. Hashimoto, Zeta functions of finite graphs and representations of p-adic groups, in: Adv. Stud. Pure Math., vol. 15, Academic Press, New York, 1989, pp. 211–280.
[5] Y. Ihara, On discrete subgroups of the two by two projective linear group over p-adic fields, J. Math. Soc. Japan 18 (1966) 219–235.
[6] H.M. Stark, A.A. Terras, Zeta functions of finite graphs and coverings, Adv. Math. 121 (1996) 124–165.
[7] T. Sunada, L-Functions in Geometry and Some Applications, in: Lecture Notes in Math., vol. 1201, Springer-Verlag, New York, 1986, pp. 266–284.
[8] T. Sunada, Fundamental Groups and Laplacians, Kinokuniya, Tokyo, 1988 (in Japanese).
[9] Y. Watanabe, K. Fukumizu, Graph zeta function in the Bethe free energy and loopy belief propagation, Adv. Neural Inf. Process. Syst. 22 (2010) 2017–2025.