Math. Annalen 220, 285—290 (1976) — © by Springer-Verlag 1976

An Extension of Becker's Univalence Condition*

Stephan Ruscheweyh

Abt. Mathematik, Universität Dortmund, Postfach 500500, D-4600 Dortmund 50, Federal Republic of Germany

Let $f(z)=\sum_1^\infty a_k z^k$, $a_1=1$, be analytic in the unit disc $\Delta:=\{z\,|\,|z|<1\}$. J. Becker [2, 3, 4] and L. V. Ahlfors [1], in connection with their work on quasiconformal extensions of univalent analytic functions, have shown that the condition

$$\left| c|z|^2+(1-|z|^2)\frac{zf''(z)}{f'(z)} \right| \leqq 1, \qquad z\in\Delta, \tag{1}$$

with a complex constant c is sufficient for the univalence of f in Δ. As a consequence of certain results concerning subordination chains of multivalent functions in Δ D. J. Hallenbeck and A. E. Livingston [6] were able to generalize (1). The aim of the present paper is to further extend this condition.

Theorem 1. *Let $s=\alpha+i\beta$, $\alpha>0$. Let $f(z)$ be analytic in Δ, $f(0)=f'(0)-1=0$. Assume that for a certain $c\in\mathbb{C}$ and all $z\in\Delta$*

$$\left| c|z|^2+s-\alpha(1-|z|^2)\left\{ s\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-s)\frac{zf'(z)}{f(z)} \right\} \right| \leqq M \tag{2}$$

with

$$M=\begin{cases} \alpha|s|+|s+c|(\alpha-1), & 0<\alpha<1, \\ |s|, & \alpha\geqq 1. \end{cases}$$

Then $f(z)$ is univalent in Δ.

Note that the case $s=1$ (with c replaced by $-1-c$) is (1). The cases $s=1/p$, $p\in\mathbb{N}$, are in [6].

From Theorem 1 we shall deduce the following interesting Corollary.

Corollary 1. *Let $f(z)$ be analytic in Δ, $f(0)=f'(0)-1=0$, and let γ, $q\in\mathbb{R}$ be related by*

$$0<\cos\gamma\leqq \begin{cases} 1/2, & -1<q\leqq 0, \\ 1/(2+4q), & q>0. \end{cases} \tag{3}$$

* Dedicated to Prof. Dr. Dr. h.c. Ernst Peschl on the occasion of his seventieth birthday.

Then f is univalent in Δ if

$$\operatorname{Re} e^{i\gamma}\left\{1+z\frac{f''(z)}{f'(z)}+q\frac{zf'(z)}{f(z)}\right\}>0, \qquad z\in\Delta. \tag{4}$$

If, in addition, $f''(0)=0$, (3) *can be replaced by*

$$0<\cos\gamma\leqq\begin{cases}1, & -1<q\leqq 0,\\ 1/(1+2q), & q>0.\end{cases} \tag{5}$$

The sufficiency of (3) and (5) (for $q=0$) has recently been established by J. Pfaltzgraff [7] and V. Singh (unpublished) respectively. The case $q\in\mathbb{N}$ in (3) is due to D. J. Hallenbeck and A. E. Livingston [6].

Although M. S. Robertson [9] has shown that (3) for $q=0$ cannot be improved, the conditions (3), (5) are not best possible in general. This will be discussed in a forthcoming paper of R. Jankovics.

In the last section we shall consider univalence conditions for functions $f(z)=z+\sum_1^\infty a_k z^{-k}$ analytic in $\Delta':=\{z\,|\,|z|>1\}$.

Theorem 2. *Let* $s=\alpha+i\beta$, $\alpha\geqq 1$. *Let* $f(z)=z+\frac{b}{z}+\dots$ *be analytic in Δ' and fulfil*

$$\left|i\beta+(1-|z|^2)\alpha\left\{(1-s)\left(1-\frac{zf'(z)}{f(z)}\right)-s\frac{zf''(z)}{f'(z)}\right\}\right|\leqq\alpha|s|-|\beta|(\alpha-1), \tag{6}$$

$z\in\Delta'$. *Then f is univalent in Δ'.*

The case $s=1$ is due to J. Becker [3]. Theorem 2 has an interesting limiting case which can be shown to represent a univalence condition too[1]:

Corollary 2. *Let* $f(z)=z+\frac{b}{z}+\dots$ *be analytic in Δ' and let*

$$(|z|^2-1)\left|1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right|\leqq 1, \qquad z\in\Delta'. \tag{7}$$

Then f(z) is univalent in Δ'.

For the proof of our Theorems we shall use a variant of Becker's method. Hence we could include statements concerning quasiconformal extensions into $\mathbb{C}$ of those functions f which satisfy somewhat stronger conditions than (2) or (6) as long as we use suitable subordination chains. (7), however, does not arise directly from such a chain and thus it remains an open problem whether, for instance, a function $f(z)=z+\frac{b}{z}+\dots$, with

$$(|z|^2-1)\left|1-\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}\right|\leqq k<1, \qquad z\in\Delta',$$

admits a quasiconformal extension into $\mathbb{C}$.

[1] I wish to thank the referee for this remark.

Proof of Theorem 1. The main tool in our development is the following Lemma due to Ch. Pommerenke [8]. Δ_ϱ denotes the disc $\{z \mid |z|<\varrho\}$.

Lemma 1. *Let $0<\varrho_0\leqq 1$. Let $f(z,t)=a_1(t)z+\dots$, $a_1(t)\neq 0$, be analytic for each $t\in I:=[0,\infty)$ in $z\in\Delta_{\varrho_0}$ and locally absolutely continuous in I, locally uniformly with respect to Δ_{ϱ_0}. For almost all $t\in I$ suppose*

$$\frac{\partial}{\partial t} f(z,t)=zf'(z,t)h(z,t)\,,\qquad z\in\Delta_{\varrho_0}\,, \tag{8}$$

where $h(z,t)$ is analytic in Δ and satisfies $\operatorname{Re} h(z,t)>0$, $z\in\Delta$. If $|a_1(t)|\to\infty$ for $t\to\infty$ and if $f(z,t)/a_1(t)$ forms a normal family in Δ_{ϱ_0}, then for each $t\in I$ $f(z,t)$ can be continued analytically in Δ and gives an univalent function.

In the proof of Theorem 1 we have to treat the case $c=0$ separately. We postpone this and assume $c\neq 0$. It is easily verified that the condition (2) implies $f(z)\cdot f'(z)\neq 0$, $0<|z|<1$. For $t\in I$ put formally

$$f(z,t):=f(e^{-st}z)\left\{1-\frac{\alpha}{c}(e^{2t}-1)\frac{e^{-st}f'(e^{-st}z)}{f(e^{-st}z)}\right\}^s.$$

The two relations $|c+s|\leqq|s|$, which follows from (2), and $\operatorname{Re} s>0$ imply $\alpha/c\notin[0,\infty)$. Hence we can find a $\varrho_0^*>0$ such that with $F(z)=zf'(z)/f(z)$

$$1-\frac{\alpha}{c}(e^{2t}-1)F(e^{-st}z)\neq 0\,,\qquad z\in\Delta_{\varrho_0^*}\,,\qquad t\in I\,.$$

Furthermore $|f'(0,t)|=\left|\left\{\left(1+\frac{\alpha}{c}\right)e^{-t}-\frac{\alpha}{c}e^{t}\right\}^s\right|$ which is nonvanishing in I and tends to infinity for $t\to\infty$ once we have choosen a fixed branch for these numbers. Thus $f(z,t)/f'(0,t)$ forms a normal family of analytic functions in Δ_{ϱ_0}, $\varrho_0=\varrho_0^*/2$. A straightforward calculation gives

$$\frac{\frac{\partial}{\partial t} f(z,t)}{zf'(z,t)}=s\,\frac{1+P(e^{-st}z,t)}{1-P(e^{-st}z,t)}:=h(z,t)\,,$$

where

$$P(x,t)=\frac{c}{\alpha}e^{-2t}+1+(e^{-2t}-1)H_s(x)$$

and

$$H_s(z)=s\left(1+z\frac{f''(z)}{f'(z)}\right)+(1-s)\frac{zf'(z)}{f(z)}\,. \tag{9}$$

Since $P(z,t)$ is analytic in Δ it remains to show $\operatorname{Re} h(z,t)>0$, $z\in\Delta$. This condition is equivalent to $|\alpha P(e^{-st}z,t)+i\beta|\leqq|s|$, $z\in\Delta$, $t\in I$.

We have

$$|\alpha P+i\beta|\leqq(1-e^{-2t})\left|\frac{ce^{-2\alpha t}+s}{1-e^{-2\alpha t}}-\alpha H_s(e^{-st}t)\right|$$

$$+\left|(ce^{-2t\alpha}+s)\frac{1-e^{-2t}}{1-e^{-2\alpha t}}-(ce^{-2t}+s)\right|$$

$$=m_1+m_2\,,$$

say. (2) implies for $|z|\leqq 1$

$$\left|\frac{c|e^{-st}z|^2+s}{1-|e^{-st}z|^2}-\alpha H_s(e^{-st}z)\right|\leqq\frac{M}{1-|e^{-st}z|^2}\leqq\frac{M}{1-e^{-2\alpha t}}\,.$$

From the maximum principle, applied to the function

$$\frac{ce^{-2\alpha t}+s}{1-e^{-2\alpha t}}-\alpha H_s(e^{-st}z)\,,$$

we obtain

$$m_1\leqq\frac{1-e^{-2t}}{1-e^{-2\alpha t}}M\,.$$

On the other hand

$$m_2=|c+s|\left|1-\frac{1-e^{-2t}}{1-e^{-2\alpha t}}\right|\,.$$

Let $h(t)=\dfrac{1-e^{-2t}}{1-e^{-2\alpha t}}$. From the relations $(t\in I)$

$$\frac{1}{\alpha}<h(t)\leqq 1,\qquad a>1\,,$$

$$1\leqq h(t)<\frac{1}{\alpha},\qquad 0<\alpha\leqq 1\,,$$

we conclude $m_1+m_2\leqq|s|$ which is our assertion.

It remains to consider (2) for $c=0$ which may happen for $\operatorname{Re}s\geqq 1$ only. In this case (2) implies

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}+\left(\frac{1}{s}-1\right)\frac{zf'(z)}{f(z)}\right\}>0,\qquad z\in\Delta\,.$$

By a theorem of P.J. Eenigenburg *et al.* [5] this condition is sufficient for the univalence (in fact spirallikeness) of f.

For the proof of Corollary 1 we require the following well-known Lemma.

Lemma 2. *Let* $p(z)=1+a_nz^n+\ldots$ *be analytic in* Δ *and let* $\operatorname{Re}p(z)>0$, $z\in\Delta$. *Then*

$$\left|p(z)-1-\frac{2|z|^{2n}}{1-|z|^{2n}}\right|\leqq\frac{2|z|^n}{1-|z|^{2n}},\qquad z\in\Delta\,.$$

Proof of Corollary 1. Let $s=1/(q+1)$ and $H_s(z)$ as in (9). Let $f(z)=z+a_2z^2+\dots$. Then for

$$p(z):=\frac{e^{i\gamma}H_s(z)-i\sin\gamma}{\cos\gamma}=1+\frac{e^{i\gamma}}{\cos\gamma}a_2(s+1)z+\dots$$

we have $p'(0)=0$ if and only if $f''(0)=0$. Condition (4) implies that $p(z)$ is analytic in Δ and fulfils $\operatorname{Re}p(z)>0$, $z\in\Delta$. With $(c+s):=\frac{2}{n}se^{-i\gamma}\cos\gamma$, $n=1,2$, we obtain from Lemma 2

$$\begin{aligned}\left|\frac{(c+s)|z|^2}{1-|z|^2}-s(H_s(z)-1)\right| &\\ &\leqq s|\cos\gamma|\left\{\left|\frac{2|z|^{2n}}{1-|z|^{2n}}-(p(z)-1)\right|+\left|\frac{2|z|^{2n}}{1-|z|^{2n}}-\frac{2}{n}\frac{|z|^2}{1-|z|^2}\right|\right\}\\ &\leqq\frac{2s}{n}\frac{|\cos\gamma|}{1-|z|^2}.\end{aligned}$$

Thus $f(z)$ is univalent in Δ whenever

$$\frac{2}{n}s|\cos\gamma|\leqq\begin{cases}s^2+\frac{2}{n}s|\cos\gamma|(s-1), & 0<s\leqq 1,\\ s\qquad\qquad, & s>1,\end{cases}$$

which is equivalent to (3) and (5) respectively.

Proof of Theorem 2. For $z\in\Delta$, $t\in I$, let formally

$$\begin{aligned}f(z,t)&:=\frac{1}{f(e^{st}/z)}\left\{1-(1-e^{-2t})\frac{e^{st}}{z}\frac{f'(e^{st}/z)}{f(e^{st}/z)}\right\}^{-s}\\ &=e^{st}z\{1-2sbe^{2t(1-s)}z^2+(2s-1)be^{-2st}z^2+\dots\}\end{aligned}$$

such that $f(z,t)/f'(0,t)$ can be defined as a normal family of analytic functions in a small disc Δ_{ϱ_0}. We use Lemma 1 to show that $f(z,0)=\dfrac{1}{f\left(\frac{1}{z}\right)}$ is univalent in Δ. Since

$$\frac{\frac{\partial}{\partial t}f(z,t)}{zf'(z,t)}=s\frac{1+P(e^{st}/z,t)}{1-P(e^{st}/z,t)}$$

with

$$P(x,t)=(e^{2t}-1)\left\{(1-s)\left(\frac{xf'(x)}{f(x)}-1\right)+s\frac{xf''(x)}{f'(x)}\right\}$$

we are left with the inequality

$$|\alpha P(e^{st}/z,t)+i\beta|\leqq|s|,\qquad z\in\Delta,\qquad t\in I. \tag{10}$$

Following the lines of the proof of Theorem 1 one can deduce that (6) implies (10).

Proof of Corollary 2. (7) implies $f(z)\cdot f'(z)\neq 0$, $z\in\Delta'$. Let $\varrho>1$ be arbitrary but fixed. It will be enough to show that $f_\varrho(z):=f(\varrho z)/\varrho$ is univalent in Δ'. The entity

$$(|z|^2-1)\left|1-\frac{zf'_\varrho(z)}{f_\varrho(z)}\right|$$

remains bounded for $|z|\to\infty$ and vanishes on $|z|=1$. Hence it is bounded by a certain constant $M<\infty$ on Δ'. Also

$$\left|1-\frac{zf'_\varrho(z)}{f_\varrho(z)}+\frac{zf''_\varrho(z)}{f'_\varrho(z)}\right|\leqq\frac{1}{|\varrho z|^2-1},\qquad z\in\Delta',$$

by (7). With $s:=\varrho^2M/(\varrho^2-1)$ we obtain for $z\in\Delta'$

$$(|z|^2-1)\left|\frac{1}{s}\left(1-\frac{zf'_\varrho(z)}{f_\varrho(z)}\right)-1-\frac{zf''_\varrho(z)}{f'_\varrho(z)}+\frac{zf'_\varrho(z)}{f_\varrho(z)}\right|\leqq\frac{M}{s}+\frac{|z|^2-1}{|\varrho z|^2-1}\leqq 1$$

which, by Theorem 2, proves the univalence of $f_\varrho(z)$ in Δ'.

References

1. Ahlfors, L. V.: Sufficient conditions for Q. C. extensions. Ann. Math. Studies **79**, Princeton, 1974
2. Becker, J.: Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen. J. reine angew. Math. **255**, 23—43 (1972)
3. Becker, J.: Löwnersche Differentialgleichung und Schlichtheitskriterien. Math. Ann. **202**, 321—335 (1973)
4. Becker, J.: Über die Lösungsstruktur einer Differentialgleichung in der konformen Abbildung. J. reine angew. Math. (to appear)
5. Eenigenburg, P. J., Miller, S. S., Mocanu, P. T., Reade, M. O.: On a subclass of Bazilevič functions. Proc. Amer. Math. Soc. **45**, 88—92 (1974)
6. Hallenbeck, D. J., Livingston, A. E.: Subordination chains and p-valent functions. Preprint 1975
7. Pfaltzgraff, J. A.: Univalence of the integral of $f'(z)^\lambda$. London Jour. of Math. (to appear)
8. Pommerenke, Ch.: Über die Subordination analytischer Funktionen. J. reine angew. Math. **218**, 159—173 (1965)
9. Robertson, M. S.: Univalent functions $f(z)$ for which $zf'(z)$ is spirallike. Michigan Math. J. **16**, 97—101 (1969)

(Received November 26, 1975)