On a Property of Singular Integrals with Even Positive Kernels

St. Ruscheweyh

*Mathematisches Institut der Universität Würzburg, Am Hubland
8700 Würzburg, West Germany*

AND

K.-J. Wirths

*Technische Universität Braunschweig Pockelsstr. 14
3300 Braunschweig, West Germany*

1

Let **K** be the set of kernels

$$\chi(\rho, u) : A_\chi \times |-\pi, \pi| \to \mathbb{R} \qquad (A_\chi \subset \mathbb{R})$$

with:

   (i) $\chi(\rho, \cdot) \in C_{2\pi}$, even, non-negative,

   (ii) $\int_{-\pi}^{\pi} \chi(\rho, u)\, du = 2\pi$, $\rho \in A_\chi$.

   (iii)   There exists an accumulation point $\rho_0 \in \mathbb{R} \cup \{-\infty, \infty\}$ of A_χ such that, for every $\delta \in (0, \pi)$,

$$\int_{\delta \leqslant |u| \leqslant \pi} \chi(\rho, u)\, du = o(1), \qquad \rho \to \rho_0.$$

For $f \in L_{2\pi}^1$ (real or complex valued) and $\chi \in \mathbf{K}$, $\rho \in A_\chi$, we define

$$I_\rho(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - u)\chi(\rho, u)\, du.$$

Let $f \in C_{2\pi}$ real valued, $\int_{-\pi}^{\pi} f(u)\, du > 0$. Then if $f \geqslant 0$ on $|-\pi, \pi|$ we have

$$|I_\rho(g \cdot f, x)| \leqslant I_\rho(f, x), \qquad x \in |-\pi, \pi|, \tag{1}$$

for every complex valued $g \in L_{2\pi}^1$ with $|g(u)| \leqslant 1$, $u \in \mathbb{R}$. Clearly this bound preserving property is also sufficient for f to be non-negative (for each $\rho \in A_\chi$). In the present paper we deal with the following related conditions:

$$|I_\rho(e^{iu}f(u), x)| \leqslant |I_\rho(f, x)|, \qquad x \in |-\pi, \pi|,$$
$$\text{for all } \rho \in A_\chi. \tag{2}$$

It appears to us that (2), for a large variety of kernels in **K**, is also sufficient for $f \in C_{2\pi}$, $\int_{-\pi}^{\pi} f(u)\,du > 0$, to be non-negative. Note that this conclusion is trivial if strict inequality is required in (2).

We shall establish this conjecture—occasionally under further restrictions for f—for a number of kernels, among them the kernels of Poisson, Fejér, de la Vallée Poussin and Weierstrass. In the last section we treat the kernel

$$P_2(\rho, u) = \frac{1-\rho^2}{1+\rho^2} \left(\frac{1-\rho^2}{1+\rho^2 - 2\rho \cos u}\right)^2,$$
$$A_{P_2} = (0, 1), \qquad \rho_0 = 1. \tag{3}$$

We have not been able, however, to derive the general condition (if there is any) to be imposed on f or $\chi \in \mathbf{K}$ to make our conjecture valid. It should be pointed out that the desired conclusion fails to hold for suitably chosen f and $\chi \in \mathbf{K}$ with "small" supports.

2

Let $b_k(\rho)$ be the Fourier coefficients of $\chi \in \mathbf{K}$,

$$\chi(\rho, u) \sim \sum_{k=-\infty}^{\infty} b_k(\rho)\, e^{iku} \tag{4}$$

such that $b_0(\rho) \equiv 1$, $b_k(\rho) = b_{-k}(\rho) \in \mathbb{R}$ for $k \in \mathbb{N}$. For $j \in \mathbb{Z}$ we define

$$\Delta_2 b_j(\rho) = b_j(\rho) - 2b_{j+1}(\rho) + b_{j+2}(\rho)$$

and for $m > 1$

$$\Delta_{2m} b_j = \Delta_{2m-2} b_j - 2\Delta_{2m-2} b_{j+1} + \Delta_{2m-2} b_{j+2}.$$

THEOREM 1. *Let $\chi \in \mathbf{K}$ such that*

(i) $\chi(\rho, \cdot) \in C_{2\pi}^2$, $\rho \in A_\chi$,

(ii) $\sum_{k=1}^{\infty} |\Delta_2 b_{k-1}(\rho)| = O((1 - b_1(\rho)))$, $\rho \to \rho_0$,

(iii) $\Delta_2 b_{k-1}(\rho) = o(1 - b_1(\rho))$, $\rho \to \rho_0$, $k \in \mathbb{N}$.

Then if (2) holds for $f \in C_{2\pi}$ with $\int_{-\pi}^{\pi} f(u)\,du > 0$ we have $f \geqslant 0$ in $|-\pi, \pi|$.

Proof. Without loss of generality we assume $I_{\rho_n}(f, x_n) = 0$ for a sequence of pairwise disjoint $\rho_n \in A_\chi$, $n \in \mathbb{N}$, with $\rho_n \to \rho_0$ and corresponding numbers x_n (otherwise the conclusion is trivial for the approximate identity χ). Hence $I_{\rho_n}(e^{iu}f(u), x_n) = 0$, $n \in \mathbb{N}$. If

$$f(u) \sim \sum_{k=-\infty}^{\infty} a_k e^{iku}, \qquad a_k = \overline{a_{-k}}.$$

then since $\chi \in C_{2\pi}^2$

$$\operatorname{Re} I_{\rho_n}(f, x_n) = a_0 + 2 \sum_{k=1}^{\infty} b_k(\rho_n) \operatorname{Re}(a_k e^{ikx_n}) = 0$$

and

$$\operatorname{Re}[e^{-ix_n} I_{\rho_n}(e^{iu}f(u), x_n)]$$

$$= a_0 b_1(\rho_n) + \sum_{k=1}^{\infty} (b_{k-1}(\rho_n) + b_{k+1}(\rho_n)) \operatorname{Re}(a_k e^{ikx_n}) = 0.$$

This implies

$$a_0(b_1(\rho_n) - 1) + \sum_{k=1}^{\infty} (\Delta_2 b_{k-1}(\rho_n)) \operatorname{Re}(a_k e^{ikx_n}) = 0$$

for $n \in \mathbb{N}$. Using (ii), (iii) of the assumption and the fact $a_k \to 0$, $k \to \infty$, we obtain

$$\sum_{k=1}^{\infty} \frac{\Delta_2 b_{k-1}(\rho_n)}{1 - b_1(\rho_n)} \cdot \operatorname{Re}(a_k e^{ikx_n}) = o(1), \qquad n \to \infty,$$

which gives the contradiction

$$0 = a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\, du > 0.$$

For Fejér's kernel

$$F(n, u) = \frac{1}{n+1} \left(\frac{\sin((n+1)(u/2))}{\sin(u/2)} \right)^2 = 1 + 2 \sum_{k=1}^{n} \left(1 - \frac{k}{n+1} \right) \cos ku,$$

$A_F = \mathbb{N}$, $\rho_0 = \infty$, and for Poisson's kernel

$$P(\rho, u) = \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos u} = 1 + 2 \sum_{k=1}^{\infty} \rho^k \cos ku,$$

$A_p = (0, 1)$, $\rho_0 = 1$, the assumptions of Theorem 1 are fulfilled and our conjecture is therefore proved in these cases.

3

For Korovkin-type kernels $\chi \in \mathbf{K}$, i.e., those with

$$\frac{1 - b_2(\rho)}{1 - b_1(\rho)} = 4 + o(1), \qquad \rho \to \rho_0,$$

we have for $k \in \mathbb{N}$

$$\frac{\varDelta_2 b_{k-1}(\rho)}{b_1(\rho) - 1} = 2 + o(1), \qquad \rho \to \rho_0,$$

and Theorem 1 is not applicable. For these kernels we do not have a general method to establish our conjecture. We can prove it, however, for those Korovkin-type kernels which admit an infinite trigonometric asymptotic expansion of $I_\rho(f, x)$ and real-analytic f.

Let $\mathbf{K}'$ denote the class of those $\chi \in \mathbf{K}$ for which

$$\varDelta_{2m+2}(b_{-m-1}(\rho)) = o(\varDelta_{2m}(b_{-m}(\rho))), \qquad \rho \to \rho_0$$

holds for $m \in \mathbb{N}$. We make use of the following Lemma which is easily deduced along the lines in [3, pp. 73–75] and can also be extracted from the results in [2].

LEMMA. *Let $\chi \in \mathbf{K}'$ and $f \in C_{2\tau}^{2m}$, $f^{(j)}(x_0) = 0$, $j = 0, 1,..., 2m - 2$. Then for $\rho \to \rho_0$*

$$I_\rho(f, x_0) = \frac{(-1)^m}{(2m)!} f^{(2m)}(x_0) \varDelta_{2m}(b_{-m}(\rho)) + o(\varDelta_{2m}(b_{-m}(\rho))). \tag{5}$$

THEOREM 2. *Let $\chi \in \mathbf{K}'$ and $f \in C_{2\pi}^{\infty}$ real analytic with $\int_{-\pi}^{\pi} f(u)\, du > 0$. Then the conditions (2) imply $f \geqslant 0$ on $[-\pi, \pi]$.*

Proof. We show that every zero of f is of even multiplicity which implies the assertion. In fact, if $f^{(i)}(x_0) = 0$ for $0 \leqslant j \leqslant 2m - 2$ and $f^{(2m-1)}(x_0) \neq 0$, a combination of (5) and (2) gives for $\rho \to \rho_0$

$$|2mi f^{(2m-1)}(x_0) + f^{(2m)}(x_0) + o(1)| \leqslant |f^{(2m)}(x_0) + o(1)|$$

which implies the contradiction $f^{(2m-1)}(x_0) = 0$.

It follows from the results in $|2|$ that the de la Valleé Poussin kernel

$$V(n, u) = \frac{(n!)^2}{(2n)!} \left(2 \cos \frac{u}{2} \right)^{2n}, \qquad n \in \mathbb{N},$$

is in $\mathbf{K}'$. The same is true for the Weierstrass kernel

$$W(\rho, u) = 1 + 2 \sum_{k=1}^{\infty} e^{-\rho k^2} \cos ku, \qquad \rho \to 0 + ,$$

as an iterated application of l'Hospital's rule to

$$\frac{\Delta_{2m-2} b_{-m-1}(\rho)}{\Delta_{2m} b_{-m}(\rho)}, \qquad \rho \to 0 + , m \in \mathbb{N}$$

shows.

4

In this concluding section we establish our conjecture for the kernel

$$P_2(\rho, u) = \frac{1 - \rho^2}{1 + \rho^2} \left(\frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos u} \right)^2$$

$$= 1 + 2 \sum_{k=1}^{\infty} \left(\frac{1 - \rho^2}{1 + \rho^2} k + 1 \right) \rho^k \cos ku.$$

$A_{p_2} = (0, 1)$, $\rho_0 = 1$, which does not belong to $\mathbf{K}'$ but is of Korovkin-type. It is known $|1, \text{pp. } 205–210|$ that for real or complex valued functions $h \in C_{2\pi}$ we have

$$I_\rho(h, x) = \frac{1 - |z|^2}{1 + |z|^2} (zH_z + \bar{z}H_{\bar{z}}) + H, \qquad z = \rho e^{ix},$$

where H denotes the harmonic function in $|z| < 1$ with the continuous boundary values h. Now if F denotes the analytic function in $|z| < 1$ with $F(0) \in \mathbb{R}$ and $\operatorname{Re} F$ harmonic with the boundary values f it is clear that the harmonic function H with the boundary values $e^{iu}f(u)$ is given by

$$2H(z) = zF(z) + \overline{\left(\frac{F(z)}{z} \right)} + \left(z - \frac{1}{\bar{z}} \right) F(0)$$

such that condition (2) reads in this particular case

$$|(1 - |z|^2)(z^2 F' + \overline{F'}) + 2z(F + \overline{F})|$$
$$\leqslant |(1 - |z|^2)(zF' + \overline{zF'}) + (1 + |z|^2)(F + \overline{F})|$$

for $|z| < 1$. A rearrangement leads to the equivalent inequality

$$(1 - |z|^2)|F'(z)| \leqslant 2\,|\operatorname{Re} F(z)|, \qquad |z| < 1. \tag{6}$$

Equation (6) is well known to hold for analytic F with $F(0) > 0$, $\operatorname{Re} F(z) > 0$ in $|z| < 1$. It remains to show the sufficiency of (6) for $\operatorname{Re} F(z) \geqslant 0$ if F is analytic in $|z| < 1$, $F(0) > 0$. Assume $\operatorname{Re} F(z_0) = 0$ for a certain $|z_0| < 1$. Then it follows from (6)

$$\lim_{z \to z_0} (1 - |z|^2) \frac{|F'(z)|}{|F(z) - F(z_0)|} \leqslant 2$$

which is impossible.

REFERENCES

1. K. W. BAUER AND ST. RUSCHEWEYH. "Differential Operators for Partial Differential Equations and Function Theoretic Applications," Lecture Notes in Mathematics No. 791, Springer-Verlag. Berlin/New York. 1980.
2. G. BLEIMANN. "Über trigonometrische und fraktionierte Taylorformeln und deren Anwendungen in der Approximationstheorie," dissertation. Technische Universität Aachen, 1979.
3. P. L. BUTZER AND R. J. NESSEL. "Fourier-Analysis and Approximation." Bd. I. Birkhäuser, Basel, 1971.