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Some Inequalities for Starlike Functions

By

St. Ruscheweyh, Würzburg, and **V. Singh,** Patiala, India

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Abstract. For normalized starlike univalent functions in the unit disc we derive upper and lower estimates for certain differential expressions of n-th order (including $|zf'(z)/f(z)|$ for $n = 1$) in terms of $|f(z)|$. Our results generalize and/or improve earlier ones by Twomey and Singh. The operators and methods applied come from the theory of the Peschl—Bauer differential equation.

1. Introduction

Let $\mathbb{D}$ be the unit disc $\{z: |z| < 1\}$ and S^* the family of normalized starlike univalent functions in $\mathbb{D}$, i. e. $f \in S^*$ iff f analytic in $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1$, and $\operatorname{Re}[zf'(z)/f(z)] > 0$ in $\mathbb{D}$. The aim of this note is to obtain new inequalities for $g(z) = zf'(z)/f(z)$ and certain combinations of its derivatives up to the n-th order in terms of $|f(z)|$. We mention two previous results.

Theorem A. (Twomey [3]) *For $f \in S^*$ and $z \in \mathbb{D}$ we have* $(r = |z|)$

$$\left|\frac{zf'(z)}{f(z)}\right| \leqslant 1 + \frac{r \log[(1+r)^2 |f(z)|/r]}{(1-r)\log\dfrac{1+r}{1-r}}. \tag{1}$$

Theorem B. (V. Singh [2]) *Under the same assumptions we have*

$$\frac{1-r^2}{r}|f(z)| \leqslant \operatorname{Re}\frac{zf'(z)}{f(z)} \leqslant \frac{1-r}{1+r} + \frac{2r\log[(1+r)^2|f(z)|/r]}{(1-r^2)\log\dfrac{1+r}{1-r}}. \tag{2}$$

We shall give a new simple proof for the right hand side of (2) and show that (2) contains an improvement of (1), at least for small r. Furthermore, our proof extends to certain differential operators which are related to the class of n-harmonic functions. These are the solutions of the so-called Peschl—Bauer equation

$$(1-|z|^2)^2 w_{z\bar{z}} - n(n+1)w = 0, \quad n \in \mathbb{N}. \tag{3}$$

For a complete account of that theory see K. W. BAUER and ST. RUSCHEWEYH [1]. We shall need only the following results:

Theorem C. *w is a real n-harmonic function in $\mathbb{D}$ iff there exists an analytic function g in $\mathbb{D}$ such that*

$$w = \operatorname{Re} S_n g \tag{4}$$

with

$$(S_n g)(z) = \sum_{k=0}^{n} \binom{n+k}{n} \left(\frac{r^2}{1-r^2}\right)^k \frac{(z^n g)^{(n-k)}}{z^k (n-k)!}.$$

Theorem D. *w is a positive n-harmonic function in $\mathbb{D}$ with $w(0) = 1$ iff* (4) *holds with* $\operatorname{Re} g(z) > 0$ *in $\mathbb{D}$, $g(0) = 1$. If g has the Herglotz representation*

$$g(z) = \int_0^{2\pi} \frac{1 + z e^{it}}{1 - z e^{it}} d\mu(t) \tag{5}$$

with a certain probability measure μ on $[0, 2\pi]$ *then*

$$w(z) = \int_0^{2\pi} \left(\frac{1-r^2}{|1 - z e^{it}|^2}\right)^{n+1} d\mu(t). \tag{6}$$

In particular,

$$\left(\frac{1-r}{1+r}\right)^{n+1} \leqslant w(z) \leqslant \left(\frac{1+r}{1-r}\right)^{n+1}, \quad z \in \mathbb{D}, \tag{7}$$

for any such solution.

Although the following theorem deals with starlike functions it is clear that the estimate (8) can as well be considered as an improvement over (7) for positive n-harmonic functions. We use the abbreviation $A = ((1+r)/(1-r))^2$.

Theorem 1: *Let $f \in S^*$. Then for $z \in \mathbb{D}$, $r = |z|$, we have*

$$L_n \leqslant \operatorname{Re}\left(S_n \frac{zf'}{f}\right)(z) \leqslant U_n, \tag{8}$$

where

$$L_n = A^{\frac{n+1}{2}} \left\{1 - \frac{A^{n+1} - 1}{\log A} \log\left(\frac{(1-r)^2}{r} |f(z)|\right)\right\}^{-1},$$

$$U_n = A^{-\frac{n+1}{2}} \left\{1 + \frac{A^{n+1} - 1}{\log A} \log\left(\frac{(1+r)^2}{r} |f(z)|\right)\right\}.$$

The upper bounds are sharp (only) for the functions

$$f(z) = \frac{z}{(1 - \varepsilon z)^{2\lambda}(1 + \varepsilon z)^{2-2\lambda}}, \quad 0 \leqslant \lambda \leqslant 1, \, |\varepsilon| = 1, \tag{9}$$

and for $z = \bar{\varepsilon} r$. *For the lower bounds equality occurs only for the functions* (9) *with* $\lambda = 0$ *or* $\lambda = 1$.

Theorem 1 holds also for $n = 0$ (with $S_0 g \equiv g$). The lower bound L_0, however, is worse than the one in Theorem B. In the general case we have not been able to establish a lower bound of similar type. The upper bounds in Theorem 1 admit the following interesting refinement:

Theorem 2: *Let* $f \in S^*$. *Then for* $z \in \mathbb{D}$ *the sequence*

$$\frac{U_n - \operatorname{Re}\left(S_n \dfrac{zf'}{f}\right)(z)}{A^{n+1} \log A^{n+1} - A^{n+1} + 1}, \quad n = 0, 1, \ldots$$

is positive and decreasing.

For easier reference we mention the explicit expression for $n = 1$ of the operator $S_n \dfrac{zf'}{f}$. We have

$$\left(S_1 \frac{zf'}{f}\right)(z) = \frac{zf'}{f}\left(1 + \frac{zf''}{f'} - \frac{zf'}{f}\right) + \frac{1 + r^2}{1 - r^2} \cdot \frac{zf'}{f}. \tag{10}$$

An obvious adaption of the method to prove Theorem 1 applies also to the case of spiral-like functions. We mention just the following special case ($n = 0$).

Theorem 3: *Let* f *be spiral-like of type* α, *i.e.* f *analytic in* $\mathbb{D}$, $f(0) = 0, f'(0) = 1$, $\operatorname{Re}[e^{i\alpha} zf'(z)/f(z)] > 0$ *in* $\mathbb{D}$. *Then*

$$\operatorname{Re} e^{i\alpha} \frac{zf'(z)}{f(z)} \leqslant \cos\alpha \frac{1 - r}{1 + r} + \frac{2r \operatorname{Re}\left\{e^{i\alpha} \log\left((1 + r)^2 \dfrac{f(z)}{z}\right)\right\}}{(1 - r^2) \log \dfrac{1 + r}{1 - r}}.$$

Finally, we shall show that Theorem B contains an improvement of Theorem A:

Theorem 4: *For $f \in S^*, z \in \mathbb{D}$, we have*

$$\left|\frac{zf'(z)}{f(z)}\right|^2 \leqslant \left(\frac{1-r}{1+r}\right)^2 + \frac{4r(1+r^2)}{(1-r^2)^2 \log\frac{1+r}{1-r}} \log\left[\frac{(1+r)^2}{r}|f(z)|\right]. \quad (11)$$

(11) improves upon (1) whenever

$$\log\left[\frac{(1+r)^2}{r}|f(z)|\right] \leqslant \frac{2}{r}\left(\frac{1-r}{1+r}\right)^2 \log\frac{1+r}{1-r}, \quad (12)$$

in particular, if $0 < r < .29$.

2. Proofs

We use the following well-known representations: $f \in S^*$ iff there exists a probability measure μ on $[0, 2\pi]$ such that

$$\frac{zf'(z)}{f(z)} = \int_0^{2\pi} \frac{1 + ze^{it}}{1 - ze^{it}}\, d\mu(t), \quad (13)$$

$$\log\frac{f(z)}{z} = \int_0^{2\pi} \log\frac{1}{(1 - ze^{it})^2}\, d\mu(t). \quad (14)$$

Next we observe that $(x-1)/\log x$ is strictly increasing for $x > 1$. Hence, for x in the range

$$1 \leqslant x \leqslant A = \left(\frac{1+r}{1-r}\right)^2 \quad (15)$$

we have

$$x^{n+1} \leqslant 1 + \frac{A^{n+1}-1}{\log A}\log x. \quad (16)$$

Now let

$$x = \frac{(1+r)^2}{|1 - ze^{it}|^2} \quad (17)$$

such that (15) is fulfilled for $0 \leqslant t \leqslant 2\pi$. For $f \in S^*$ we find μ as in (13) and an integration of (16) with x as in (17) with respect to μ yields

$$\int_0^{2\pi} \left(\frac{(1+r)^2}{|1-ze^{it}|^2}\right)^{n+1} d\mu(t) \leqslant 1 + \frac{A^{n+1}-1}{\log A} \int_0^{2\pi} \log\frac{(1+r)^2}{|1-ze^{it}|^2}\, d\mu(t) \quad (18)$$

Comparing with (5), (6) we see that the left hand side of (18) is $A^{\frac{n+1}{2}} \operatorname{Re}\left(S_n \frac{zf'}{f}\right)(z)$. A multiplication of (18) with $A^{-\frac{n+1}{2}}$ and using the real part of (14) gives the upper estimate in Theorem 1. For the lower bound we let $x = |1 - z e^{it}|^2 (1-r)^{-2}$ instead of (17). Again x fulfills (15) for all t and integration gives

$$\int_0^{2\pi} \left(\frac{|1-z e^{it}|^2}{(1-r)^2}\right)^{n+1} d\mu(t) \leqslant 1 - \frac{A^{n+1}-1}{\log A} \int_0^{2\pi} \log \frac{(1-r)^2}{|1-z e^{it}|^2}\, d\mu(t). \quad (19)$$

Using the Cauchy—Schwarz inequality we get

$$1 = \int_0^{2\pi} d\mu \leqslant \int_0^{2\pi} \left(\frac{(1-r)^2}{|1-z e^{it}|^2}\right)^{n+1} d\mu(t) \cdot \int_0^{2\pi} \left(\frac{|1-z e^{it}|^2}{(1-r)^2}\right)^{n+1} d\mu(t). \quad (20)$$

If we insert this into (19) and make the same identifications as before the lower bound turns out.

Concerning the sharpness of these estimates it suffices to mention that equality in (16) holds only for $x = 1$ and $x = A$. Hence we have equality in (18) iff μ is concentrated in two points whose difference is π and if z is choosen accordingly. This leads immediately to the functions (9). For the lower bound, the use of the Cauchy—Schwarz inequality reduces the cases of equality to just 1-point measures and again the result follows. This completes the proof of Theorem 1.

We now turn to Theorem 2. In view of Theorem 1 it suffices to show that for $z \in \mathbb{D}$ the inequality

$$U_n - \operatorname{Re}\left(S_n \frac{zf'}{f}\right)(z) \geqslant \alpha_n \left(U_{n+1} - \operatorname{Re}\left(S_{n+1} \frac{zf'}{f}\right)(z)\right) \quad (21)$$

holds where

$$\alpha_n = \frac{A^{n+1} \log A^{n+1} - A^{n+1} + 1}{A^{n+2} \log A^{n+2} - A^{n+2} + 1}, \quad n = 0, 1, \dots .$$

We prove that the functions

$$\frac{x^{n+1} - 1 - \alpha_n (x^{n+2} - 1)}{\log x}$$

are strictly increasing in $1 \leqslant x \leqslant A$. Once this is done, the proof of (21) and therefore Theorem 2 runs on exactly the same lines as in Theorem 1. By differentiation we deduce that our claim is equivalent to the statement

$$\alpha_n \leqslant \frac{x^{n+1}\log x^{n+1} - x^{n+1} + 1}{x^{n+2}\log x^{n+2} - x^{n+2} + 1} = \lambda(x)$$

in $1 \leqslant x \leqslant A$. But $\lambda(A) = \alpha_n$ so that we only need to show that λ strictly decreases for $x > 1$. The proof of this fact is straightforward; we therefore omit the details.

The idea of the proof of Theorem 3 is the same as the one used for Theorem 1 ($n = 0$) and needs no further explanation.

Proof of Theorem 4. Since $g = zf'/f$ satisfies $g(0) = 1$, $\operatorname{Re} g > 0$ in $\mathbb{D}$ we have the well-known inequality

$$\left|\frac{zf'}{f} - \frac{1+r^2}{1-r^2}\right| \leqslant \frac{2r}{1-r^2}.$$

Hence

$$\left|\frac{zf'(z)}{f(z)}\right|^2 \leqslant 2\frac{1+r^2}{1-r^2}\operatorname{Re}\frac{zf'(z)}{f(z)} - 1$$

and (11) follows immediately from Theorem B. A lengthy calculation shows that indeed (12) ensures that the bound in (11) is better than the one in Theorem A. Because of the distortion theorem in S^* we know

$$\log(1+r)^2 \left|\frac{f(z)}{z}\right| \leqslant 2\log\frac{1+r}{1-r}$$

and we conclude that (12) is fulfilled for all r with $\left(\frac{1-r}{1+r}\right)^2 > r$ or $1 - 3r - r^2 - r^3 > 0$. This holds for $0 < r < .29$.

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St. Ruscheweyh
Mathematisches Institut
Universität Würzburg
D-8700 Würzburg, Federal Republic of Germany

V. Singh
Department of Mathematics
Punjabi University
Patiala 147002, India