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On the Order of Starlikeness of Hypergeometric Functions

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1. INTRODUCTION

For $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, let

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} $$

denote the hypergeometric function. This function is analytic for $z \in \mathbb{D}$ where $\mathbb{D}$ is the unit disc $\{z: |z| < 1\}$. In this paper we shall estimate the order of starlikeness of the functions

$$ u(z) = z\ {}_2F_1(a, b; c; \rho z) \tag{1} $$

for certain combinations of the parameters a, b, c, ρ. An analytic function f in $\mathbb{D}$ is called starlike of order $\gamma < 1$ if and only if $f(0) = 0$, $f'(0) = 1$, and

$$ \operatorname{Re} \frac{zf'(z)}{f(z)} > \gamma, \qquad z \in \mathbb{D}. $$

S_γ^* denotes the set of these functions. Previous results of this type are due to Merkes and Scott [4] and Lewis [3]. Theorem A is a slightly generalized version of Lewis' theorem [6, p. 60].

THEOREM A. *Let $a, b, c \in \mathbb{R}$, $a + \frac{1}{2} \geq |b + \frac{1}{2}|$. Then*

$$ z\ {}_2F_1(1 + a + b, 1 + b + ic; 1 + a + ic; z) \in S^*_{(1-a-b)/2}. $$

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Further, from the theory of prestarlike functions (compare [6, p. 61]), one obtains

THEOREM B. *Let* $a, b, c \in \mathbb{R}$, $a \leqslant b \leqslant c$. *Then*

$$z\,{}_2F_1(a, b; c; z) \in S^*_{1-a/2}.$$

The reason for a more thorough investigation of this problem is that the functions $u(z)$ play a central role in the discussion of convolution properties of classes of starlike (and prestarlike) functions of various orders. This is due to the fact that in terms of convolution (Hadamard product) we have

$$z\,{}_2F_1(a, b; c; z) = \frac{z}{(1-z)^a} * \frac{z}{(1-z)^b} * \overset{(-1)}{\frac{z}{(1-z)^c}},$$

where $\overset{(-1)}{f}$ denotes the solution of

$$f * \overset{(-1)}{f} = \frac{z}{1-z}, \quad \overset{(-1)}{f}(0) = 0.$$

The functions

$$s_\gamma(z) = \frac{z}{(1-z)^{2-2\gamma}}, \qquad \gamma < 1,$$

are distinguished members of S^*_γ. One such theorem is

THEOREM C [6, p. 56]. *Let* $\frac{1}{2} \leqslant \alpha \leqslant \beta < 1$, $f \in S^*_\alpha$, $g \in S^*_\beta$. *Then* $f * g \in S^*_\gamma$ *where* γ *is the order of starlikeness of*

$$z\,{}_2F_1(2-2\alpha, 2-2\beta; 1; z) = s_\alpha * s_\beta.$$

Note that Theorem B implies $\gamma \geqslant \beta$, but this is not sharp. One of the theorems in the present paper improves this estimate for γ although the precise value remains unknown.

To obtain our results we employ two different methods: the first one makes use of a recent paper of Ruscheweyh and Wilken [7] dealing with the range of $1/{}_2F_1(a, b; c; z)$ and gives sharp estimates when applicable. The second method is a refined version of the idea of Merkes and Scott using continued fractions. These theorems are stated and proved in Section 2. In Section 3 applications to certain operators acting on starlike functions are given and in the concluding Section 4 we discuss the confluent case and prove some new convolution properties of the exponential and related functions.

The formulas on hypergeometric functions used in the sequel can be found in [1]. For the theory of continued fractions we refer to Wall [10] and a complete description of the necessary convolution theory is in [6].

2. The Main Theorems

THEOREM 1. *Let* $a, b, \rho \in \mathbb{R}$, $0 < \rho \leqslant 1$, $a > 0$, *and* $-1 \leqslant \rho b \leqslant 1 + \rho a$. *Then*

$$u(z) = z\ {}_2F_1(a, b; a+1; \rho z) \in S_\gamma^*$$

with

$$\gamma = 1 - a + \left[\int_0^1 t^{a-1} \left(\frac{1+\varepsilon\rho}{1+\varepsilon\rho t}\right)^b dt\right]^{-1}, \qquad \varepsilon = \operatorname{sgn} b.$$

The value of γ *is best possible.*

Proof. We start from the relation

$$z\ {}_2F_1'(a, b; c; z) = -a\ {}_2F_1(a, b; c; z) + a\ {}_2F_1(a+1, b; c; z).$$

Because

$${}_2F_1(a+1, b; a+1; z) = \frac{z}{(1-z)^b}$$

we get

$$\frac{z\ {}_2F_1'(a, b; a+1; z)}{{}_2F_1(a, b; a+1; z)} = -a + \frac{a}{(1-z)^b\ {}_2F_1(a, b; a+1; z)}. \tag{2}$$

For $0 < B < C$ we have the integral representation

$${}_2F_1(A, B; C; z) = \frac{\Gamma(C)}{\Gamma(B)\,\Gamma(C-B)} \int_0^1 t^{B-1}(1-t)^{C-B-1} \frac{dt}{(1-tz)^A}.$$

Using this and the relation

$$(1-z)^b\ {}_2F_1(a, b; a+1; z) = (1-z)\ {}_2F_1(a-b+1, 1; a+1; z)$$

one easily verifies the existence of probability measures μ_j on $[0, 1]$ such that

$$(1-\rho z)^b\ {}_2F_1(a, b; a+1; \rho z) = \begin{cases} \displaystyle\int_0^1 \left(\frac{1-\rho z}{1-t\rho z}\right)^b d\mu_1, & a > 0, \\ \displaystyle\int_0^1 \frac{1-\rho z}{1-t\rho z}\, d\mu_2, & 0 \leqslant b \leqslant a+1, \\ \displaystyle\int_0^1 (1-\rho z)(1-t\rho z)^{b-a-1}\, d\mu_3, & a > 0. \end{cases} \tag{3}$$

The following two lemmas are in [7].

LEMMA 1. *For* $0 \leqslant t \leqslant 1$ *let* $g(z, t)$ *be such that*

(i) $g(\cdot, t)$ *and* $1/g(\cdot, t)$ *are typically real and convex univalent in* $\mathbb{D}$,

(ii) $g(0, t) = 1$, $-1 < g'(0, t) < 0$.

Let μ *be a probability measure on* $[0, 1]$ *and let*

$$g(z) = \int_0^1 g(z, t)\, d\mu.$$

Then

$$\frac{1}{g(-r)} \leqslant \operatorname{Re} \frac{1}{g(z)} \leqslant \frac{1}{g(r)}, \qquad |z| \leqslant r < 1.$$

LEMMA 2. *Let* $g(z, t)$ *be one of the functions*

$$\left(\frac{1 - \rho\varepsilon z}{1 - t\rho\varepsilon z}\right)^{\alpha}, \qquad (1 - \rho z)(1 - t\rho z)^{\beta}$$

where $\varepsilon = \operatorname{sgn} \alpha$, $0 < \rho \leqslant 1$, $|\alpha\rho| < 1$, $0 \leqslant \beta\rho \leqslant 1 - \rho$. *Then* $g(z, t)$ *fulfills the assumptions of Lemma* 1.

A combination of (2), (3), and Lemmas 1 and 2 shows that

$$\operatorname{Re} \frac{zu'(z)}{u(z)} = 1 + \operatorname{Re} \frac{\rho z\, {}_2F_1'(a, b; a+1; \rho z)}{{}_2F_1(a, b; a+1; \rho z)}$$

takes its minimum in $|z| \leqslant 1$ at $z = -\operatorname{sgn} b$. The assertion follows now from (2). It is clear that this result is best possible in every case.

COROLLARY 1. *Let* $a, b, \rho \in \mathbb{R}$, $0 < \rho \leqslant 1$, $0 \leqslant a - 1 \leqslant b \leqslant a$. *Then*

$$u(z) = z\, {}_2F_1(1, b; a; \rho z) \in S_\gamma^*$$

with

$$\gamma = 2 - a - (1 + b - a) \frac{\rho}{1 + \rho} + \left[\int_0^1 t^{a-2} \left(\frac{1+\rho}{1+t\rho}\right)^{a-b} dt \right]^{-1}.$$

The value of γ *is best possible.*

Proof. Using a transformation of ${}_2F_1$ we deduce

$$u(z) = \frac{z}{(1 - \rho z)^{1+b-a}}\, {}_2F_1(a - 1, a - b; a; \rho z).$$

If $1+b-a\geqslant 0$, $a\geqslant 1$, $0\leqslant b-a\leqslant a$, the same methods as in the proof of Theorem 1 show that $\mathrm{Re}[zu'(z)/u(z)]$ assumes its minimum at $z=-1$. The result follows immediately.

Our next theorem deals with a more general situation. The estimates, however, are not best possible in general.

THEOREM 2. *Let* $0\leqslant a\leqslant b\leqslant c$, $0<\rho\leqslant 1$. *Then*

$$z\,{}_2F_1(a,b;c;\rho z)\in S_\gamma^*$$

with

$$\gamma=1-\frac{ab\rho(1+c+\rho c-\rho a)}{(c+b)(1+2c-a)+(c-b)(1+a)}.$$

If $c\geqslant a+b$ *then* γ *can be replaced by*

$$\tilde{\gamma}=(c-a-b)\,\frac{\rho}{1+\rho}+1-\frac{(c-b)(c-a)(1+c+\rho a)}{(2c-b)(1+c+\rho a)+\rho b(1+c-a)},$$

which, for some values of the parameters, is $>\gamma$.

For the proof we need the following lemma [10, pp. 46, 283].

LEMMA 3. *Let* $0<g_j<1$, $j\in\mathbb{N}$, *and* f *be represented by the continued fraction*

$$f(z)=\frac{g_1}{1-}\;\frac{(1-g_1)\,g_2z}{1-}\;\frac{(1-g_2)\,g_3z}{1-}\cdots. \tag{4}$$

Then f *is analytic in* $\mathbb{D}$ *and fulfills the subordinations*

$$f\prec\frac{g_1}{1+(1-g_1)\,z},\qquad \frac{1-f}{1-zf}\prec\frac{1-g_1}{1-zg_1}. \tag{5}$$

Proof of Theorem 2. Let $u(z)=z\,{}_2F_1(a,b;c;z)$, $s(z)=zu'(z)/u(z)$. The following relation is easily established:

$$s(z)=1+\frac{az}{1-z}\left[1-\frac{c-b}{c}\,\frac{{}_2F_1(a+1,b;c+1;z)}{{}_2F_1(a,b;c;z)}\right]. \tag{6}$$

The continued fraction expansion

$$\frac{{}_2F_1(a+1,b;c+1;z)}{{}_2F_1(a,b;c;z)}=\frac{1}{1-}\;\frac{(1-g_0)\,g_1z}{1-}\;\frac{(1-g_1)\,g_2z}{1-}\cdots \tag{7}$$

with

$$g_{2j} = \frac{c-b+j}{c+2j}, \qquad g_{2j+1} = \frac{c-a+j}{c+2j+1}, \qquad j = 0, 1, \ldots,$$

is due to Gauss (compare [10, p. 339]). Note that under our assumptions $g_j \in (0, 1)$, $j \in \mathbb{N}$. Hence the function f in (4) is analytic in $\mathbb{D}$ and a combination with (7) yields

$$\frac{c-b}{c} \frac{{}_2F_1(a+1, b; c+1; z)}{{}_2F_1(a, b; c; z)} = \frac{c-b}{c-bzf(z)}.$$

From a little manipulation we then get

$$s(z) = 1 + \frac{abz}{1-z} \frac{1-zf(z)}{c-bzf(z)}$$

and thus

$$\frac{1-f(z)}{1-zf(z)} = \frac{c-bz}{cz-bz} + \frac{ab}{(b-c)} \frac{1}{s(z)-1}. \tag{8}$$

An application of the second subordination in (7) yields

$$\left| \frac{1-f(z)}{1-zf(z)} - \frac{1-g_1}{1-\rho^2 g_1^2} \right| \leqslant \frac{\rho g_1(1-g_1)}{1-\rho^2 g_1^2} \tag{9}$$

for $|z| \leqslant \rho$. Inserting (8) into (9) we obtain restrictions for the range of $s(z)$, $|z| \leqslant r$. A careful examination of this range yields the lower bound for $\operatorname{Re} s(z)$ as described in Theorem 2. We omit the simple technical details. This proves our assertion about γ. The claim concering $\tilde{\gamma}$ follows in the same way as Corollary 1 was derived from Theorem 1.

The ranges of Theorems 1 and 2 overlap. Thus we can get an impression about the quality of the estimates in Theorem 2 when compared with the sharp results of Theorem 1. For example, the precise order of starlikeness of $z\,{}_2F_1\,(1, 1; 2; z)$ is $1/(2 \log 2) = 0.7213...$ while Theorem 2 gives $5/7 = 0.7142....$ Hence the approximation in Theorem 2 appears to be reasonable.

3. Application to Starlike Functions

The basic tool for our applications is the following lemma.

LEMMA 4 [6, p. 119]. *Let $\alpha \leqslant \beta \leqslant 1$. If f is analytic in $\mathbb{D}$ and satisfies*

$$f * \frac{z}{(1-z)^{2-2\alpha}} \in S_\beta^*$$

then $f * g \in S_\beta^*$ for every $g \in S_\alpha^*$.

THEOREM 3. *Let* $0 \leqslant b \leqslant a+1$, $a>0$, *and*

$$\gamma = 1 - a + \left[2^b \int_0^1 t^{a-1}(1+t)^{-b}\, dt \right]^{-1}.$$

Then, for $g \in S^*_{1-b/2}$ *we have*

$$az^{1-a} \int_0^z t^{a-2} g(t)\, dt \in S^*_\gamma. \tag{10}$$

This result is sharp w.r.t. the value of γ.

Proof. Let

$$f(z) = a \int_0^1 t^{a-2} \frac{tz}{1-tz}\, dt.$$

From Theorem 1 we deduce

$$f * \frac{z}{(1-z)^b} = a \int_0^1 t^{a-2} \frac{tz}{(1-tz)^b} = z\, {}_2F_1(a, b; a+1; z) \in S^*_\gamma$$

and γ is the best constant with this property. As a consequence of Theorem B we conclude $\gamma \geqslant 1 - b/2$. Thus Lemma 4 applies to this situation. Note that the function (10) is just $f * g$. Theorem 3 contains two well-known special cases:

(i) $a = 1$, $0 \leqslant b \leqslant 2$, which gives the order of starlikeness of functions convex of order $1 - b/2$. This problem was studied by several authors and finally solved by Wilken and Feng [11].

(ii) $a = b$, which corresponds—after a little manipulation—to the order of starlikeness in the Mocanu classes; see Miller, Mocanu, and Reade [5].

THEOREM 4. *Let* $0 < b \leqslant a+1$, $a > 0$, *and* γ *as in Theorem* 3. *Then*

$$\frac{\Gamma(a+1)}{\Gamma(b)\,\Gamma(a-b+1)} \int_0^1 t^{b-2}(1+t)^{a-b}\, g(tz)\, dt \in S^*_\gamma$$

for every $g \in S^*_{1-a/2}$. *The result is sharp.*

Proof. This time we use the represenation

$$z\,{}_2F_1(a, b; a+1; z) = \frac{\Gamma(a+1)}{\Gamma(b)\,\Gamma(a-b+1)} \int_0^1 t^{b-2}(1+t)^{a+b} \times \frac{tz}{1-tz}\, dt * \frac{z}{(1-z)^a}.$$

The conclusion follows again from Theorem 1 and Lemma 4 since $\gamma \geqslant 1 - a/2$ by Theorem B.

The following two theorems are consequences of Corollary 1 obtained in exactly the same manner as Theorems 3 and 4 were deduced from Theorem 1. We omit the details.

THEOREM 5. *Let* $0 < a - 1 < b < a$ *and*

$$\gamma = \frac{1}{2}(3 - a - b) + \left[2^{a-b} \int_0^1 t^{a-2}(1+t)^{b-a}\, dt\right]^{-1}.$$

Then

$$\frac{\Gamma(a)}{\Gamma(b)\,\Gamma(a-b)} \int_0^1 t^{b-2}(1-t)^{a-b-1}\, g(tz)\, dt \in S_\gamma^*$$

for every $g \in S_{1/2}^*$. *The result is sharp.*

THEOREM 6. *Let* $0 < a - 1 \leqslant b \leqslant a$ *and* γ *as in Theorem* 5. *Then*

$$(a-1) \int_0^1 (1-t)^{a-2}\, \frac{g(tz)}{t}\, dt \in S_\gamma^*$$

for every $g \in S_{1-b/2}^*$. *The result is sharp.*

Finally, we use Theorem 2 in connection with Theorem C.

THEOREM 7. *Let* $\frac{1}{2} \leqslant \alpha \leqslant \beta < 1$, $f \in S_\alpha^*$, $g \in S_\beta^*$. *Then* $f * g \in S_\gamma^*$ *where*

$$\gamma = 1 - \frac{(1-\alpha)(1-\beta)(1+2\beta)}{\alpha + 2\beta - 2\alpha\beta}.$$

For $\alpha + \beta \leqslant \frac{3}{2}$, *a better estimate is*

$$\gamma = \alpha + \beta - \frac{1}{2} - \frac{(\alpha - 1/2)(\beta - 1/2)(2-\beta)}{\alpha + \beta - \alpha\beta}.$$

We just mention that the second case comes from $\tilde{\gamma}$ in Theorem 2 which is better for this range of the parameters.

4. THE CONFLUENT CASE

The confluent hypergeometric function ${}_1F_1$ is defined by the relation

$${}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}$$

for a, $c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, and $z \in \mathbb{D}$. It is related to ${}_2F_1$ through the following limit which exists uniformly on compact subsets of $\mathbb{D}$:

$$ {}_1F_1(a; c; bz) = \lim_{\rho \to 0} {}_2F_1(a, b/\rho; c; \rho z). \tag{11} $$

Since the sets S_γ^* are compact in the corresponding topology we immediately deduce from Theorem 1:

THEOREM 8. *Let* $a > 0$, $0 < b \leqslant 1$, *and*

$$ \gamma = 1 - a + \left[\int_0^1 t^{a-1} e^{b(1-t)}\, dt \right]^{-1}. $$

Then $z\ {}_1F_1(a; a+1; bz) \in S_\gamma^*$. *This result is sharp.*

Proof. Replace b by b/ρ in Theorem 1 and perform the limit as in (11).

We mention two special cases of this result. First let $a = \frac{1}{2}$. Since for the error function erf(z) we have

$$ \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}}\, z\ {}_2F_1\left(\frac{1}{2}; \frac{3}{2}; -z^2\right) $$

we obtain

COROLLARY 2. $(\sqrt{\pi}/2) \operatorname{erf}(z) \in S_\gamma^*$ *with* $\gamma = [\int_0^1 e^{1-t^2}\, dt]^{-1} = 0.4925\ldots$. *The result is sharp.*

Proof. From Theorem 8 we see that $g(z) = \sqrt{z}(\sqrt{\pi}/2) \operatorname{erf}(\sqrt{z}) = z\ {}_2F_1(\frac{1}{2}; \frac{3}{2}; -z) \in S_\delta^*$ with

$$ \delta = \frac{1}{2} + \left[\int_0^1 t^{-1/2} e^{1-t}\, dt \right]^{-1} = \frac{1}{2} + \left[2 \int_0^1 e^{1-t^2}\, dt \right]^{-1}. $$

But then

$$ \operatorname{Re} \frac{z \operatorname{erf}'(z)}{\operatorname{erf}(z)} = \operatorname{Re} \frac{2z^2 g'(z^2)}{g(z^2)} - 1 \geqslant 2\delta - 1 = \gamma, \qquad z \in \mathbb{D}. $$

Remark. Kreyszig and Todd [2] proved that the radius of univalence of erf is 1.574.... Note that Corollary 2 implies that erf is *not* convex univalent in $\mathbb{D}$.

COROLLARY 3. The (convex univalent) function $e^z - 1$ is starlike of order $1/(e-1) = 0.5819\ldots$.

Proof. Apply Theorem 8 with $a = b = 1$.

We conclude this paper with a new convolution property of the exponential function. This result is not directly related to the other theorems but is of similar taste as the theorems of Section 3.

THEOREM 9. *Let* $n \in \mathbb{N}$, $f_n = ze^{z^n/n}$. *Then for every* $g \in S^*_{1/2}$ *we have* $f_n * g \in S^*_0$.

Remark. The functions f_n solve many extremal problems in the class $T \subset S^*_0$ of normalized functions f restricted by

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \qquad z \in \mathbb{D}.$$

It is not unlikely that Theorem 9 extends to the whole class T but our present methods are not strong enough to decide this question. For other unusual convolution properties in T see [6, p. 82].

For the proof we need two lemmas which we state in a suitable form. Let $\mathscr{H}$ denote the class of analytic functions H in $\mathbb{D}$ with $H(0) = 1$ and $\operatorname{Re}[e^{i\alpha}H(z)] > 0$ in $\mathbb{D}$ for a certain $\alpha = \alpha(H)$.

LEMMA 5 (Sheil-Small, Silverman, and Silvia [9]). *Let f be analytic in* $\mathbb{D}$, $f(0) = 0$, $f'(0) = 1$, *such that for every* $t \in \mathbb{R}$ *there exist* $h \in S^*_{1/2}$, $H \in \mathscr{H}$ *with*

$$\frac{zf'(z) + itf(z)}{1 + it} = h(z)\, H(z).$$

Then $f * g \in S^*_0$ *for every* $g \in S^*_{1/2}$.

LEMMA 6 (Ruscheweyh and Schwittek [8]). *Let* $f \in S^*$, $p \in S^*_{1/2}$. *Then there exists* $x \in \mathbb{C}$, $|x| = 1$, *such that*

$$(1 - xz)\frac{f(z)}{p(z)} \in \mathscr{H}.$$

Proof of Theorem 9. Let $n = 1$. In order to apply Lemma 5 we have to show

$$e^z\left(1 + \frac{z}{1 + it}\right) = \frac{h(z)}{z}\, H(z) \tag{12}$$

with $h \in S^*_{1/2}$, $H \in \mathscr{H}$ depending on $t \in \mathbb{R}$. But

$$p(z) = \frac{z}{1 + \dfrac{z}{1 + it}}, \qquad h(z) = \frac{z}{1 - xz} \tag{13}$$

are both in $S^*_{1/2}$ and (12) follows from Lemma 6. For arbitrary $n \in \mathbb{N}$ we get from (12), (13)

$$e^{z^n/n}\left(1+\frac{z^n}{1+it}\right)=(1-xz^n)^{-1/n}\,H_n(z) \tag{14}$$

with

$$H_n(z)=H(z^n)^{1/n}\left(1+\frac{z^n}{1+it}\right)^{(n-1)/n}.$$

We conclude $H_n \in \mathscr{H}$ and because

$$\frac{z}{(1-xz^n)^{1/n}} \in S^*_{1/2}$$

another application of Lemma 5 completes the proof.

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