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Universally prestarlike functions as convolution multipliers

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Abstract Universally prestarlike functions (of order $\alpha \leq 1$) in the slit domain $\Lambda := \mathbb{C}\backslash[1, \infty]$ have recently been introduced in Ruscheweyh et al. (Israel J Math, to appear). This notation generalizes the corresponding one for functions in the unit disk $\mathbb{D}$ (and other circular domains in $\mathbb{C}$). In this paper we study the behaviour of universally prestarlike functions under the Hadamard product. In particular it is shown that these function classes (with α fixed), are closed under convolution, and that their members, as Hadamard multipliers, also preserve the prestarlikeness (of the same order) of functions in arbitrary circular domains containing the origin.

Keywords Universally prestarlike functions · Universally convex functions · Convolution invariance · Bernardi-Libera transform

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1 Introduction

1.1 Prestarlike functions

Let $\mathcal{H}(\Omega)$ denote the set of analytic functions in a domain Ω. If Ω contains the origin, $\mathcal{H}_0(\Omega)$ stands for the set of functions $f \in \mathcal{H}(\Omega)$ with $f(0) = 1$. We also use the notation $\mathcal{H}_1(\Omega) := \{zf \ : \ f \in \mathcal{H}_0(\Omega)\}$. In the special case that Ω is the unit disk $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$ we use the abbreviations $\mathcal{H}, \mathcal{H}_0, \mathcal{H}_1$, respectively.

A function $f \in \mathcal{H}_1$ is called *starlike of order* α (with $\alpha < 1$) if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geq \alpha, \quad z \in \mathbb{D},$$

and the set of such functions is denoted by $\mathcal{S}_\alpha$. Finally, a function $f \in \mathcal{H}_1$ is called *prestarlike of order* α (or $f \in \mathcal{R}(\alpha)$) if

$$\frac{z}{(1-z)^{2-2\alpha}} * f(z) \ \in \ \mathcal{S}_\alpha, \tag{1.1}$$

where '$*$' stands for the Hadamard product of two formal power series:

$$g(z) = \sum_{k=0}^{\infty} g_k z^k, \ h(z) = \sum_{k=0}^{\infty} h_k z^k \ \Rightarrow (g * h)(z) := \sum_{k=0}^{\infty} g_k h_k z^k.$$

For certain reasons one also introduces the set $\mathcal{R}_1$ to consist of the functions $f \in \mathcal{H}_1$ with

$$\operatorname{Re} \frac{f(z)}{z} \geq \frac{1}{2}, \quad z \in \mathbb{D}.$$

Prestarlike functions have a number of interesting geometric properties. For instance, the set $\mathcal{C}$ of univalent functions in $\mathcal{H}_1$, mapping $\mathbb{D}$ onto convex domains, equals $\mathcal{R}_0$, and obviously we also have $\mathcal{R}_{1/2} = \mathcal{S}_{1/2}$. We refer to Ruscheweyh [2] and Sheil-Small [5] for a description of the essentials of the theory of prestarlike functions. Two crucial properties are given in the following lemmas.

Lemma 1.1 *For* $\alpha < \beta \leq 1$ *we have* $\mathcal{R}_\alpha \subset \mathcal{R}_\beta$.

Lemma 1.2 *For* $\alpha \leq 1$ *the set* $\mathcal{R}_\alpha$ *is closed under convolution, i.e.*

$$f, g \in \mathcal{R}_\alpha \quad \Rightarrow \quad f * g \ \in \ \mathcal{R}_\alpha.$$

To define prestarlike functions intrinsically we use the operators

$$\left(D^\beta f\right)(z) := \frac{z}{(1-z)^\beta} * f, \quad \beta \geq 0.$$

Then one can see that a function $f \in \mathcal{H}_1$ is prestarlike of order $\alpha \leq 1$ if and only if

$$z \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{R}_1.$$

1.2 Universally prestarlike functions

In [3] the notion of prestarlike functions has been extended from the unit disk to other disks and half-planes containing the origin. Let Ω be such a disk or half-plane. Then there are two unique parameters, $\gamma \in \mathbb{C}\backslash\{0\}$ and $\rho \in [0, 1]$, such that

$$\Omega = \left\{ w_{\gamma,\rho}(z) \ : \ z \in \mathbb{D} \right\} =: \Omega_{\gamma,\rho},$$

where

$$w_{\gamma,\rho}(z) := \frac{\gamma z}{1 - \rho z}.$$

Note that $1 \notin \Omega_{\gamma,\rho}$ if and only if $|\gamma + \rho| \leq 1$.

Definition 1.1 Let $\alpha \leq 1$ and $\Omega = \Omega_{\gamma,\rho}$ for some admissible pair (γ, ρ). A function $f \in \mathcal{H}_1(\Omega)$ is called *prestarlike of order* α *in* Ω if

$$f_{\gamma,\rho}(z) := \frac{1}{\gamma} f(w_{\gamma,\rho}(z)) \in \mathcal{R}_\alpha.$$

The set of these functions f is denoted by $\mathcal{R}_\alpha(\Omega)$.

Let Λ be the slit domain $\mathbb{C}\backslash[1, \infty)$ (the slit being along the positive real axis). In this paper we are mainly concerned with the following objects (see [3, Definition. 1.2]).

Definition 1.2 Let $\alpha < 1$. A function $f \in \mathcal{H}_1(\Lambda)$ is called *universally prestarlike of order* α if and only if f is prestarlike of order α in all sets $\Omega_{\gamma,\rho}$ for which $1 \notin \Omega_{\gamma,\rho}$. The set of these functions is denoted by $\mathcal{R}_\alpha^{\mathrm{u}}$.

Let $\mathcal{M}$ denote the set of probability measures on the interval [0, 1], and set

$$\mathcal{T} := \left\{ \int_0^1 \frac{d\mu(t)}{1 - tz} \ : \ \mu \in \mathcal{M} \right\}.$$

Clearly $\mathcal{T} \in \mathcal{H}_0(\Lambda)$ and $z \cdot \mathcal{T} \subset \mathcal{R}_1$. For reasons that will soon become clear we define $\mathcal{R}_1^{\mathrm{u}} := z \cdot \mathcal{T}$. Note that $\mathcal{T}$ is the set of generating functions of Hausdorff moments; it is very likely that the theory of universally prestarlike functions and, in particular, their behaviour under the Hadamard product will have an impact on the study of Hausdorff Moment Problems in other contexts as well.

The main result in [3] was

Theorem 1.1 *Let* $\alpha \leq 1$ *and* $f \in \mathcal{H}_1(\Lambda)$. *Then* $f \in \mathcal{R}_\alpha^{\mathrm{u}}$ *if and only if*

$$z \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{R}_1^{\mathrm{u}}.$$

In the sequel frequent use will be made of the following abbreviations.

(i) For $\alpha \leq 1$ set

$$e_\alpha(z) := \frac{1}{(1 - z)^{2-2\alpha}}.$$

(ii) Let $n \geq 0$. If a function $F(z) := \sum_{k=n}^{\infty} a_k z^k$ is analytic at $z = 0$ with $a_k \neq 0$ for $k \geq n$, then we set (formally) $F^{[-1]}(z) := \sum_{k=n}^{\infty} \frac{z^k}{a_k}$ so that (formally)

$$F(z) * F^{[-1]}(z) = \frac{z^n}{1-z}.$$

From Theorem 1.1 we obtain an explicit representation for the members in $\mathcal{R}_\alpha^{\mathrm{u}}$.

Corollary 1.1 [3] *A function $f \in \mathcal{H}_1(\Lambda)$ is universally prestarlike of oder $\alpha < 1$ if and only if there exists $\mu \in \mathcal{M}$ such that*

$$\frac{f(z)}{z} = e_\alpha^{[-1]}(z) * \exp\left(\int_0^1 \log e_\alpha(tz)\, d\mu(t)\right).$$

For some general information concerning universally prestarlike functions, in particular about universally *convex* functions ($f \in \mathcal{R}_0^{\mathrm{u}}$) and universally *starlike* functions ($f \in \mathcal{R}_{1/2}^{\mathrm{u}}$), see [3].

1.3 The main results

The principal theorem of this paper is

Theorem 1.2 *Let Ω be a circular domain containing the origin, $\alpha \leq 1$, and let $f \in \mathcal{R}_\alpha(\Omega)$, $F \in \mathcal{R}_\alpha^{\mathrm{u}}$. Then $f * F \in \mathcal{R}_\alpha(\Omega)$.*

And this has the following immediate consequence.

Corollary 1.2 *Let $\alpha \leq 1$. Then the set $\mathcal{R}_\alpha^{\mathrm{u}}$ is closed under convolution, i.e.*

$$f, g \in \mathcal{R}_\alpha^{\mathrm{u}} \quad \Rightarrow \quad f * g \in \mathcal{R}_\alpha^{\mathrm{u}}.$$

In view of these results and Lemma 1.2 one may be inclined to believe that $\mathcal{R}_\alpha(\Omega)$ is closed under convolution for any circular disk Ω containing the origin (which would obviously contain both, Theorem 1.2 and Corollary 1.2). This is not true. Actually, the unit disk $\mathbb{D}$ is the only circular domain with this property (note that Λ is not circular). We will come back to this question in a forthcoming paper.

1.4 Two applications to convex univalent functions

From the large set of possible applications of Theorem 1.2 we mention only two typical ones. The first one is a generalization of a well-known result concerning the Bernardi-Libera integral transform of convex univalent functions.

Theorem 1.3 (Bernardi-Libera transform) *Let Ω be a circular domain containing the origin, and $g \in \mathcal{R}_0(\Omega)$ (i.e. g univalently convex in Ω). Then, for $b \geq 0$, the functions*

$$F_b(z) := \frac{b+1}{z^b} \int_0^z t^{b-1} g(t)\, dt, \quad z \in \Omega,$$

have the same property. In particular, if g is universally convex (i.e. $g \in \mathcal{R}_0^{\mathrm{u}}$), then F_b is universally convex as well.

This has been known for many years for $\Omega = \mathbb{D}$, and in that case it holds for $\operatorname{Re} b \geq 0$ instead of $b \geq 0$, see for instance [2, Theorem. 2.21]. This, however, cannot be true in our more general situation. A result equivalent to Theorem 1.3, but only for the range $b \in [0, 1]$, has previously been obtained by Selinger [7]. However, the context discussed in [7] is completely different.

Theorem 1.4 *Let Ω be a circular domain containing the origin. If g is convex univalent in Ω then, for $0 \leq x < y \leq 1$, the function*

$$F_{x,y}(z) := \int_0^z \frac{g(xt) - g(yt)}{xt - yt}\, dt, \quad z \in \Omega,$$

has the same property. In particular, if g is universally convex (i.e. $g \in \mathcal{R}_0^{\mathrm{u}}$), then $F_{x,y}$ is universally convex as well.

Generalizations of Theorems 1.3 and 1.4 to other choices of the multipliers and other values of α are possible, but have not yet been studied in detail. Compare, however [3, Sect. 1.6].

1.5 A general convolution theorem

The results in this paper are based on the following generalization of a convolution theorem of Sheil-Small [6, Theorem 5.10].

We call a function $H(z, \zeta)$ a *kernel* if, for each $0 < r < 1$, it is analytic in $|z| < r < 1$, $|\zeta| < 1 + \delta(r)$ where $\delta(r) > 0$. If H is a kernel then

$$\Phi(f)(z) := H(z, \zeta) *_\zeta f(\zeta)|_{\zeta=1}$$

is a continuous linear operator on $\mathcal{H}(\mathbb{D})$ (equipped with the topology of compact convergence). On the other hand, if Φ is such an operator, then

$$H(z, \zeta) := \Phi\left(\frac{1}{1 - \zeta z}\right)$$

is a kernel, compare [6, Sect. 5.1].

From now on, to simplify the notation, we shall work with the classes $\mathcal{R}'_\alpha$ and $\mathcal{S}'_\alpha$ instead of $\mathcal{R}_\alpha$ and $\mathcal{S}_\alpha$, where $f \in \mathcal{R}'_\alpha$ if and only if there exists a constant $c \neq 0$ such that $cf \in \mathcal{R}_\alpha$, and similarly for $\mathcal{S}'_\alpha$.

Theorem 1.5 *Let $\alpha < 1$. For the kernel $H(z, \zeta)$ with its associated operator Φ assume the following two properties.*

(i) *For $|\zeta| = 1$:*

$$P_\zeta(z) := \frac{1}{\zeta} H(z, \zeta) \in \mathcal{R}'_\alpha, \tag{1.2}$$

(ii) *For $z \in \mathbb{D}$:*

$$Q_z(\zeta) := e_\alpha(z) * \frac{1}{z} H(z, \zeta) \in \mathcal{S}'_\alpha. \tag{1.3}$$

Then

$$\Phi(\mathcal{R}'_\alpha) \subset \mathcal{R}'_\alpha.$$

The proof of this theorem will be given in the concluding Sect. 4.

2 Proof of Theorem 1.2

2.1 Construction of the operator Φ

Let $\Omega = \Omega_{\gamma,\rho}$ where $\gamma \in \mathbb{C}\backslash\{0\}$ and $0 \le \rho \le 1$. For f, F as in the assumption we have (see Definition 1.1)

$$f(w) = \gamma f_{\gamma,\rho}\left(\frac{w}{\gamma + \rho w}\right), \quad w \in \Omega,$$

where $f_{\gamma,\rho} \in \mathcal{R}_\alpha$. In the sequel we shall use the notation $*_x$, $*_y$, etc. if the convolution is to be taken with respect to the variable x, y, etc. We then have

$$\begin{aligned}\frac{1}{\gamma}F(w) * f(w) &= F(w) *_w f_{\gamma,\rho}\left(\frac{w}{\gamma+\rho w}\right)\\ &= F(w) *_w f_{\gamma,\rho}(\zeta) *_\zeta \left.\frac{1}{1-\frac{w\zeta}{\gamma+\rho w}}\right|_{\zeta=1}\\ &= F(w) *_w f_{\gamma,\rho}(\zeta) *_\zeta \left.\left(1 + \frac{\zeta}{\zeta-\rho}\,\frac{\frac{\zeta-\rho}{\gamma}w}{1-\frac{\zeta-\rho}{\gamma}w}\right)\right|_{\zeta=1}\end{aligned}$$

so that

$$\frac{1}{\gamma}F(w) * f(w) = \frac{\zeta}{\zeta-\rho}\,F\left(\frac{\zeta-\rho}{\gamma}w\right) *_\zeta f_{\gamma,\rho}(\zeta)\bigg|_{\zeta=1}.$$

Writing $w = w(z) := w_{\gamma,\rho}(z)$ we arrive at

$$\begin{aligned}\frac{1}{\gamma}\left(F(w) *_w f(w)\right)|_{w=w(z)} &= \frac{\zeta}{\zeta-\rho}F\left(\frac{(\zeta-\rho)z}{1-\rho z}\right) *_\zeta f_{\gamma,\rho}(\zeta)\bigg|_{\zeta=1}\\ &= \Phi(f_{\gamma,\rho})(z),\end{aligned}$$

where Φ is the operator belonging to the kernel

$$H(z,\zeta) := \frac{\zeta}{\zeta-\rho}F\left(\frac{(\zeta-\rho)z}{1-\rho z}\right).$$

Note that for $z \in \mathbb{D}$ and $|\zeta| \le 1$ we have $(\zeta-\rho)z/(1-\rho z) \in \Lambda$.

2.2 Verification of the assumptions

We now show that this function H satisfies the assumptions of Theorem 1.5.

2.2.1 We have

$$P_\zeta(z) = \frac{1}{\zeta-\rho}F\left(\frac{(\zeta-\rho)z}{1-\rho z}\right) \in \mathcal{R}_\alpha \subset \mathcal{R}'_\alpha,$$

since, for $|\zeta| = 1$, the disk $\Omega_{\zeta-\rho,\rho}$ is contained in Λ, so that the definition for $F \in \mathcal{R}^{\mathrm{u}}_\alpha$ applies.

2.2.2 The proof of (1.3) is less immediate. We make use of the representation for F given in Corollary 1.1, namely

$$F(w) = (w\, e_\alpha(w))^{[-1]} *_w w e^{G(w)},$$

where

$$G(w) = \int_0^1 \log e_\alpha(tw)\, d\mu(t).$$

We then have

$$\begin{aligned} Q_z(\zeta) &= e_\alpha(z) *_z \frac{\zeta}{z(\zeta-\rho)} F\left(\frac{(\zeta-\rho)z}{1-\rho z}\right) \\ &= e_\alpha(z) *_z \left[F(w) *_w \frac{\zeta}{\zeta-\rho} \frac{w\frac{\zeta-\rho}{1-\rho z}}{1-w\frac{(\zeta-\rho)z}{1-\rho z}} \right]\Bigg|_{w=1} \\ &= e_\alpha(z) *_z \left[(w\, e_\alpha(w))^{[-1]} *_w w e^{G(w)} *_w \frac{w\frac{\zeta}{1-\rho z}}{1-w\frac{(\zeta-\rho)z}{1-\rho z}} \right]\Bigg|_{w=1} \\ &= \zeta\, w\, q(\zeta, z, w) *_w w e^{G(w)}\Big|_{w=1} \end{aligned}$$

where

$$\begin{aligned} q(\zeta, z, w) &= e_\alpha(z) *_z \frac{1}{1-(\rho+(\zeta-\rho)w)z} *_w e_\alpha^{[-1]}(w) \\ &= e_\alpha((\rho+(\zeta-\rho)w)z) *_w e_\alpha^{[-1]}(w) \\ &= \left(e_\alpha(\rho z)\, e_\alpha\left(\frac{(\zeta-\rho)z}{1-\rho z} w\right)\right) *_w e_\alpha^{[-1]}(w) \\ &= e_\alpha(\rho z) \frac{1}{1-w\frac{(\zeta-\rho)z}{1-\rho z}}, \end{aligned}$$

so that

$$Q_z(\zeta) = \zeta\, e_\alpha(\rho z)\, \exp\left(G\left(\frac{(\zeta-\rho)z}{1-\rho z}\right)\right).$$

Now $Q_z(0) = 0$ and $Q_z'(0) \neq 0$ hold for $z \in \mathbb{D}$ and, furthermore, we have

$$\begin{aligned} \frac{\zeta\, Q_z'(\zeta)}{Q_z(\zeta)} &= 1 + \frac{\zeta z}{1-\rho z}\, G'\left(\frac{(\zeta-\rho)z}{1-\rho z}\right) \\ &= 1 + (2-2\alpha) \int_0^1 \frac{\zeta t z}{1-\rho z - t(\zeta-\rho)z} d\mu(t). \end{aligned}$$

Therefore (1.3) is established once the relation

$$\operatorname{Re} \frac{\zeta z t}{1-\rho z - t(\zeta-\rho)z} \geq -\frac{1}{2}$$

has been verified for $\zeta, z \in \mathbb{D}$, and $t, \rho \in [0, 1]$. This, however, is easily done.

2.3 Application of Theorem 1.5

Theorem 1.5 now implies that

$$\frac{1}{\gamma}\,(F(w) *_w f(w))\,|_{w=w(z)} = \Phi(f_{\gamma,\rho})(z) \in \mathcal{R}'_\alpha,$$

so that it only remains to show that

$$\Phi(f_{\gamma,\rho})'(0) = 1.$$

But this follows readily from the relation

$$\Phi(f_{\gamma,\rho})'(0) = \left(\zeta F'(0) \ *_\zeta \ f_{\gamma,\rho}(\zeta)\right)\big|_{\zeta=1}$$

and the normalisation valid for $f_{\gamma,\rho}$ and F.

3 Proof of Theorems 1.3 and 1.4

Both these theorems are special cases of Theorem 1.5: they are dealing with the convolutions of $g \in \mathcal{R}_0(\Omega)$ with

$$\phi_b(z) = (b+1)z \int_0^1 \frac{t^b}{1-tz}\,dt$$

in Theorem 1.3 and

$$\phi_{x,y}(z) = \int_0^z \frac{dt}{(1-xt)(1-yt)}$$

in Theorem 1.4. A simple calculation shows that

$$\phi_b(z) = z\,{}_2F_1(1, b+1, b+2, z) = \frac{b+1}{b}({}_2F_1(1, b, b+1, z) - 1)$$

where ${}_2F_1$ stands for the hypergeometric function (see [1, Sect. 15]), and [3, Theorem 1.9] implies $\phi_b \in \mathcal{R}_0^{\mathrm{u}}$. That $\phi_{x,y}$ belongs to $\mathcal{R}_0^{\mathrm{u}}$ follows from the representation given in Corollary 1.1, with $\alpha = 0$ and μ a discrete measure with equal jumps at $t = x$ and $t = y$. □

4 Proof of Theorem 1.5

4.1 A Lemma

The following lemma is a special case of a general result due to Sheil-Small [6, Theorem 5.10]. The notation has been adjusted to our present one.

Lemma 4.1 *Let $\alpha < 1$. For the kernel $F(z,\zeta)$ with its associated operator Ψ assume the following two properties.*

(i) *For each $|\zeta| = 1$:*

$$A_\zeta(z) := z\,\Psi\,(e_\alpha(\zeta z)) \in \mathcal{S}'_\alpha.$$

(ii) *For each* $z \in \mathbb{D}$:

$$B_z(\zeta) := \zeta\ F(z, \zeta) \in \mathcal{R}'_\alpha.$$

Then, for any $f \in \mathcal{S}'_\alpha$, *we have* $z\ \Psi(f(z)/z) \in \mathcal{S}'_\alpha$.

4.2 About $H(z, \zeta)$

We write

$$H(z, \zeta) = \sum_{k=0}^{\infty} h_k(\zeta) z^k,$$

where $h_k(\zeta)$ are analytic in $\mathbb{D}$. From (1.2) we find

$$h_0(\zeta) \equiv 0, \quad h_1(\zeta) = \zeta \eta(\zeta),$$

with $\eta(\zeta) \neq 0$ in $\mathbb{D}$. And (1.3), written as

$$Q_z(\zeta) = e_\alpha(z) *_z \frac{H(z, \zeta)}{z} = \sum_{k=0}^{\infty} \frac{(2-2\alpha)_k}{k!} h_{k+1}(\zeta) z^k \in \mathcal{S}'_\alpha, \quad z \in \mathbb{D},$$

implies

$$h_k(0) = 0, \quad k \in \mathbb{N}.$$

4.3 A Proposition

Proposition 4.1 *Let* Φ *be the operator defined in Theorem 1.5. Then the operator* Σ *defined by*

$$\Sigma(f)(z) := z\, e_\alpha(z) *_z \Phi\left((z\, e_\alpha(z))^{[-1]} *_z f(z)\right),$$

with its associated kernel $S(z, \zeta)$ *has the following two properties*:

(i) *For* $|\zeta| = 1$:

$$M_\zeta(z) := \Sigma\,(z\ e_\alpha(\zeta\ z)) \in \mathcal{S}'_\alpha.$$

(ii) *For* $z \in \mathbb{D}$:

$$N_z(\zeta) := \frac{1}{z} S(z, \zeta) \in \mathcal{R}'_\alpha. \tag{4.1}$$

Proof The relations (1.1) and (1.2) imply for $|\zeta| = 1$:

$$\begin{aligned} M_\zeta(z) &= z\ e_\alpha(z) *_z \left[H(z, x) *_x (x\ e_\alpha(x))^{[-1]} *_x (x\ e_\alpha(x\,\zeta)) \right]\Big|_{x=1} \\ &= z\ e_\alpha(z) *_z H(z, x) *_x \left.\frac{x}{1 - x\zeta}\right|_{x=1} \\ &= z\ e_\alpha(z) *_z \frac{1}{\zeta} H(z, \zeta) \in \mathcal{S}'_\alpha. \end{aligned}$$

And for $z \in \mathbb{D}$ the condition (1.3) yields

$$N_z(\zeta) = e_\alpha(z)\ *_z \frac{1}{z} H(z, \zeta) *_\zeta (\zeta\ e_\alpha(\zeta))^{[-1]} \in \mathcal{R}'_\alpha.$$

□

4.4 Construction of Ψ

Now let

$$S(z,\zeta)=\sum_{k=0}^{\infty}\sigma_k(z)\zeta^k.$$

Since from above we have for $|\zeta|=1$

$$\begin{aligned}M_\zeta(z) &= (S(z,x)*_x \zeta\ e_\alpha(x\zeta))|_{x=1}\\ &=\sum_{k=0}^{\infty}\sigma_k(z)\left(x^k *_x x\ e_\alpha(x\zeta)\right)\Big|_{x=1}\\ &=\sum_{k=1}^{\infty}\sigma_k(z)\frac{(2-2\alpha)_{k-1}}{(k-1)!}\zeta^{k-1}\in\mathcal{S}'_\alpha\end{aligned}$$

it is clear that $\sigma_k(0)=0$ for $k\geq 1$, and therefore, for arbitrary $f(z)=\sum_{k=0}^{\infty}f_k z^k\in\mathcal{H}(\mathbb{D})$,

$$\Sigma(zf(z))(0)=\sum_{k=1}^{\infty}\sigma_k(0)f_{k-1}=0.$$

Hence

$$\Psi(f):=\frac{1}{z}\Sigma(z\,f)$$

is a continuous linear operator on $\mathcal{H}(\mathbb{D})$. For the corresponding kernel F we get

$$\begin{aligned}F(z,\zeta) &=\Psi\left(\frac{1}{1-\zeta z}\right)=\frac{1}{\zeta z}\Sigma\left(\frac{\zeta z}{1-\zeta z}\right)\\ &=\frac{1}{\zeta z}\Sigma\left(\frac{1}{1-\zeta z}-1\right)=\frac{1}{\zeta z}S(z,\zeta).\end{aligned}$$

In the last step we made use of the fact that $\Sigma(1)=S(z,0)=0$, as can be seen from (4.1).

4.5 Conclusion

The results of Sects. 4.3 and 4.4 now show that

(i) for each $|\zeta|=1$ the function $A_\zeta(z):=z\ \Psi(e_\alpha(\zeta\ z))$ belongs to $\mathcal{S}'_\alpha$,
(ii) for each $z\in\mathbb{D}$ the function $B_z(\zeta):=\zeta\ F(z,\zeta)$ belongs to $\mathcal{R}'_\alpha$.

Applying Lemma 4.1 to this situation we find that $z\Psi(f(z)/z)=\Sigma(f)\in\mathcal{S}'_\alpha$ for any $f\in\mathcal{S}'_\alpha$, and therefore $\Phi(\mathcal{R}'_\alpha)\subset\mathcal{R}'_\alpha$, the assertion.

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