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On a conjecture of S.P. Robinson ☆

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Abstract

In his thesis, S.P. Robinson made a conjecture concerning the polynomials

$$P_n^{\beta}(z) = 1 + \sum_{k=1}^{n} \prod_{j=0}^{k-1} \frac{n-j}{\beta+n+j} z^k, \quad n \in \mathbb{N},\ \beta \geqslant 1,$$

namely that zP_n^{β} is prestarlike of order $(3-\beta)/2$. These polynomials are closely related to the de la Vallée Poussin means (the case $\beta = 1$). We prove this conjecture in a more general form and show that these functions constitute a sort of two-dimensional subordination chain. These results are then compared with similar ones for Cesáro means of various orders.

Keywords: Convex univalent; Prestarlike; Subordination; de la Vallée Poussin means; Cesáro means

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1. Introduction

In their celebrated paper [2] G. Pólya and I.J. Schoenberg studied, among other things, the *de la Vallée Poussin means*

$$V_n(f,z) := (V_n * f)(z),$$

where

$$V_n(z) := \frac{1}{\binom{2n}{n}} \sum_{k=1}^{n} \binom{2n}{n+k} z^k, \quad n \in \mathbb{N}, \tag{1.1}$$

and f is analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. Here $*$ stands for the Hadamard product of two analytic functions in $\mathbb{D}$. For $F(z) = \sum_{k=0}^{\infty} a_k z^k$ and $G(z) = \sum_{k=0}^{\infty} b_k z^k$ analytic in $\mathbb{D}$ then $(f * g)(z) := \sum_{k=0}^{\infty} a_k b_k z^k$. Let $\mathcal{H}(\mathbb{D})$ denote the set of analytic functions in $\mathbb{D}$, $\mathcal{H}_0(\mathbb{D})$ the subset of $\mathcal{H}(\mathbb{D})$ whose members f are normalised by $f(0) = 0$, $f'(0) = 1$, and $\mathcal{C}$ the set of convex univalent functions $f \in \mathcal{H}_0(\mathbb{D})$.

The following property of the de la Vallée Poussin means was established in [2, Theorem 2]:

$$f \in \mathcal{C} \quad \Leftrightarrow \quad \forall n \in \mathbb{N}\colon \quad \frac{n+1}{n} V_n * f \in \mathcal{C}.$$

Furthermore, it was conjectured that $f \in \mathcal{C}$ implies

$$V_1 * f \prec \cdots \prec V_n * f \prec V_{n+1} * f \prec \cdots \prec f, \quad n \geqslant 2, \tag{1.2}$$

where $\prec$ stands for the subordination of functions in $\mathcal{H}(\mathbb{D})$. This, in an even more general form, has been established by Ruscheweyh and Suffridge [8]. There it was observed that the de la Vallée Poussin means can be represented as

$$V_n(z) = \frac{nz}{n+1}\, {}_2F_1(1, 1-n, 2+n, -z), \quad n \in \mathbb{N},$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Thus, the functions

$$V_\lambda(z) = \frac{\lambda z}{\lambda+1}\, {}_2F_1(1, 1-\lambda, 2+\lambda, -z), \quad \lambda > 0, \tag{1.3}$$

constitute a continuous extension of the de la Vallée Poussin means. The main result was:

Theorem 1 [8]. *The functions $\frac{\lambda+1}{\lambda} V_\lambda(z)$ belong to $\mathcal{C}$ for $\lambda > 0$, and form a convex subordination chain, i.e., for each $f \in \mathcal{C}$ one has*

$$V_{\lambda_1} * f \prec V_{\lambda_2} * f \prec f, \quad 0 < \lambda_1 < \lambda_2. \tag{1.4}$$

Note that the system $\{V_\lambda\colon \lambda > 0\}$ forms an approximate identity, i.e.,

$$V_\lambda \prec \frac{z}{1-z}, \quad \lambda > 0, \quad \text{with} \lim_{\lambda \to \infty} V_\lambda = \frac{z}{1-z}.$$

S.P. Robinson, in his thesis [3] (see also T. Sheil-Small [9, p. 301]), introduced the polynomials

$$P_n^\beta(z) = 1 + \sum_{k=1}^{n} \prod_{j=0}^{k-1} \frac{n-j}{\beta+n+j} z^k, \quad n \in \mathbb{N},$$

where $1 \leqslant \beta \in \mathbb{R}$ and made an interesting conjecture concerning their properties. To state this we use the notion of prestarlike functions [4]: a function $f \in \mathcal{H}_0(\mathbb{D})$ is called prestarlike of order $\alpha < 1$ if $f * \frac{z}{(1-z)^{2-2\alpha}}$ is starlike of order α. It is called prestarlike of order 1 if $\operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}$, $z \in \mathbb{D}$. We denote these sets by $\mathcal{R}_\alpha$. Note that $\mathcal{R}_0 = \mathcal{C}$.

Conjecture 2 (Robinson). *For $\beta \geqslant 1$ and $n \in \mathbb{N}$ we have $zP_n^\beta \in \mathcal{R}_{\frac{3-\beta}{2}}$.*

Note that in its original version this conjecture looks slightly different because of different notations used in [3,9]. However, the form given above is completely equivalent to the original.

Robinson verified the cases $\beta = 1, 3$ and showed that $zP_n^\beta \in \mathcal{R}_1$ for all choices of n, β. Actually, we have

$$V_n(z) = \frac{n}{n+1} zP_{n-1}^3(z) = P_n^1(z) - 1, \quad n \in \mathbb{N}, \tag{1.5}$$

so that the case $\beta = 3$ is just the known statement about the de la Vallée Poussin means, see above. In the sequel we prove the above conjecture, again in a more general setup, actually extending Theorem 1 to this new situation. We define

$$P_\lambda^\beta(z) := {}_2F_1(1, -\lambda, \lambda + \beta, -z), \quad \lambda \geqslant -1,\ \beta \geqslant 1, \tag{1.6}$$

and easily verify that this definition agrees with Robinson's for $\lambda = n \in \mathbb{N}$. Our main results are then:

Theorem 3. *For $\beta \geqslant \max\{1, -2\lambda\}$ we have $zP_\lambda^\beta \in \mathcal{R}_{\frac{3-\beta}{2}}$.*

Note that this result improves even the corresponding portion of Theorem 1: since, according to Theorem 3, we have $zP_{\lambda-1}^3 \in \mathcal{R}_0 = \mathcal{C}$ for $\lambda \geqslant -\frac{1}{2}$ and since (1.5) holds for real λ as well, it is clear that V_λ is convex univalent also for $-\frac{1}{2} \leqslant \lambda < 0$ (and not only for $\lambda > 0$). The subordination portion of Theorem 1, however, does not extend to that range of λ.

Theorem 4. *For $0 \leqslant \lambda_1 \leqslant \lambda_2$ and $1 \leqslant \beta_2 \leqslant \beta_1$ we have*

$$P_{\lambda_1}^{\beta_1} \prec P_{\lambda_2}^{\beta_2}.$$

These theorems have the following corollary.

Corollary 5. *Let $\beta \geqslant 1$ and $f \in \mathcal{R}_{\frac{1-\beta}{2}}$. Then*

$$\frac{\lambda + \beta}{\lambda} P_\lambda^\beta * f \in \mathcal{R}_{\frac{1-\beta}{2}}, \quad \lambda \geqslant -\frac{1}{2}, \tag{1.7}$$

$$P_{\lambda_1}^\beta * f \prec P_{\lambda_2}^\beta * f \prec f, \quad 0 \leqslant \lambda_1 \leqslant \lambda_2, \tag{1.8}$$

$$\lim_{\lambda \to \infty} P_\lambda^\beta * f = f. \tag{1.9}$$

Note that the case $\beta = 1$ of Corollary 5 is essentially Theorem 1, so that this provides a substantial generalization of the previous results. The corollary cannot be extended to cases corresponding to $\beta < 1$. This follows from the fact that a necessary condition for that would be $P_\lambda^\beta \prec \frac{1}{1-z}$, which is not true since for those values of β we have $P_\lambda^\beta(1) < \frac{1}{2}$.

It is not clear whether these generalized de la Vallée Poussin means (with $\lambda > 0$, $\beta > 1$) share more of the very strong properties of the classical de la Vallée Poussin means (like general periodic variation diminution). There are some hints in that direction, but nothing concrete is known so far.

Section 2 gives the proofs of Theorems 3, 4, while in Section 3 we discuss briefly certain similarities of our results with some results and conjectures on Cesáro means of varying orders.

2. Proofs

Lemma 6 [4, Theorem 2.12]. *Let $a, b \in \mathbb{R}$ satisfy $2a + 1 \geqslant |2b + 1|$. Then*

$$z\,{}_2F_1(1, 1+b, 1+a, z) \in \mathcal{R}_{\frac{1-a-b}{2}}.$$

Proof of Theorem 3. In view of Lemma 6 one defines $a := \lambda + \beta - 1$ and $b := -\lambda - 1$. Then $\frac{1}{2}(1 - a - b) = \frac{1}{2}(3 - \beta)$, and the lemma's condition on a, b becomes $2\lambda + 2\beta - 1 \geqslant |2\lambda + 1|$ which is the same as $\beta \geqslant \max\{1, -2\lambda\}$. □

For the proof of Theorem 4 we need a few results from convolution theory.

Lemma 7 [4, Theorem 2.1].

(i) *For $\alpha \leqslant 1$ let $f, g \in \mathcal{R}_\alpha$. Then $f * g \in \mathcal{R}_\alpha$.*
(ii) *For $\alpha < \beta \leqslant 1$ we have $\mathcal{R}_\alpha \subset \mathcal{R}_\beta$.*

Lemma 8 [7]. *Let $F, G \in \mathcal{C}$ and $f \prec F$, $g \prec G$. Then $f * g \prec F * G$.*

Lemma 9. *Let f be convex univalent in $\mathbb{D}$, and $zg \in \mathcal{R}_1$. Then $g * f \prec f$.*

Lemma 10 [1]. *Let a_k, $k \geqslant 0$, be a monotonically decreasing, convex sequence of non-negative numbers with $a_0 = 1$. Then $f(z) := z\sum_{k=0}^{\infty} a_k z^k \in \mathcal{R}_1$.*

Proof of Theorem 4. We note that

$$\frac{\lambda}{\lambda + \beta} z P_{\lambda-1}^{\beta+2}(z) = P_\lambda^\beta(z) - 1,$$

holds for $\lambda > 0$, $\beta \geqslant 1$. From this and Theorem 3 it becomes clear that for the same parameters the functions P_λ^β are convex univalent in $\mathbb{D}$.

We first prove the case $\lambda_1 = \lambda_2 =: \lambda$, i.e., $P_\lambda^{\beta_1} \prec P_\lambda^{\beta_2}$. We have

$$P_\lambda^{\beta_1} = P_\lambda^{\beta_2} * {}_2F_1(1, \beta_2 + \lambda, \beta_1 + \lambda, z).$$

In view of Lemma 9 and the first part of this proof it is sufficient to prove $z\,{}_2F_1(1, \beta_2+\lambda, \beta_1+\lambda, z) \in \mathcal{R}_1$. But Lemma 6 shows that this function is in $\mathcal{R}_{\frac{1}{2}(1-2\lambda)}$ which is a subset of $\mathcal{R}_1$ by Lemma 7. It remains to prove $P_{\lambda_1}^{\beta} \prec P_{\lambda_2}^{\beta}$ for $\lambda \geqslant 1$ and $0 < \lambda_1 < \lambda_2$.

We write

$$\begin{aligned} P_\lambda^\beta(z) &= {}_2F_1(1, -\lambda, \lambda+\beta, -z] \\ &= {}_2F_1(1, -\lambda, \lambda+1, -z) * {}_2F_1(1, \lambda+1, \lambda+\beta, z) \\ &= \big(V_\lambda(z)+1\big) * {}_2F_1(1, \lambda+1, \lambda+\beta, z), \end{aligned}$$

so that we are left with the assertion

$$\big(V_{\lambda_1}(z)+1\big) * {}_2F_1(1, \lambda_1+1, \lambda_1+\beta, z) \prec \big(V_{\lambda_2}(z)+1\big) * {}_2F_1(1, \lambda_2+1, \lambda_2+\beta, z).$$

Now $V_{\lambda_1}(z)+1 \prec V_{\lambda_2}(z)+1$ by Theorem 1. Then, in view of Lemma 8, it only remains to prove

$${}_2F_1(1, \lambda_2+1, \lambda_2+\beta, z) \text{ is convex univalent in } \mathbb{D}, \tag{2.1}$$

and

$${}_2F_1(1, \lambda_1+1, \lambda_1+\beta, z) \prec {}_2F_1(1, \lambda_2+1, \lambda_2+\beta, z). \tag{2.2}$$

We have

$${}_2F_1(1, \lambda_2+1, \lambda_2+\beta, z) = 1 + \frac{\lambda_2+1}{\lambda_2+\beta} z\,{}_2F_1(1, \lambda_2+2, \lambda_2+\beta+1, z),$$

and setting $b := \lambda_2+1$, $a := \lambda_2+\beta$ we find $2a+1-|2b+1| = 2\beta_2-2 \geqslant 0$, so that Lemmas 6 and 7 give

$$z\,{}_2F_1(1, \lambda_2+2, \lambda_2+\beta+1, z) \in \mathcal{R}_{-\frac{1}{2}(2\lambda_2+\beta)} \subset \mathcal{R}_0 = \mathcal{C}.$$

This proves (2.1).

Next we observe that

$${}_2F_1(1, \lambda_1+1, \lambda_1+\beta, z) = H(z) * {}_2F_1(1, \lambda_2+1, \lambda_2+\beta, z),$$

where

$$H(z) = \sum_{k=0}^{\infty} \frac{(\lambda_1+1)_k}{(\lambda_2+1)_k} \frac{(\lambda_2+\beta)_k}{(\lambda_1+\beta)_k} z^k.$$

From Lemma 9 we see that (2.2) follows if $zH(z) \in \mathcal{R}_1$, which remains to be established. We set

$$a_k := \frac{(\lambda_1+1)_k}{(\lambda_2+1)_k} \frac{(\lambda_2+\beta)_k}{(\lambda_1+\beta)_k}, \quad k = 0, 1, \ldots,$$

so that $a_0 = 1$ and $a_k > 0$ for all k. Furthermore,

$$\frac{a_{k+1}}{a_k} = 1 - \frac{(\beta-1)(\lambda_2-\lambda_1)}{(\beta+k+\lambda_1)(1+k+\lambda_2)} \leqslant 1,$$

which means the a_k are decreasing, and

$$\begin{aligned}&a_k - 2a_{k+1} + a_{k+2}\\&\quad = a_k \frac{(\beta-1)(\lambda_2-\lambda_1)(2(k+\lambda+1)+\beta(1+\lambda_2-\lambda_1))}{(\beta+k+\lambda_1)(1+k+\lambda_2)(\beta+k+\lambda_1+1)(2+k+\lambda_2)} > 0,\end{aligned}$$

so that this sequence is convex. □

This proof was much easier than the one of Theorem 1 in [8] because we could reduce a major portion of the present situation to the one solved there. It might be of interest that the functions $w(z) = P_\lambda^\beta(z)$ are the solutions of the ODE

$$zw'(z) + \frac{\beta - 1 + \lambda(1-z)}{1+z} w(z) = \frac{\beta+\lambda-1}{1+z}, \qquad w(0) = 1. \tag{2.3}$$

This relation, for $\beta = 1$, was instrumental in the proof of Theorem 1 in [8].

As far as Corollary 5 is concerned: the first two assertions follow from Theorems 3, 4 and Lemmas 7, 8. The last one is a consequence of Robinson's observation that the P_n^β, for β fixed, form an approximate identity, which obviously extends to the continuous case with $\lambda > 0$.

3. Cesáro means

The Cesáro means of order $\beta > 0$, in their function theoretic version, are convolutions of analytic functions f in $\mathbb{D}$ with

$$s_n^\beta(z) = \sum_{k=0}^{n} \frac{\binom{n-k+\beta}{n-k}}{\binom{n+\beta}{n}} k^k, \quad n \in \mathbb{N}.$$

It is not difficult to see that

$$s_n^\beta(z) = {}_2F_1(1, -n, -n-\beta, z), \tag{3.1}$$

which should be compared with the representation

$$P_\lambda^\beta(z) := {}_2F_1(1, -\lambda, \lambda+\beta, -z), \quad \lambda \geqslant -1,\ \beta \geqslant 1, \tag{3.2}$$

of (1.6). In [5] it was shown that

$$zs_n^\beta \in \mathcal{R}_{\frac{3-\beta}{2}},$$

which is similar to Theorem 3. And it was conjectured that

$$s_n^\alpha \prec s_m^\beta, \quad n \leqslant m,\ 2 \leqslant \beta < \alpha, \tag{3.3}$$

which reminds of Theorem 4. This conjecture is still open as far as the part $\alpha \neq \beta$ is concerned (for $\alpha = \beta$ we proved it in [6]). The following partial result narrows the gap a bit.

Theorem 11. *For $\beta \geqslant 3$, $\alpha \geqslant \beta + 1$, and $n \in \mathbb{N}$ we have $s_n^\alpha \prec s_n^\beta$.*

Proof. Since $s_n^\beta(z) - 1 = \frac{n}{n+\beta} z s_{n+1}^\beta$, it follows from the result mentioned above that s_n^β is convex univalent in $\mathbb{D}$ for $\beta \geqslant 3$. Now $s_n^\alpha = s_n^\beta * H(z)$, where

$$H(z) = \sum_{k=0}^{[n+\beta+1]} \frac{(-n-\alpha)_k}{(-n-\beta)_k} z^k.$$

Set $k_0 = [n+\beta+1]$. It is easily checked that the numbers

$$c_k := \frac{(-n-\alpha)_k}{(-n-\beta)_k}, \quad 0 \leqslant k \leqslant k_0 + 1,$$

form a decreasing convex sequence, of which all but the last one are non-negative and the last one is non-positive. From this it is clear that the sequence

$$h_k := \begin{cases} c_k, & 0 \leqslant k \leqslant k_0, \\ 0, & k_0 < k, \end{cases}$$

is a decreasing convex sequence of non-negative numbers with $h_0 = 1$, and therefore fulfils the assumptions of Lemma 10. The Taylor coefficients of $H(z)$ are exactly the h_k and a combination of Lemmas 10 and 9 leads to the desired conclusion. □

We point out that also for the Cesáro means we have a differential equation like (2.3): $w = s_n^\beta$ is the solution of

$$zw'(z) - \left(\frac{\beta+1}{1-z} + n\right) w(z) = -\frac{\beta+n+1}{1-z}, \quad w(0) = 1.$$

It is an interesting question what one can say about the corresponding situation in the cases of the parameters which constitute the link between the generalized de la Vallée Poussin means and the Cesáro means as suggested by the relations (3.1) and (3.2), respectively. In this context one should also mention the very challenging second conjecture of S.P. Robinson (see [9, p. 300–301]) for Cesáro means.

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