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New Pólya–Schoenberg type theorems ☆

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ABSTRACT

In this paper the theory of Hadamard product multipliers is extended from the unit disk in the complex plane to arbitrary so-called disk-like domains, i.e. such domains which are the union of disks or half-planes, all containing the origin. In such a domain, say Ω, we define (the class $\mathcal{R}_\alpha^d(\Omega)$ of) generalized prestarlike functions of order $\alpha \leqslant 1$ and ask for Hadamard multipliers g analytic at $z=0$ for which $f \in \mathcal{R}_\alpha^d(\Omega)$ implies $g * f \in \mathcal{R}_\alpha^d(\Omega)$. We prove that such a multiplier necessarily has to be analytic in

$$\Omega^* := \left\{ \frac{u}{v} \colon u \in \Omega,\ v \in \mathbb{C} \setminus \Omega \right\}.$$

In many cases (we prove this for all proper disks containing the origin) we actually find that $\mathcal{R}_\alpha^d(\Omega^*)$ is the precise description of the set of all such multipliers. For these disks, Ω_γ say, the domains Ω_γ^* turn out to be bounded by the outer loops of certain Limaçons of Pascal. The parameter γ is related to the characteristic $q(\Omega_\gamma) = (1-\gamma)/(1+\gamma) := r/s$ of the disk, where r is the shortest distance of the origin to the boundary of that disk, and s the largest. Large subclasses of $\mathcal{R}_\alpha^d(\Omega^*)$ are being explicitly determined. For the case $\gamma = 0$, i.e. $\Omega_\gamma = \Omega_\gamma^* = \mathbb{D}$, this result coincides with an old one by Ruscheweyh and Sheil-Small, previously conjectured by G. Pólya and I.J. Schoenberg. The notion of the characteristic of a disk (containing the origin) is then extended to general disk-like domains, and some multipliers are identified for those general classes $\mathcal{R}_\alpha^d(\Omega)$. The previously determined class of 'universally prestarlike functions', defined in the slit-domain $\mathbb{C} \setminus [1, \infty]$, is identified as the class of 'universal multipliers' for $\mathcal{R}_\alpha^d(\Omega)$ in any disk-like domain Ω.

1. Introduction

1.1. General notation

Let $\mathbb{C}$ be the set of complex numbers and $\mathfrak{P}_0$ the system of domains $\Omega \subset \mathbb{C}$ containing the origin 0. For $\Omega \in \mathfrak{P}_0$ we write $\mathcal{H}(\Omega)$ for the family of analytic functions in Ω, and $\mathcal{H}_0(\Omega)$ respectively $\mathcal{H}_1(\Omega)$ are the subsets of functions $f \in \mathcal{H}(\Omega)$ satisfying $f(0) = 1$ respectively $f(0) = f'(0) - 1 = 0$.

A circular domain is an open disk or half-plane in $\mathbb{C}$. Throughout this paper we write $\mathfrak{D}_0$ for the set of circular domains containing the origin 0, i.e. $\mathfrak{D}_0 \subset \mathfrak{P}_0$. For a domain $\Omega \in \mathfrak{P}_0$ we let $\mathfrak{D}_0(\Omega)$ denote the subset of all circular domains in $\mathfrak{D}_0$ which are contained in Ω. $\mathbb{D}$ stands for the unit disk $\{z \in \mathbb{C} \colon |z| < 1\}$.

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For two functions $f, g \in \mathcal{H}(\{0\})$ with $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $g(z) = \sum_{k=0}^{\infty} g_k z^k$ we define the Hadamard product (or convolution) $f * g$ by $(f * g)(z) := \sum_{k=0}^{\infty} f_k g_k z^k$. Clearly $f * g \in \mathcal{H}(\{0\})$ as well.

Let $Q \subset \mathcal{H}_1(\Omega)$ for some $\Omega \in \mathfrak{P}_0$. A function $g \in \mathcal{H}_1(\{0\})$ is called a *multiplier for* Q (or $g \in \mathcal{M}(Q)$) if

$$\forall_{f \in Q}: \quad g * f \in Q. \tag{1.1}$$

Remark 1.1. Note that, by definition, the set $\mathcal{M}(Q)$ is closed under convolution, i.e.

$$g, h \in \mathcal{M}(Q) \quad \Rightarrow \quad g * h \in \mathcal{M}(Q). \tag{1.2}$$

1.2. The principal example: Convex univalent functions in $\mathbb{D}$

Let $\mathcal{C}(\mathbb{D})$ be the family of normalized convex univalent functions in the unit disk $\mathbb{D}$, i.e. $f \in \mathcal{C}(\mathbb{D})$ if and only if $f \in \mathcal{H}_1(\mathbb{D})$ with f univalent in $\mathbb{D}$ and $f(\mathbb{D})$ convex.

An old result of Ruscheweyh and Sheil-Small [6], previously conjectured by Pólya and Schoenberg [3], states

$$\mathcal{M}\big(\mathcal{C}(\mathbb{D})\big) = \mathcal{C}(\mathbb{D}). \tag{1.3}$$

Remark 1.2. It is a classical result due to Study (later refined by Heins and Pommerenke) that every convex univalent function in a circular domain D maps any circular subdomain $D' \subset D$ also onto a convex domain, compare Theorem 1.3 below. We are going to use this fact while passing from circular domains to other configurations.

Definition 1.1. A domain $\Omega \in \mathfrak{P}_0$ is called *disk-like* (*with respect to the origin*) if

$$\Omega = \bigcup_{D \in \mathfrak{D}_0(\Omega)} D.$$

Note that disk-like domains (w.r.t. the origin) are starlike (w.r.t. the origin), since for any $z \in \Omega$ there exists $D \in \mathfrak{D}_0(\Omega)$ such that $z \in D$ and therefore $[0, z] \subset D \subset \Omega$. The slit-domain $\Lambda := \mathbb{C} \setminus [1, \infty)$ is disk-like and important for the theory to be developed in this paper.

Definition 1.2. Let $\Omega \in \mathfrak{P}_0$ be disk-like and $f \in \mathcal{H}_1(\Omega)$. Then f is called *disk-convex* if f maps every $D \in \mathfrak{D}_0(\Omega)$ univalently onto a convex domain.

Remark 1.3. Note that the origin 0 of the complex plane plays a special role in Definition 1.1. That has to do with our main subject, namely the study of multipliers. It might be interesting to study a more general class of disk-like domains and disk-convex functions, without this restriction. This, however, is not our aim here. All disk-like domains used in the sequel will be disk-like w.r.t. the origin. For the sake of brevity we call them just 'disk-like'.

An interesting example for a disk-convex mapping defined in Λ is

$$v(z) := \frac{8}{z}(1 - \sqrt{1-z}) - 4,$$

which maps Λ onto the disk $4\mathbb{D}$ (see [8]). This v is, up to normalizations, the inverse of the Koebe function. The identity function $w(z) = z$ is also disk-convex in Λ, but $w(\Lambda)$ is obviously not convex. On the other hand, the following result holds (see Theorem 1.5 for a generalization).

Theorem 1.1. *Let Ω be a disk-like domain w.r.t. the origin, and f disk-convex in Ω. Then f is univalent in Ω and $f(\Omega)$ is starlike with respect to the origin.*

As far as convex and disk-convex mappings are concerned, we are dealing with the following questions.

Problem 1.1. For Ω disk-like let $\mathcal{C}(\Omega)$ be the set of functions in $\mathcal{H}_1(\Omega)$ which are disk-convex in Ω. Describe $\mathcal{M}(\mathcal{C}(\Omega))$.

Problem 1.2. Describe disk-like domains Ω such that the set of disk-convex functions in Ω is closed under convolution.

Note that (1.3) solves Problem 1.1 for $\Omega = \mathbb{D}$, and with respect to Problem 1.2 the same result shows that $\mathbb{D}$ is a domain with the desired property.

As we shall see, the solution of Problem 1.1 for $\Omega = D \in \mathfrak{D}_0$ will produce a set of disk-like domains having the property described in Problem 1.2. It is perhaps surprising, that none of these domains, except for $\mathbb{D}$, is circular or even convex.

1.3. Prestarlike functions

Suffridge [10] (see also Lewis [2] and Ruscheweyh [4]) extended (1.3) in various directions, mainly by introducing and studying the classes $\mathcal{R}_\alpha$ of so-called prestarlike functions of order $\alpha \in (-\infty, 1]$ in $\mathbb{D}$. The set $\mathcal{C}(\mathbb{D})$ is embedded in this scheme: $\mathcal{C}(\mathbb{D}) = \mathcal{R}_0$. See [5] for a general reference concerning prestarlike functions.

For $f \in \mathcal{H}_1(\{0\})$ and $\beta \geqslant 0$ set

$$\big(D^\beta f\big)(z) := \frac{z}{(1-z)^\beta} * f.$$

Note that, in particular,

$$\big(D^0 f\big)(z) = z, \qquad \big(D^1 f\big)(z) = f(z),$$

and

$$D^{n+1} f = \frac{z}{n!}\big(z^{n-1} f\big)^{(n)}, \quad n \in \mathbb{N}.$$

Let

$$\mathcal{P} := \left\{F \in \mathcal{H}_0(\mathbb{D})\colon \operatorname{Re} f(z) > \frac{1}{2},\ z \in \mathbb{D}\right\}. \tag{1.4}$$

Definition 1.3. Let $\alpha \leqslant 1$ and $f \in \mathcal{H}_1(\mathbb{D})$. Then $f \in \mathcal{R}_\alpha$ if and only if

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{P}. \tag{1.5}$$

Note that this, for $\alpha = 0$, reduces to the common condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D},$$

for a function f to be convex univalent in $\mathbb{D}$ and that $f \in \mathcal{R}_1$ if and only if $f(z)/z \in \mathcal{P}$.

For our purposes the following two facts (see e.g. [5, Theorem 2.1]) about prestarlike functions are the most relevant ones.

Lemma 1.1. *For $\alpha < \beta \leqslant 1$ we have $\mathcal{R}_\alpha \subset \mathcal{R}_\beta$.*

Lemma 1.2. *For $\alpha \leqslant 1$ we have $\mathcal{M}(\mathcal{R}_\alpha) = \mathcal{R}_\alpha$.*

Note that Lemma 1.2, for $\alpha = 0$, is (1.3).

1.4. Generalized prestarlike functions

Generalized prestarlike functions have been introduced in [8]. They are translations of prestarlike functions into arbitrary circular domains in $\mathfrak{D}_0$ in a specific way.

Let $\Omega \neq \mathbb{C}$ be in $\mathfrak{D}_0$. Then there are two *unique* parameters $\tau \in \mathbb{C} \setminus \{0\}$ and $\gamma \in [0, 1]$ such that

$$\Omega = \big\{w_{\tau,\gamma}(z)\colon z \in \mathbb{D}\big\},$$

where

$$w_{\tau,\gamma}(z) := \frac{\tau z}{1 + \gamma z}.$$

We write $\Omega_{\tau,\gamma}$ for this Ω.

Definition 1.4. Let $\alpha \leqslant 1$ and $\Omega = \Omega_{\tau,\gamma}$ for some admissible pair (τ, γ). A function $f \in \mathcal{H}_1(\Omega)$ is called *prestarlike of order α in Ω* (written as $f \in \mathcal{R}_\alpha(\Omega)$) if

$$f_{\tau,\gamma}(z) := \frac{1}{\tau} f\big(w_{\tau,\gamma}(z)\big) \in \mathcal{R}_\alpha.$$

The following theorem will be crucial later on. Its proof is in Section 2.

Theorem 1.2. *Let $\alpha \leqslant 1$ and $D, D' \in \mathfrak{D}_0$ with $D \subset D'$. Then $\mathcal{R}_\alpha(D') \subset \mathcal{R}_\alpha(D)$.*

Next we extend the notion of disk-convexity to generalized prestarlike functions.

Definition 1.5. Let Ω be a disk-like domain, and $\alpha \leqslant 1$. A function $f \in \mathcal{H}_1(\Omega)$ is called *disk-prestarlike of order* α if $f \in \mathcal{R}_\alpha(D)$ for every $D \in \mathfrak{D}_0(\Omega)$. We denote the set of these functions by $\mathcal{R}_\alpha^d(\Omega)$.

Theorem 1.2 can be used to simplify the above definition by using the notion of maximal disks for disk-like domains.

Definition 1.6. Let Ω be disk-like. A disk $D \in \mathfrak{D}_0(\Omega)$ is called *maximal* (w.r.t. Ω) if $D \subset D' \in \mathfrak{D}_0(\Omega)$ implies $D = D'$.

Note that every disk $D \in \mathfrak{D}_0(\Omega)$ is contained in some maximal disk of Ω.

Corollary 1.1. *Let Ω be a disk-like domain, and $\alpha \leqslant 1$. A function $f \in \mathcal{H}_1(\Omega)$ is disk-prestarlike of order α if $f \in \mathcal{R}_\alpha(D)$ for every maximal $D \in \mathfrak{D}_0(\Omega)$.*

The definition and Lemma 1.1 imply

Theorem 1.3. *Let $\alpha \leqslant 1$ and let Ω, Ω' be disk-like with $\Omega \subset \Omega'$. Then $\mathcal{R}_\alpha^d(\Omega') \subset \mathcal{R}_\alpha^d(\Omega)$.*

Theorem 1.4. *Let $\alpha < \beta \leqslant 1$ and let Ω be disk-like. Then $\mathcal{R}_\alpha^d(\Omega) \subset \mathcal{R}_\beta^d(\Omega)$.*

Note that this implies that disk-convex functions are in particular disk-prestarlike of order $\frac{1}{2}$. We have the following extension of Theorem 1.1, to be established in Section 2.

Theorem 1.5. *Let Ω be a disk-like domain and $f \in \mathcal{R}_\alpha^d(\Omega)$ for some $\alpha \leqslant \frac{1}{2}$. Then f is univalent in Ω and $f(\Omega)$ is starlike with respect to the origin.*

We mention in passing that, although the original concept of prestarlike functions of order α has its roots in the theory of starlike functions of the same order in the unit disk, a result corresponding to Theorem 1.2 does not seem to exist if one replaces *prestarlike* by *starlike* in Definition 1.4.

The complex plane $\mathbb{C}$ is disk-like, but the only function in $\mathcal{R}_\alpha^d(\mathbb{C})$ is the identity function. So we always assume that the disk-like domains we are dealing with are not $\mathbb{C}$. Then, since disk-like domains are starlike with respect to the origin, they have one or more rays of the form $\{tx\colon\ t \geqslant 1\}$ in their complement. Assume that a disk-like Ω omits the line segment $[1, \infty)$. Then $\Omega \subset \Lambda$ and therefore, in view of Theorem 1.3, we obtain

$$\mathcal{R}_\alpha^d(\Lambda) \subset \mathcal{R}_\alpha^d(\Omega). \tag{1.6}$$

The functions $f \in \mathcal{R}_\alpha^d(\Lambda)$ have been studied in [8] already, in particular, it was shown that

$$\mathcal{M}\big(\mathcal{R}_\alpha^d(\Lambda)\big) = \mathcal{R}_\alpha^d(\Lambda), \quad \alpha \leqslant 1, \tag{1.7}$$

and, moreover,

$$\mathcal{R}_\alpha^d(\Lambda) \subset \mathcal{M}\big(\mathcal{R}_\alpha^d(D)\big), \quad D \in \mathfrak{D}_0,\ \alpha \leqslant 1,$$

which implies that for any disk-like domain Ω we have

$$\mathcal{R}_\alpha^d(\Lambda) \subset \mathcal{M}\big(\mathcal{R}_\alpha^d(\Omega)\big), \quad \alpha \leqslant 1. \tag{1.8}$$

In [8] the functions in $\mathcal{R}_\alpha^d(\Lambda)$ have been called *universally prestarlike of order* α, which seems to be well justified in view of (1.7) and (1.8).

In generalization of Problems 1.1 and 1.2 we are actually interested in some answers regarding the following problems.

Problem 1.3. For $\alpha \leqslant 1$ and Ω disk-like describe $\mathcal{M}(\mathcal{R}_\alpha^d(\Omega))$.

Problem 1.4. Describe disk-like domains Ω such that the set $\mathcal{R}_\alpha^d(\Omega)$ is closed under convolution for (some) $\alpha \leqslant 1$.

1.5. General considerations

A basic property of disk-like domains — shown in Section 2 — is as follows.

Lemma 1.3. *Let Ω be disk-like, and $\frac{1}{x} \in \mathbb{C} \setminus \Omega$. Then*

$$f_x(z) := \frac{z}{1-xz} \in \mathcal{R}_\alpha^d(\Omega), \quad \alpha \leqslant 1. \tag{1.9}$$

For any disk-like Ω we define its adjoint set

$$\Omega^* := \left\{ \frac{u}{v} \colon u \in \Omega,\ v \in \mathbb{C} \setminus \Omega \right\}.$$

It is easy to verify that Ω^* is also disk-like with $\Omega^* \subset \Lambda$. And in Section 3.1 we shall prove

Theorem 1.6. *For each disk-like Ω and $\alpha \leqslant 1$ we have*

$$\mathcal{M}\big(\mathcal{R}_\alpha^d(\Omega)\big) \subset \mathcal{H}_1\big(\Omega^*\big). \tag{1.10}$$

This has the following immediate but still surprising consequence, which is a first approximation towards a solution of Problem 1.4.

Corollary 1.2. *A necessary condition for a disk-like domain Ω to have the property described in Problem 1.4 is that it satisfies $\Omega = \Omega^*$. In particular, such Ω must satisfy $1 \in \partial\Omega$ and $\Omega \subset \Lambda$.*

Note that $\mathbb{D}^* = \mathbb{D}$, in accordance with Lemma 1.2.

The multiplier problem for disk-like domains $\Omega \neq \mathbb{C}$ can be reduced to the cases $\Omega \subset \Lambda$ with $1 \in \partial\Omega$. This is an obvious consequence of the following lemma. For a proof see Section 3.1.

Lemma 1.4. *Let Ω be disk-like and $\sigma \neq 0$. Then for $\sigma\Omega := \{\sigma w\colon\ w \in \Omega\}$ we have*

$$\mathcal{M}\big(\mathcal{R}_\alpha^d(\Omega)\big) = \mathcal{M}\big(\mathcal{R}_\alpha^d(\sigma\Omega)\big).$$

This means that in the case of the 'disk-like' circular domains $\Omega_{\tau,\gamma}$ we can reduce our attention to the cases $\tau := \gamma + 1$, and therefore to

$$\Omega_\gamma := \Omega_{\gamma+1,\gamma} = \left\{ \frac{(\gamma+1)z}{1+\gamma z} \colon z \in \mathbb{D} \right\}, \quad \gamma \in [0,1].$$

Note that Ω_γ is a disk in $\mathfrak{D}_0$, symmetric to the real axis, with $1 \in \partial\Omega_\gamma$, growing from the unit disk ($\gamma = 0$) up to the half-plane $\Omega_1 := \{z\colon\ \operatorname{Re} z < 1\}$. Of course, $\Omega_\gamma \subset \Lambda$. It is our first goal to study the sets $\mathcal{M}(\mathcal{R}_\alpha(\Omega_\gamma))$.

1.6. Limaçons of Pascal and the main theorem

We first identify Ω_γ^*. For $0 \leqslant \gamma < 1$ let Π_γ denote the Limaçon of Pascal consisting of the points $w \in \mathbb{C}$ with

$$\gamma^2|1-w|^2 = \big(1-|w|\big)^2.$$

Then Π_γ consists of two loops, an outer one, say Π_γ^o, and an inner one, say Π_γ^i, defined by

$$w \in \Pi_\gamma^o \quad \Leftrightarrow \quad |w| - \gamma|1-w| = 1, \tag{1.11}$$

$$w \in \Pi_\gamma^i \quad \Leftrightarrow \quad |w| + \gamma|1-w| = 1. \tag{1.12}$$

Note that

$$\Pi_\gamma^o = \Pi_\gamma \setminus \mathbb{D} \quad \text{and} \quad w \in \Pi_\gamma^i \quad \Leftrightarrow \quad \frac{1}{w} \in \Pi_\gamma^o. \tag{1.13}$$

The following fact will be established in Section 3.2.

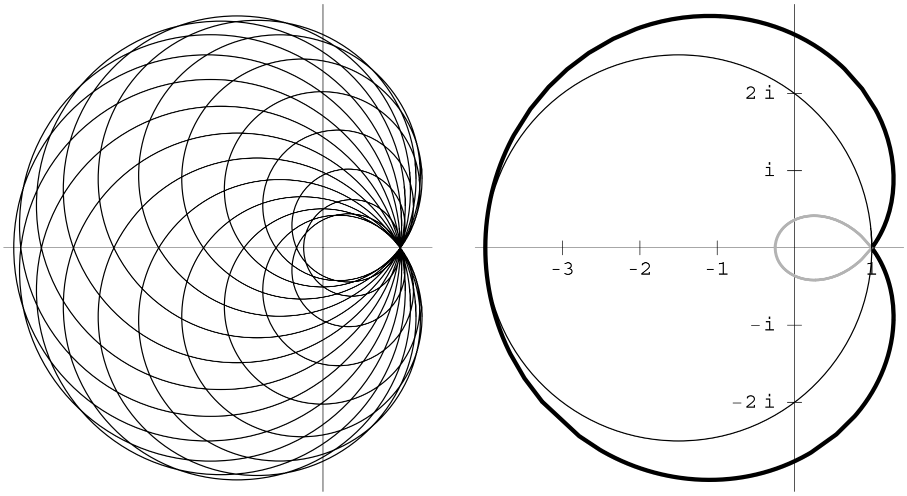

Fig. 1. Ω_γ^* and Π_γ, $\gamma = .4$.

Theorem 1.7. *For* $0 \leqslant \gamma < 1$ *we have*

$$\Omega_\gamma^* = \frac{\gamma - 1}{\gamma + 1}(\Omega_\gamma)^2, \tag{1.14}$$

and $\partial\Omega_\gamma^* = \Pi_\gamma^o$. *For* $\gamma = 1$ *we have*

$$\Omega_1^* = \mathbb{C} \setminus [1, \infty) = \Lambda. \tag{1.15}$$

We mention in passing that a parametric form of $\partial\Omega_\gamma^*$ is

$$\partial\Omega_\gamma^* = \left\{ \frac{\gamma^2 - 1}{(e^{it \arccos\gamma} - \gamma)^2} \colon -1 \leqslant t \leqslant 1 \right\}. \tag{1.16}$$

The drawing on the left-hand side of Fig. 1 shows Ω_γ^*, $\gamma = .4$, as a union of its maximal disks

$$\Delta_\gamma(x) := \left\{ \frac{(1 + \gamma x)z}{(1 + \gamma z)x} \colon z \in \mathbb{D} \right\}, \quad x \in \partial\mathbb{D} \setminus \{-1\} \tag{1.17}$$

(see the definition of Ω_γ^*). The thick curve on the right is the whole of Π_γ, $\gamma = .4$, with $\Pi_\gamma^o = \partial\Omega_\gamma^*$ in black, and Π_γ^i in grey. The inscribed circle is the boundary of Ω_γ. It will turn out later that the convex domain

$$L_\gamma := \{ w \colon \gamma|1 - w| + |w| < 1 \}, \tag{1.18}$$

bounded by Π_γ^i, is, quite surprisingly, also significant for this theory, see Section 1.9.

The main result of this paper solves Problem 1.3 for circular domains. Its proof is in Section 3.4.

Theorem 1.8. *For* $0 \leqslant \gamma \leqslant 1$ *and* $\alpha \leqslant 1$ *we have*

$$\mathcal{M}(\mathcal{R}_\alpha(\Omega_\gamma)) = \mathcal{M}(\mathcal{R}_\alpha^d(\Omega_\gamma)) = \mathcal{R}_\alpha^d(\Omega_\gamma^*) \subset \mathcal{R}_\alpha(\Omega_\gamma). \tag{1.19}$$

And this obviously implies a partial solution to Problem 1.4.

Corollary 1.3. $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$ *is closed under convolution for* $0 \leqslant \gamma \leqslant 1$ *and* $\alpha \leqslant 1$.

1.7. About the members of $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$

In order to get a more or less explicit representation for the members of $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$ we introduce the sets $\mathcal{P}_\gamma$ as follows. Recall the definition of $\mathcal{P}$ from (1.4).

Definition 1.7. For $0 \leqslant \gamma \leqslant 1$ let $\mathcal{P}_\gamma$ consist of the functions $F \in \mathcal{H}_0(\Omega_\gamma^*)$ satisfying

$$z \mapsto \frac{1}{1+\gamma z} F\left(\frac{(1+\gamma x)z}{(1+\gamma z)x}\right) \in \mathcal{P}, \quad |x| = 1. \tag{1.20}$$

Note that $\mathcal{P}_0 = \mathcal{P}$, and in [8] it has (essentially) been shown that $\mathcal{P}_1$ consists of the functions

$$F(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}_0(\Lambda), \tag{1.21}$$

whose coefficient sequence $\{c_k\}$ is completely monotone.

Using this notation we obtain (see Section 3.3)

Theorem 1.9. *Let $\gamma \in [0,1]$, $\alpha \leqslant 1$. A function $f \in \mathcal{H}_1(\Omega_\gamma^*)$ belongs to $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$ if and only if*

$$\frac{D^{3-2\alpha} f}{D^{2-2\alpha} f} \in \mathcal{P}_\gamma. \tag{1.22}$$

Note that $\mathcal{R}_1^d(\Omega_\gamma^*) = z\mathcal{P}_\gamma$.

Corollary 1.4. *Let $\gamma \in [0,1]$, $\alpha < 1$. A function $f \in \mathcal{H}_1(\Omega_\gamma^*)$ belongs to $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$ if and only if there exists a function $F \in \mathcal{P}_\gamma$ such that*

$$\left(D^{2-2\alpha} f\right)(z) = z \exp\left((2-2\alpha) \int_0^z \frac{F(t)-1}{t}\, dt\right). \tag{1.23}$$

It is easily checked that for $0 \leqslant \gamma \leqslant 1$ we have

$$w \in \overline{L_\gamma} \quad \Rightarrow \quad \frac{1}{1-wz} \in \mathcal{P}_\gamma, \tag{1.24}$$

and since $\mathcal{P}_\gamma$ is a convex set we conclude that

$$\mathcal{T}_\gamma := \overline{\mathrm{co}}\left\{\frac{1}{1-wz}\colon w \in \overline{L_\gamma}\right\} \subset \mathcal{P}_\gamma, \tag{1.25}$$

where $\overline{\mathrm{co}}(A)$ stands for the closed convex hull of a set A in some linear space. This leads to the following 'explicit' representation for some members of $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$:

Corollary 1.5. *Let $\alpha < 1$, $0 \leqslant \gamma < 1$ and $\mu \in W(\overline{L_\gamma})$. Then f defined by*

$$\left(D^{2-2\alpha} f\right)(z) = z \exp\left(\int_{\overline{L_\gamma}} \log \frac{1}{(1-wz)^{2-2\alpha}}\, d\mu\right) \tag{1.26}$$

belongs to $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$.

Here and in the sequel we denote by $W(A)$ the set of Borel probability measures on sets $A \subset \mathbb{C}$.

Note that, for $\alpha = \frac{1}{2}$, the left-hand side of (1.26) is just $f(z)$. Also for $\alpha < \frac{1}{2}$, using the integral representation for the hypergeometric function (see e.g. [1, 15.3.1]), the representation (1.26) can be replaced by an 'even more' explicit one:

$$f(z) = (1-2\alpha) z \int_0^1 (1-\tau)^{-2\alpha} \exp\left(\int_{\overline{L_\gamma}} \log \frac{1}{(1-w\tau z)^{2-2\alpha}}\, d\mu\right) d\tau. \tag{1.27}$$

In particular, choosing $\mu(t)$ as a discrete measure, we find $f(z) \in \mathcal{R}_\alpha^d(\Omega_\gamma^*)$ if

$$f(z) = \begin{cases} z \prod_{k=1}^n \frac{1}{(1-w_k z)^{\lambda_k}}, & \alpha = \frac{1}{2}, \\ (1-2\alpha) z \int_0^1 (1-\tau)^{-2\alpha} \prod_{k=1}^n \frac{1}{(1-w_k \tau z)^{\lambda_k}}\, d\tau, & \alpha < \frac{1}{2}, \end{cases}$$

if $w_k \in \overline{L_\gamma}$ and $\lambda_k > 0$ holds for $k = 1, \ldots, n \in \mathbb{N}$, where $\sum_{k=1}^n \lambda_k = 2 - 2\alpha$.

1.8. Further consequences

The claims made in this subsection will be verified in Section 3.5. First we will show that

$$\Omega_\gamma^* \subset \frac{1+\gamma}{1-\gamma}\mathbb{D}. \tag{1.28}$$

It then follows by definition that

$$\mathcal{R}_\alpha\left(\frac{1+\gamma}{1-\gamma}\mathbb{D}\right) = \mathcal{R}_\alpha^d\left(\frac{1+\gamma}{1-\gamma}\mathbb{D}\right) \subset \mathcal{R}_\alpha^d\big(\Omega_\gamma^*\big),$$

and this implies

Theorem 1.10. *Let $\alpha \leqslant 1$, $0 \leqslant \gamma < 1$, $q = \frac{1-\gamma}{1+\gamma}$ and $g \in \mathcal{R}_\alpha(\mathbb{D})$. Then*

$$\frac{1}{q}g(qz) \in \mathcal{M}\big(\mathcal{R}_\alpha(\Omega_\gamma)\big).$$

The constant q is best possible.

Note that this, for $\gamma = 0$, is Lemma 1.2. We also have

Theorem 1.11. *For $0 \leqslant \gamma < \delta \leqslant 1$ and $\alpha \leqslant 1$ we have $\Omega_\gamma^* \subset \Omega_\delta^*$, and therefore $\mathcal{R}_\alpha^d(\Omega_\delta^*) \subset \mathcal{R}_\alpha^d(\Omega_\gamma^*)$.*

This has an interesting application. For any (bounded) disk $D \in \mathfrak{D}_0$ we set $q(D) := r/R$, where r is the shortest distance from the origin to ∂D, and R the largest.

Definition 1.8. For a bounded disk-like Ω we set

$$q(\Omega) := \inf\big\{q(D)\colon\ D \in \mathfrak{D}_0(\Omega) \text{ maximal}\big\}.$$

For every unbounded disk-like domain Ω we set $q(\Omega) := 0$. We call $q(\Omega)$ the *dl-characteristic* of Ω.

Clearly, Ω_γ as well as Ω_γ^* have the dl-characteristic $(1-\gamma)/(1+\gamma)$.

Theorem 1.12. *Assume that Ω is a disk-like domain of dl-characteristic $(1-\gamma)/(1+\gamma)$, where $0 \leqslant \gamma \leqslant 1$. Then, for $\alpha \leqslant 1$, we have*

$$\mathcal{R}_\alpha^d\big(\Omega_\gamma^*\big) \subset \mathcal{M}\big(\mathcal{R}_\alpha^d(\Omega)\big).$$

In particular, $\Omega^* \subset \Omega_\gamma^*$.

This implies, in particular, a generalization of Theorem 1.10.

Corollary 1.6. *Let Ω be disk-like with dl-characteristic $q = q(\Omega) > 0$ and let $\alpha \leqslant 1$. Then, for $g \in \mathcal{R}_\alpha(\mathbb{D})$, we have*

$$\frac{1}{q}g(qz) \in \mathcal{M}\big(\mathcal{R}_\alpha^d(\Omega)\big). \tag{1.29}$$

We mention in passing that for an unbounded disk-like domain we actually have $\Omega^* = \Omega_1^* = \Lambda$. It is not likely that in all such cases we have $\mathcal{R}_\alpha^d(\Lambda) = \mathcal{M}(\mathcal{R}_\alpha^d(\Omega))$. However, since the latter statement is true for Ω_1 as well as for $\Omega = \Lambda$, we state the following as a problem.

Problem 1.5. Is it true that we have $\mathcal{R}_\alpha^d(\Lambda) = \mathcal{M}(\mathcal{R}_\alpha^d(\Omega))$ for every disk-like domain such that $\mathfrak{D}_0(\Omega)$ contains a half-plane?

Note that every unbounded disk-like domain actually contains a half-plane, but not necessarily with the origin in its interior. As an example take for Ω the union of $\mathbb{D}$ with the left half-plane. There are reasons to doubt that the answer to the question above is affirmative in that case.

1.9. More about $\mathcal{P}_\gamma$

The convex and compact set $\mathcal{P}_\gamma$ is obviously crucial for the theory presented here: more information about this set immediately translates into information for members of $\mathcal{R}_\alpha^d(\Omega_\gamma^*)$.

For $\gamma = 0$ we immediately deduce $\mathcal{T}_0 = \mathcal{P}_0 = \mathcal{P}$ (Herglotz formula). A limiting argument for $\gamma \to 1$, using (1.21) and the well-known solution of Hausdorff's moment problem, yields

$$\mathcal{T}_1 := \left\{ f \in \mathcal{H}_0(\Lambda):\ f(z) = \int_0^1 \frac{d\mu(t)}{1-tz},\ \mu \in W\big([0,1]\big) \right\} = \mathcal{P}_1.$$

Unfortunately, for $0 < \gamma < 1$ we do not have complete descriptions for $\mathcal{P}_\gamma$ available. However we can prove one related result.

Theorem 1.13. *The functions $E_{\gamma,w}(z) := \frac{1}{1-wz}$, $w \in \partial L_\gamma$, are extreme points of $\mathcal{P}_\gamma$.*

The proof of this theorem is based on Theorem 1.14 below. Let

$$A_\gamma := \big\{ f'(0):\ f \in \mathcal{P}_\gamma \big\}, \qquad B_\gamma := \big\{ f'(0):\ f \in \mathcal{T}_\gamma \big\}.$$

Theorem 1.14. *For $0 \leqslant \gamma \leqslant 1$ we have $A_\gamma = B_\gamma = \overline{L_\gamma}$. Furthermore, the only functions in $\mathcal{P}_\gamma$ for which $f'(0) \in \partial L_\gamma$ are the functions $f = E_{\gamma,w}$, $w \in \partial L_\gamma$.*

Other properties of $\mathcal{P}_\gamma$ which might be helpful in obtaining a more complete and explicit description of the members in $\mathcal{P}_\gamma$ are as follows.

Theorem 1.15. *For $0 \leqslant \gamma \leqslant 1$ the sets $\mathcal{P}_\gamma$ are closed under convolution.*

Theorem 1.16. *For $0 \leqslant \gamma < \delta \leqslant 1$ we have $\mathcal{P}_\delta \subset \mathcal{P}_\gamma$.*

Theorem 1.17. *Let $0 \leqslant \gamma \leqslant 1$. For $f \in \mathcal{P}_\gamma$ we have*

$$\operatorname{Re}\left[\frac{\zeta}{\zeta+\gamma} \left(f\left(\frac{(\zeta+\gamma)z}{1+\gamma z} \right) - 1 \right) \right] \geqslant -\frac{1}{2}, \quad |\zeta| \leqslant 1,\ z \in \mathbb{D}. \tag{1.30}$$

Theorems 1.13–1.16 will be verified in Section 4, while the proof of Theorem 1.17 can be found in Section 3.4.

2. Disk-prestarlike functions

In this section we justify the general statements about generalized and disk-prestarlike functions made in Sections 1.4 and 1.5.

We first recall the well-known Herglotz representation of the functions in $\mathcal{P}$ (see (1.4)).

Lemma 2.1. *A function f belongs to $\mathcal{P}$ if and only if there is a measure $\mu \in W([0,2\pi])$ such that*

$$f(z) = \int_0^{2\pi} \frac{d\mu(t)}{1-e^{-it}z}.$$

Basic for our work is also the following general formula, see [8, Lemma 2.3].

Lemma 2.2. *Let $\beta \geqslant 0$, $\tau \in \mathbb{C} \setminus \{0\}$, $|\gamma| \leqslant 1$ and $f \in \mathcal{H}(\Omega_{\tau,\gamma})$. Then, in the notation of Definition* 1.4 *we find*

$$\frac{(D^{\beta+1} f_{\tau,\gamma})(z)}{(D^\beta f_{\tau,\gamma})(z)} = \frac{1}{1+\gamma z} \frac{(D^{\beta+1} f)_{\tau,\gamma}(z)}{(D^\beta f)_{\tau,\gamma}(z)}. \tag{2.1}$$

Proof of Theorem 1.2. Let D and D' be described by

$$w(z) = \frac{\tau z}{1+\gamma z}, \qquad v(z) = \frac{\tau' z}{1+\gamma' z} \quad (z \in \mathbb{D}),$$

with $\tau, \tau' \in \mathbb{C} \setminus \{0\}$ and $\gamma, \gamma' \in [0,1]$. Since $D \subset D'$ we must have $w \prec v$ (subordination) which means that there exists $\omega \in \mathcal{H}(\mathbb{D})$ satisfying

$$\omega : \mathbb{D} \mapsto \mathbb{D}, \qquad \omega(0) = 0, \qquad w = v \circ \omega.$$

Of course, we have

$$\omega(z) = \frac{(\tau/\tau')z}{1 + \frac{\gamma\tau' - \gamma'\tau}{\tau'} z},$$

and the condition above is fulfilled if and only if

$$|\gamma\tau' - \gamma'\tau| + |\tau| \leqslant |\tau'|. \tag{2.2}$$

Now assume $f \in \mathcal{R}_\alpha(D')$, or, equivalently, $\varphi(z) = \frac{1}{\tau'} f(v(z)) \in \mathcal{R}_\alpha(\mathbb{D})$, which implies

$$F := \frac{D^{3-2\alpha}\varphi}{D^{2-2\alpha}\varphi} \in \mathcal{P}.$$

We want to prove that $f \in \mathcal{R}_\alpha(D)$ or, equivalently, $\psi(z) := \frac{1}{\tau} f(w(z)) \in \mathcal{R}_\alpha(\mathbb{D})$. But we have

$$\psi(z) = \frac{1}{\tau} f\big(w(z)\big) = \frac{1}{\tau} f\big(v\big(\omega(z)\big)\big) = \frac{\tau'}{\tau}\varphi\big(\omega(z)\big),$$

so that, using Lemma 2.2, we obtain

$$\left(\frac{D^{3-2\alpha}\psi}{D^{2-2\alpha}\psi}\right)(z) = \frac{\tau'}{\tau' + (\gamma\tau' - \gamma'\tau)z}\left(\frac{D^{3-2\alpha}\varphi}{D^{2-2\alpha}\varphi}\right)\big(\omega(z)\big),$$

and it remains to show that

$$\frac{\tau'}{\tau' + (\gamma\tau' - \gamma'\tau)z} F\big(\omega(z)\big) \in \mathcal{P}.$$

In view of Lemma 2.1 it will be enough to do so for functions $F(z) = \frac{1}{1-xz}$, $|x| = 1$. Hence we are left with a proof for

$$\frac{\tau'}{\tau' + (\gamma\tau' - \gamma'\tau)z}\,\frac{1}{1 - x\omega(z)} = \frac{1}{1 + \frac{\gamma\tau' - \gamma'\tau - x\tau}{\tau'} z} \in \mathcal{P}.$$

This, however, is an immediate consequence of (2.2). $\square$

Proof of Theorem 1.5. That $f(\Omega)$ is starlike is obvious, since it is the union of starlike disks. Now assume that $f(z_1) = f(z_2)$ for two distinct points $z_1, z_2 \in \Omega$. Let $z_j \in D_j$, $j = 1, 2$, where D_j are maximal disks of Ω. Then $D_1 \neq D_2$ because f is univalent in each maximal disk. The line-segment $[0, f(z_1)]$ has a pre-image γ_j in each of the D_j, connecting 0 with z_j, respectively. However, since f is univalent at least in a neighborhood of the origin, these pre-images must coincide at least there. But they are analytic arcs, and therefore, they have to coincide forever. This implies $z_1 = z_2$, a contradiction to the assumption. f must be univalent in Ω. $\square$

Proof of Lemma 1.3. Clearly $f_x \in \mathcal{H}_1(\Omega)$. Let $\Omega_{\tau,\gamma} \in \mathfrak{D}_0(\Omega)$. Then

$$\frac{1}{\tau} f_x\big(w_{\tau,\gamma}(z)\big) = \frac{z}{1 - (x\tau - \gamma)z}.$$

But, by assumption, $\frac{1}{x} \notin \Omega$, which means, in particular, that $\frac{1}{x} \notin \Omega_{\tau,\gamma}$, or

$$\frac{1}{x} \neq \frac{\tau z}{1 + \gamma z}, \quad z \in \mathbb{D}.$$

This implies that $|\gamma - x\tau| \leqslant 1$. However, it is well known (and easily checked) that the functions $\frac{z}{1-\zeta z}$ with $|\zeta| \leqslant 1$ belong to $\mathcal{R}_\alpha(\mathbb{D})$ for every $\alpha \leqslant 1$. $\square$

3. The multiplier problem

3.1. General facts

In this subsection we verify the remaining claims of Section 1.5.

Proof of Theorem 1.6. Let $g \in \mathcal{M}(\mathcal{R}_\alpha^d(\Omega))$. Then, by Lemma 1.3 and the definition, we find that

$$z \mapsto (g * f_x)(z) = \frac{1}{x} g(xz) \in \mathcal{R}_\alpha^d(\Omega), \quad \frac{1}{x} \notin \Omega.$$

This means that g has to be analytic at all points of the form $u = xz$ with $\frac{1}{x} \notin \Omega$, $z \in \Omega$, i.e. for all $u \in \Omega^*$. □

Proof of Lemma 1.4. Note that $f \in \mathcal{R}_\alpha^d(\sigma\Omega)$ if and only if $z \mapsto f(\sigma z) \in \mathcal{R}_\alpha^d(\Omega)$. Now $g \in \mathcal{M}(\mathcal{R}_\alpha^d(\sigma\Omega))$ if and only if for all $f \in \mathcal{R}_\alpha^d(\sigma\Omega)$ we have $(f * g)(z) \in \mathcal{M}(\mathcal{R}_\alpha^d(\sigma\Omega))$ or, equivalently, $z \mapsto (f * g)(\sigma z) = f(\sigma z) * g(z) \in \mathcal{R}_\alpha^d(\Omega)$. Hence $g \in \mathcal{M}(\mathcal{R}_\alpha^d(\Omega))$. The other direction follows similarly. □

3.2. The Limaçons of Pascal

Recall the description of Ω_γ given at the end of Section 1.5. Then it is easily seen, that for $0 \leqslant \gamma < 1$ we have

$$\Omega_\gamma^* = \bigcup_{|x|=1} \left\{ \frac{(1+\gamma x)z}{(1+\gamma z)x} \colon z \in \mathbb{D} \right\} \tag{3.1}$$

(note that the choices of x with $|x| > 1$ will not further contribute to the set).

Proof of Theorem 1.7. For $0 \leqslant \gamma < 1$ we have to show that

$$\Omega_\gamma^* = \left\{ \frac{(\gamma^2-1)z^2}{(1+\gamma z)^2} \colon z \in \mathbb{D} \right\}. \tag{3.2}$$

We write

$$\frac{z}{1+\gamma z} = \frac{\gamma+\zeta}{\gamma^2-1},$$

and note that $\zeta = \zeta(z)$ is an automorphism of $\mathbb{D}$. Therefore, replacing x by $1/x$ in (3.1), we see that our claim is equivalent to

$$U_\gamma := \left\{ (x+\gamma)(\zeta+\gamma) \colon |\zeta| < 1, \ |x| = 1 \right\} = \left\{ (\zeta+\gamma)^2 \colon |\zeta| < 1 \right\} =: V_\gamma. \tag{3.3}$$

Note that both, Ω_γ^* and $(\Omega_\gamma)^2$, are starlike Jordan domains, so that it will be sufficient to show that the boundary of U_γ is contained in $\overline{V_\gamma}$ and vice versa.

U_γ is the union of the disks

$$\left\{ w \colon \left| w - \gamma\left(e^{i\varphi}+\gamma\right) \right| \leqslant \left| e^{i\varphi}+\gamma \right| \right\} \in \mathfrak{D}_0, \quad \varphi \in [0, 2\pi],$$

so that the boundary of U_γ is contained in the envelope of the boundary circles which are described by

$$f(w,\varphi) := \left| w - \gamma\left(e^{i\varphi}+\gamma\right) \right|^2 - \left| e^{i\varphi}+\gamma \right|^2 = 0. \tag{3.4}$$

The points $w = w(\varphi)$ on the envelope satisfy $f_\varphi(w,\varphi) = 0$, or

$$\operatorname{Im}\left(we^{i\varphi}\right) = \left(\gamma^2-1\right)\sin\varphi,$$

which implies $w = re^{-i\varphi} + \gamma^2 - 1$, with $r \in \mathbb{R}$. Inserting this into (3.4) gives, after some calculation, $r = 2(\gamma + \cos\varphi)$ and then $w = (e^{-i\varphi}+\gamma)^2$. Hence $w \in \overline{V_\gamma}$. That $\partial V_\gamma \subset \overline{U_\gamma}$ is obvious. This proves (3.2).

A direct verification now shows that

$$\Pi_\gamma = \left\{ \frac{\gamma^2-1}{(e^{i\varphi}-\gamma)^2} \colon -\pi \leqslant \varphi \leqslant \pi \right\} \supset \partial\Omega_\gamma^*$$

and

$$\Pi_\gamma^o = \Pi_\gamma \setminus \mathbb{D} = \left\{ \frac{\gamma^2-1}{(e^{i\varphi}-\gamma)^2} \colon |\varphi| \leqslant \arccos\gamma \right\} = \partial\Omega_\gamma^*,$$

since $\mathbb{D} \subset \Omega_\gamma \subset \Omega_\gamma^*$. This also implies (1.16). The statement (1.15) is obvious. □

It is not difficult to see that the disks $\Delta_\gamma(x)$, as described in (1.17), are indeed maximal for Ω^*_γ, and that they are the only ones with that property. Less obvious is the following containment property, which will be needed for the proof of Theorem 1.11.

Lemma 3.1. *For* $0 \leqslant \gamma < \delta \leqslant 1$ *we have* $\Omega^*_\gamma \subset \Omega^*_\delta$.

Proof. Note that $\frac{\gamma+1}{\gamma-1} \in \partial\Omega^*_\gamma$, while the line segment $(\frac{\delta+1}{\delta-1}, 1)$ belongs to Ω^*_δ. Hence there exists a boundary point of Ω^*_γ in Ω^*_δ. We show that the boundaries of these two sets have only one point in common (namely the point 1), which will complete the proof. Recall that $\partial\Omega^*_\gamma = \Pi^o_\gamma$ and $\partial\Omega^*_\delta = \Pi^o_\delta$. Our claim then follows immediately from the definition of these sets in (1.11). □

3.3. About the members of $\mathcal{R}^d_\alpha(\Omega^*_\gamma)$

In this subsection we prove Theorem 1.9 and its Corollary 1.4.

Proof of Theorem 1.9. A function $f \in \mathcal{H}_1(\Omega^*_\gamma)$ belongs to $\mathcal{R}^d_\alpha(\Omega^*_\gamma)$ if it is prestarlike of order α in any maximal disk in Ω^*_γ which are the ones appearing in (3.1). Writing

$$w_x(z) := \frac{(1+\gamma x)z}{(1+\gamma z)x}$$

the condition is

$$\frac{x}{1+\gamma x} f\big(w_x(z)\big) \in \mathcal{R}_\alpha(\mathbb{D}), \quad |x| = 1. \tag{3.5}$$

Using Definition 1.4 and Lemma 2.2 this translates into

$$\frac{D^{3-2\alpha} f(w_x(z))}{D^{2-2\alpha} f(w_x(z))} = \frac{1}{1+\gamma z} \frac{D^{3-2\alpha} f}{D^{2-2\alpha} f}\big(w_x(z)\big) \in \mathcal{P}, \quad |x| = 1,$$

and this is, using Definition 1.7, exactly (1.22). □

We now make use of the general formula (see [5, p. 71])

$$D^{3-2\alpha} f = \frac{1-2\alpha}{2-2\alpha}\big(D^{2-2\alpha} f\big) + \frac{1}{2-2\alpha} z\big(D^{2-2\alpha} f\big)',$$

to obtain from (1.22) the relation

$$\frac{z(D^{2-2\alpha} f)'}{D^{2-2\alpha} f} = (2-2\alpha)\big(F(z)-1\big) + 1$$

for some $F \in \mathcal{P}_\gamma$. This is, after integration, the formula in Corollary 1.4, which is therefore also established.

3.4. Proof of the main theorem

For the proof of Theorem 1.8 we need Theorem 1.17. Therefore we prefer to include its proof at this point.

Proof of Theorem 1.17. This proof is rather indirect. With $F \in \mathcal{P}_\gamma$ we obtain via (1.23) a corresponding function $f \in \mathcal{R}^d_0(\Omega^*_\gamma)$, i.e. a disk-convex function in Ω^*_γ. Note that, for each $z \in \mathbb{D}$ the circle

$$w(\zeta) := \frac{(\zeta+\gamma)z}{1+\gamma z}, \quad |\zeta| = 1,$$

belongs to Ω^*_γ, therefore also

$$\Delta := \big\{w(\zeta)\colon \zeta \in \mathbb{D}\big\} \in \mathfrak{D}_0\big(\Omega^*_\gamma\big).$$

The disk-convex f maps Δ univalently onto a convex domain, which implies

$$\varphi(\zeta) := f\big(w(\zeta)\big), \quad \zeta \in \mathbb{D},$$

is convex univalent. Therefore,

$$0 \leqslant \mathrm{Re}\left(1+\frac{\zeta\varphi''(\zeta)}{\varphi'(\zeta)}\right)=\mathrm{Re}\left(1+\frac{\zeta z}{1+\gamma z}\frac{f''(w(\zeta))}{f'(w(\zeta))}\right)=\mathrm{Re}\left(1+2\frac{\zeta z}{1+\gamma z}\frac{F(w(\zeta))-1}{w(\zeta)}\right),$$

for $\zeta \in \mathbb{D}$. This is (1.30). □

We call a function $H(z,\zeta)$ a *kernel* if, for each r, $0<r<1$, it is analytic in $|z|<r<1$, $|\zeta|<1+\delta(r)$ where $\delta(r)>0$. If H is a kernel then

$$\Phi(f)(z) := H(z,\zeta) *_\zeta f(\zeta)|_{\zeta=1}$$

is a continuous linear operator on $\mathcal{H}(\mathbb{D})$ (equipped with the topology of compact convergence). On the other hand, if Φ is such an operator, then

$$H(z,\zeta) := \Phi\left(\frac{1}{1-\zeta z}\right)$$

is a kernel, compare [9, Section 5.1].

We use the following variant of a standard notation: a function $g \in \mathcal{H}(\mathbb{D})$ with $g(0)=0$, $g'(0)\neq 0$ is called starlike of order $\alpha<1$ ($g \in \mathcal{S}'_\alpha$) if

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)} > \alpha, \quad z \in \mathbb{D}.$$

The following result, for $\alpha<1$, is [7, Theorem 1.5]. It is a generalization of a convolution theorem of Sheil-Small [9, Theorem 5.10]. That it is also true for $\alpha=1$ is an immediate consequence of [9, Theorem 5.9]. A convolution symbol like $*_x$ means that the convolution is with respect to the variable x.

Theorem 3.1. *Let $\alpha \leqslant 1$. For the kernel $H(z,\zeta)$ with its associated operator Φ assume*

$$z \mapsto \frac{1}{\zeta}H(z,\zeta) \in \mathcal{R}_\alpha \quad (\zeta \in \partial\mathbb{D}), \tag{3.6}$$

and

$$\zeta \mapsto \frac{1}{(1-z)^{2-2\alpha}} *_z \frac{1}{z}H(z,\zeta) \in \mathcal{S}'_\alpha \quad (z \in \mathbb{D}). \tag{3.7}$$

Then

$$\Phi(\mathcal{R}_\alpha) \subset \mathcal{R}_\alpha.$$

Proof of Theorem 1.8. (i) Let $f \in \mathcal{M}(\mathcal{R}_\alpha(\Omega_\gamma))$. The functions

$$f_x(z) := \frac{z}{1-\frac{1+\gamma x}{(\gamma+1)x}z}, \quad |x|=1,$$

belong to $\mathcal{R}_\alpha(\Omega_\gamma)$, so that we must have

$$z \mapsto \frac{(\gamma+1)x}{1+\gamma x} f\left(\frac{(1+\gamma x)z}{(\gamma+1)x}\right) \in \mathcal{R}_\alpha(\Omega_\gamma), \quad |x|=1,$$

and therefore

$$z \mapsto \frac{x}{1+\gamma x} f\left(\frac{(1+\gamma x)z}{(1+\gamma z)x}\right) \in \mathcal{R}_\alpha(\mathbb{D}), \quad |x|=1. \tag{3.8}$$

But this is exactly the necessary and sufficient condition for f to belong to $\mathcal{R}^d_\alpha(\Omega^*_\gamma)$, see (3.5). This settles one direction of our claim.

(ii) Now assume $f \in \mathcal{R}^d_\alpha(\Omega^*_\gamma)$ and let $F \in \mathcal{R}_\alpha(\Omega_\gamma)$ be arbitrary. This implies that $\varphi(z) := \frac{1}{1+\gamma}F(w(z)) \in \mathcal{R}_\alpha$, where $w=w(z)=\frac{(1+\gamma)z}{1+\gamma z}$, and therefore

$$F(w) = (1+\gamma)\varphi\left(\frac{w}{1+\gamma-\gamma w}\right).$$

Note that

$$\frac{1}{1-\frac{w}{1+\gamma-\gamma w}\zeta}=1+\frac{\zeta}{\gamma+\zeta}\frac{w\frac{\gamma+\zeta}{1+\gamma}}{1-w\frac{\gamma+\zeta}{1+\gamma}}.$$

We now write

$$\begin{aligned}\frac{1}{1+\gamma}\big(f(w)*_w F(w)\big)&=f(w)*_w\varphi\left(\frac{w}{1+\gamma-\gamma w}\right)\\&=\left\{f(w)*_w\varphi(\zeta)*_\zeta\frac{1}{1-\frac{w}{1+\gamma-\gamma w}\zeta}\right\}\bigg|_{\zeta=1}\\&=\left\{f(w)*_w 1+\frac{\zeta}{\gamma+\zeta}\frac{w\frac{\gamma+\zeta}{1+\gamma}}{1-w\frac{\gamma+\zeta}{1+\gamma}}*_\zeta\varphi(\zeta)\right\}\bigg|_{\zeta=1}\\&=\left\{\frac{\zeta}{\gamma+\zeta}f\left(w\frac{\gamma+\zeta}{1+\gamma}\right)*_\zeta\varphi(\zeta)\right\}\bigg|_{\zeta=1},\end{aligned}$$

and this implies

$$\frac{1}{1+\gamma}\big(f(w)*_w F(w)\big)\big|_{w=w(z)}=\left\{\frac{\zeta}{\gamma+\zeta}f\left(\frac{(\zeta+\gamma)z}{1+\gamma z}\right)*_\zeta\varphi(\zeta)\right\}\bigg|_{\zeta=1}=\big\{H(z,\zeta)*_\zeta\varphi(\zeta)\big\}\big|_{\zeta=1},$$

where

$$H(z,\zeta):=\frac{\zeta}{\gamma+\zeta}f\left(\frac{(\zeta+\gamma)z}{1+\gamma z}\right)$$

is a kernel for an operator Φ as described above. Our proof is complete once we can establish (3.6) and (3.7) for this kernel H.

Condition (3.6) is exactly (3.8), with $x=1/\zeta$. The other condition requires a bit more work.

We recall Corollary 1.4: for our $f\in\mathcal{R}^d_\alpha(\Omega^*_\gamma)$ we have a function $g\in\mathcal{P}_\gamma$ such that

$$\frac{z}{(1-z)^{2-2\alpha}}*f(z)=ze^{G(z)},$$

where

$$G(z):=(2-2\alpha)\int_0^z\frac{g(t)-1}{t}\,dt.$$

Let $\sigma\in\mathcal{H}_1(\mathbb{D})$ be the function with the property

$$\sigma(z)*_z\frac{z}{(1-z)^{2-2\alpha}}=\frac{z}{1-z},$$

i.e. the inverse in $\mathcal{H}_1(\mathbb{D})$ with respect to convolution.

We then have

$$\begin{aligned}Q(\zeta)&:=\frac{1}{(1-z)^{2-2\alpha}}*_z\frac{\zeta}{z(\zeta+\gamma)}f\left(\frac{(\zeta+\gamma)z}{1+\gamma z}\right)\\&=\frac{1}{(1-z)^{2-2\alpha}}*_z\frac{\zeta}{z(\zeta+\gamma)}\left(f(w)*_w\frac{w\frac{(\gamma+\zeta)z}{1+\gamma z}}{1-w\frac{(\gamma+\zeta)z}{1+\gamma z}}\right)\bigg|_{w=1}\\&=\frac{1}{(1-z)^{2-2\alpha}}*_z\left(\frac{\zeta w}{1-((\zeta+\gamma)w-\gamma)z}*_w\sigma(w)*_w we^{G(w)}\right)\bigg|_{w=1}\\&=\frac{\zeta w}{(1-((\zeta+\gamma)w-\gamma)z)^{2-2\alpha}}*_w\sigma(w)*_w we^{G(w)}\bigg|_{w=1}\\&=\frac{\zeta}{(1+\gamma z)^{2-2\alpha}}\frac{w}{(1-\frac{(\zeta+\gamma)z}{1+\gamma z}w)^{2-2\alpha}}*_w\sigma(w)*_w we^{G(w)}\bigg|_{w=1}\end{aligned}$$

$$= \frac{\zeta}{(1+\gamma z)^{2-2\alpha}} \frac{w}{1-\frac{(\zeta+\gamma)z}{1+\gamma z}w} *_w we^{G(w)}\bigg|_{w=1}$$
$$= \frac{\zeta}{(1+\gamma z)^{2-2\alpha}} \exp G\left(\frac{(\zeta+\gamma)z}{1+\gamma z}\right).$$

Therefore,

$$\operatorname{Re}\frac{\zeta Q'(\zeta)}{Q(\zeta)} = \operatorname{Re}\left[1+G'\left(\frac{(\zeta+\gamma)z}{1+\gamma z}\right)\frac{\zeta z}{1+\gamma z}\right] = \operatorname{Re}\left[1+(2-2\alpha)\frac{\zeta z}{1+\gamma z}\frac{g(\frac{(\zeta+\gamma)z}{1+\gamma z})-1}{\frac{(\zeta+\gamma)z}{1+\gamma z}}\right] > \alpha,$$

where the last estimate follows immediately from Theorem 1.17. This implies $Q'(0) \neq 0$ and therefore $Q \in \mathcal{S}'_\alpha$. We conclude that

$$f(w) *_w F(w)|_{w=w(z)} \in \mathcal{R}_\alpha,$$

and since Ω_γ has only one maximal disk (namely itself) we have finally shown that $f \in \mathcal{M}(\mathcal{R}^d_\alpha(\Omega_\gamma))$. Since $1 \in \partial\Omega_\gamma$ implies $\Omega_\gamma \subset \Omega^*_\gamma$, and therefore

$$\mathcal{R}^d_\alpha(\Omega^*_\gamma) \subset \mathcal{R}^d_\alpha(\Omega_\gamma) = \mathcal{R}_\gamma(\Omega_\gamma),$$

the last part of (1.19) has also been established. □

3.5. Proofs for Section 1.8

Most of the results in this subsection are more or less obvious after the preparations from before. To see (1.28) we note that $\Omega \subset \frac{1+\gamma}{1-\gamma}\mathbb{D}$, and this, together with Theorem 1.7 gives

$$\Omega^*_\gamma = \frac{1-\gamma}{1+\gamma}(\Omega_\gamma)^2 \subset \frac{1-\gamma}{1+\gamma}\left(\frac{1+\gamma}{1-\gamma}\mathbb{D}\right)^2 = \frac{1+\gamma}{1-\gamma}\mathbb{D}.$$

Theorem 1.11 follows from Lemma 3.1 and Theorem 1.12 combines Theorem 1.11 and Lemma 1.4. Corollary 1.6 is a combination of Theorems 1.10 and 1.12.

4. Proofs for Section 1.9

We first prove (1.24). We have to show

$$\operatorname{Re}\left(\frac{1}{1+\gamma z}\frac{1}{1-w\frac{(x+\gamma)z}{1+\gamma z}}\right) > \frac{1}{2}, \quad |z|<1,\ |x|=1,\ w \in \overline{L_\gamma}.$$

Eliminating z from this expression we are left with

$$\left|\gamma(1-w) - xw\right| \leqslant 1, \quad |x|=1,\ w \in \overline{L_\gamma}.$$

After elimination of x we obtain as the final condition

$$\gamma|1-w| + |w| \leqslant 1, \quad w \in \overline{L_\gamma},$$

which follows from (1.18), with equality for all admissible choices for γ and $w \in \partial L_\gamma$. □

Proof of Theorem 1.14. From (1.16) and (1.13) we see that $\Pi^i_\gamma = \partial L_\gamma$, and this implies $B_\gamma = \overline{L_\gamma}$.

Now let $F(z) = 1 + az + \cdots \in \mathcal{P}_\gamma$. Inserting this into the definition of $\mathcal{P}_\gamma$ we get

$$g_x(z) := \frac{1}{1+\gamma z}F\left(\frac{(1+\gamma x)z}{(1+\gamma z)x}\right) = 1 + \left(a\frac{1+\gamma x}{x} - \gamma\right)z + \cdots \in \mathcal{P}, \quad |x|=1,$$

and, since $|g'(0)| \leqslant 1$ for all $g \in \mathcal{P}$ we obtain

$$|a| + \gamma|a-1| \leqslant 1. \tag{4.1}$$

Thus the possible range of a is exactly $\overline{L_\gamma}$. Since $B_\gamma \subset A_\gamma$ we finally conclude that $A_\gamma = B_\gamma$, which completes this proof. □

Proof of Theorem 1.13. We use the notations of the previous proof. If we have equality in (4.1) then, for some $|x| = 1$, the function g_x must be an extreme point of $\mathcal{P}$, i.e. $g_x(z) = (1 - \zeta z)^{-1}$ for some suitable ζ with $|\zeta| = 1$. A direct calculation shows that in this case $F(z)$ must be of the form $(1 - \omega z)^{-1}$ where

$$\omega = \frac{(1 + \gamma\rho)x}{(1 + \gamma x)\rho}, \quad \rho := -\bar{\zeta}.$$

Since $|x| = |\rho| = 1$ this implies that both, ω and $1/\omega$, belong to $\overline{\Omega_\gamma^*}$. However, since $F \in \mathcal{H}(\Omega_\gamma^*)$, we must also have $1/\omega \neq z$ for $z \in \Omega_\gamma^*$ and therefore $1/\omega \in \partial\Omega_\gamma^* = \Pi_\gamma^o$ i.e. $\omega \in \Pi_\gamma^i$ so that $F = E_{\gamma,\omega}$ as asserted.

In other words, the only functions $F \in \mathcal{P}_\gamma$ for which $F'(0)$ is on the boundary of the coefficient body A_γ are the functions $E_{\gamma,t}$. Now let φ be real and $\lambda_\varphi : \mathcal{H}(\mathbb{D}) \mapsto \mathbb{R}$ the continuous linear functional with $\lambda_\varphi(F) := \mathrm{Re}(e^{i\varphi} F'(0))$. It is now clear that

$$\max_{F \in \mathcal{P}_\gamma} \lambda_\varphi(F) = \lambda_\varphi(F_0),$$

for exactly one function $F_0 \in \mathcal{P}_\gamma$, namely for $F_0 = E_{\gamma,w}$ with a unique $w = w(\varphi) \in \partial L_\gamma$. This shows that this $E_{\gamma,w}$ is the unique support point for λ_φ in $\mathcal{P}_\gamma$, and therefore an extreme point. Since this is true for every real φ, it is clear that each of the functions $E_{\gamma,w}$, $w \in \partial L_\gamma$, is an extreme point of $\mathcal{P}_\gamma$. □

The remaining Theorems 1.15 and 1.16 are immediate consequences of the fact that $\mathcal{P}_\gamma = \mathcal{R}_1^d(\Omega_\gamma^*)/z$ and Theorems 1.3 and 1.8.

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