PRINCIPLES OF MATHEMATICAL ANALYSIS.

WALTER RUDIN

10. Integration of Differential Forms

1. Let H be a compact convex set in $\mathbb{R}^k$ with nonempty interior. Let $f \in \mathcal{C}(H)$, put $f(\mathbf{x}) = 0$ in the complement of H, and define $\int_H f$ as in Definition 10.3.

   Prove that $\int_H f$ is independent of the order in which the k integrations are carried out. *Hint:* Approximate f by functions that are continuous on $\mathbb{R}^k$ and whose supports are in H, as was done in Example 10.4.

2. For $i = 1, 2, 3, \dots$, let $\varphi_i \in \mathcal{C}(\mathbb{R}^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\, \varphi_i(y).$$

   Then f has compact support in $\mathbb{R}^2$, f is continuous except at $(0, 0)$, and

$$\int f(x, y)\, dx\, dy = 0 \quad \text{but} \quad \int f(x, y)\, dy\, dx = 1.$$

   Observe that f is unbounded in every neighbourhood of $(0, 0)$.

3. (a) If F is as in Theorem 10.7, put $\mathbf{A} = \mathbf{F}'(0)$, $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$. Then $\mathbf{F}_1'(\mathbf{0}) = I$. Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \dots \circ \mathbf{G}_1(\mathbf{x})$$

   in some neighbourhood of $\mathbf{0}$, for certain primitive mappings $\mathbf{G}_1, \dots, \mathbf{G}_n$. This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(0)\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \dots \circ \mathbf{G}_1(\mathbf{x}).$$

   (b) Prove that the mapping $(x, y) \mapsto (y, x)$ of $\mathbb{R}^2$ onto $\mathbb{R}^2$ is not the composition of any two primitive mappings, in any neighbourhood of the origin. (This shows that the flips B_i cannot be omitted from the statement of Theorem 10.7.)

4. For $(x, y) \in \mathbb{R}^2$, define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y).$$

May 20, 2005. Solutions by Erin P. J. Pearse.

◇ Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\mathbf{G}_1(x,y) = (e^x \cos y - 1, y)$$
$$\mathbf{G}_2(u,v) = (u, (1+u)\tan v)$$

are primitive in some neighbourhood of $(0,0)$.

$$\begin{aligned}\mathbf{G}_2 \circ \mathbf{G}_1(x,y) &= \mathbf{G}_2(e^x \cos y - 1, y) \\ &= \left(e^x \cos y - 1, (e^x \cos y)\frac{\sin y}{\cos y}\right) \\ &= (e^x \cos y - 1, e^x \sin y) \\ &= \mathbf{F}(x,y).\end{aligned}$$

$\mathbf{G}_1(x,y) = (g_1(x,y), y)$ and $\mathbf{G}_2(u,v) = (u, g_2(u,v)$ are clearly primitive.

◇ Compute the Jacobians of $\mathbf{G}_1, \mathbf{G}_2, \mathbf{F}$ at $(0,0)$.

$$\begin{aligned}J_{\mathbf{G}_1}(\mathbf{0}) = \ \det \mathbf{G}_1'(\mathbf{x})\Big|_{\mathbf{x}=\mathbf{0}} &= \left| \begin{matrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{matrix} \right|_{(x,y)=(0,0)} \\ &= e^x \cos y\Big|_{(x,y)=(0,0)} \\ &= 1 \cdot 1 = 1.\end{aligned}$$

$$\begin{aligned}J_{\mathbf{G}_2}(\mathbf{0}) &= \left| \begin{matrix} 1 & 0 \\ \tan v & (1+u)\sec^2 v \end{matrix} \right|_{(x,y)=(0,0)} \\ &= \ (1+u)\sec^2 v\Big|_{(x,y)=(0,0)} \\ &= 1 \cdot 1^2 = 1.\end{aligned}$$

$$\begin{aligned}J_{\mathbf{F}}(\mathbf{0}) &= \left| \begin{matrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{matrix} \right|_{(x,y)=(0,0)} \\ &= \ e^{2x}\cos^2 y + e^{2x}\sin^2 y\Big|_{(x,y)=(0,0)} \\ &= \ e^{2x}\left(\cos^2 y + \sin^2 y\right)\Big|_{(x,y)=(0,0)} \\ &= \ e^{2x}\Big|_{(x,y)=(0,0)} \\ &= 1.\end{aligned}$$

◇ Define

$$\mathbf{H}_2(x,y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u,v) = (h(u,v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ in some neighbourhood of $(0,0)$.
We have

$$\begin{aligned}\mathbf{H}_1 \circ \mathbf{H}_2(x,y) &= \mathbf{H}_1(x, e^x \sin y)\\ &= (h(x, e^x \sin y), e^x \sin y)\\ &= (e^x \cos y - 1, e^x \sin y)\end{aligned}$$

for $h(u,v) = e^u \cos(\arcsin(e^{-u}v)) - 1$, valid in the neighbourhood $\mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of $(0,0)$. Indeed, for $(u,v) = (x, e^x \sin y)$,

$$\begin{aligned}e^u \cos\left(\arcsin(e^{-u}v)\right) - 1 &= e^x \cos\left(\arcsin(e^{-x}e^x \sin y)\right) - 1\\ &= e^x \cos(\arcsin(\sin y)) - 1\\ &= e^x \cos y - 1.\end{aligned}$$

5. Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 22 of Chap. 4.)

6. Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, even smooth (infinitely differentiable). (Use Exercise 1 of Chap. 8 in the construction of the auxiliary functions φ_i.)

7. (a) Show that the simplex Q^k is the smallest convex subset of $\mathbb{R}^k$ that contains the points $\mathbf{0} = \mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_k$.

We first show that Q^k is the convex hull of the points $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$, i.e.,

$$\mathbf{x} \in Q^k \quad \iff \quad \mathbf{x} = \sum_{i=0}^{k} c_i \mathbf{e}_i, \quad \text{with } 0 \le c_i \le 1, \sum_{k=0}^{k} c_i = 1.$$

($\Rightarrow$) Consider $\mathbf{x} = (x_1, \dots, x_k) \in Q^k$. Let $c_i = x_i$ for $1 = 1, \dots, k$ and define $c_0 = 1 - \sum_{i=1}^{k} x_i$. Then $c_i \in [0,1]$ for $i = 0, 1, \dots, k$ and $\sum_{i=0}^{k} c_i = 1$. Moreover,

$$\begin{aligned}\sum_{i=0}^{k} c_i \mathbf{e}_i &= \left(1 - \sum_{i=1}^{k} x_i\right) \mathbf{e}_0 + \sum_{i=1}^{k} c_i \mathbf{e}_i \\ &= \left(1 - \sum_{i=1}^{k} x_i\right) \mathbf{0} + \sum_{i=1}^{k} x_i \mathbf{e}_i = (x_1, \dots, x_k) = \mathbf{x}.\end{aligned}$$

($\Leftarrow$) Now given $\mathbf{x}$ as a convex combination $\mathbf{x} = \sum_{i=0}^{k} c_i \mathbf{e}_i$, write $\mathbf{x} = (c_1, \dots, c_k)$. Then $0 \le c_i \le 1$ and

$$\sum_{i=0}^{k} c_i = 1 \quad \Longrightarrow \quad \sum_{i=1}^{k} c_i \le 1,$$

so $x \in Q^k$.
Now if K is a convex set containing the points $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$, convexity implies it must also contain all points $\mathbf{x} = \sum_{i=0}^{k} c_i \mathbf{e}_i$, i.e., it must contain all of Q^k.

(b) Show that affine mappings take convex to convex sets.

An affine mapping T may always be represented as the composition of a linear mapping S followed by a translation B. Translations (as congruences) obviously preserve convexity, so it suffices to show that linear mappings do.
Let S be a linear mapping. If we have a convex set K with $\mathbf{u}, \mathbf{v} \in K$, then any point between $\mathbf{u}, \mathbf{v}$ is $(1-\lambda)\mathbf{u} + \lambda \mathbf{v}$ for some $\lambda \in [0,1]$, and in the image of S, the linearity of S gives

$$S\left((1-\lambda)\mathbf{u} + \lambda \mathbf{v}\right) = (1-\lambda) S(\mathbf{u}) + \lambda S(\mathbf{v}),$$

which shows that any point between $\mathbf{u}$ and $\mathbf{v}$ gets mapped to a point between $S(\mathbf{u})$ and $S(\mathbf{v})$, i.e., convexity is preserved.

8. Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1),(3,2),(4,5),(2,4)$.

   ◇ Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, and $(0,1)$ to $(2,4)$.

   Since $T = B \circ S$, where S is linear and B is a translation, let $B(\mathbf{x}) = \mathbf{x} + (1,1)$ and find S which takes I^2 to $(0,0),(2,1),(3,4),(1,3)$. Once we fix a basis (let's use the standard basis in $\mathbb{R}^k$), then S corresponds to a unique matrix A. Then we can define S in terms of its action on the basis of $\mathbb{R}^2$:

$$A = \begin{bmatrix} | & | \\ S(\mathbf{e}_1) & S(\mathbf{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

   Then $S(\mathbf{e}_1) = S(1,0) = (2,1)^T$, and $S(\mathbf{e}_2) = S(0,1) = (1,3)^T$, as you can check. Now for $\mathbf{b} = (1,1)$, we have

$$T(\mathbf{x}) = B \circ S(\mathbf{x}) = A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 + 1 \\ x_1 + 3x_2 + 1 \end{bmatrix}.$$

   ◇ Show that $J_T = 5$.

   Let $T_1(x_1,x_2) = 2x_1 + x_2 + 1$ and $T_2(x_1,x_2) = x_1 + 3x_2 + 1$.

$$J_T = \begin{vmatrix} \frac{d}{dx_1}T_1 & \frac{d}{dx_2}T_1 \\ \frac{d}{dx_1}T_2 & \frac{d}{dx_2}T_2 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5.$$

   ◇ Use T to convert the integral

$$\alpha = \int_H e^{x-y}\,dx\,dy$$

   to an integral over I^2 and thus compute α.

   T satisfies all requirements of Thm 10.9, so we have

$$\begin{aligned} \alpha = \int_H e^{x-y}\,d\mathbf{y} &= \int_{I^2} e^{(2x_1+x_2+1)-(x_1+3x_2+1)}|5|\,d\mathbf{x} \\ &= 5\int_0^1\int_0^1 e^{x_1-2x_2}\,dx_1\,dx_2 \\ &= 5\left(\int_0^1 e^{x_1}\,dx_1\right)\left(\int_0^1 e^{-2x_2}\,dx_2\right) \\ &= 5\left[e^{x_1}\right]_0^1\left[-\tfrac{1}{2}e^{-2x_2}\right]_0^1 \\ &= 5(e^1-1)(-\tfrac{1}{2}e^{-2}+\tfrac{1}{2}) \\ &= \tfrac{5}{2}(e^1-1)(1-e^{-2}) \\ &= \tfrac{5}{2}(e^1-1-e^{-1}+e^{-2}). \end{aligned}$$

9. Define $(x, y) = T(r, \theta)$ on the rectangle

$$0 \le r \le a, \qquad 0 \le \theta \le 2\pi$$

by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Show that T maps this rectangle onto the closed disc D with center at $(0, 0)$ and radius a, that T is one-to-one in the interior of the rectangle, and that $J_T(r, \theta) = r$. If $f \in \mathcal{C}(D)$, prove the formula for integration in polar coordinates:

$$\int_D f(x, y)\, dx\, dy = \int_0^a \int_0^{2\pi} f(T(r, \theta))\, r dr\, d\theta.$$

Hint: Let D_0 be the interior of D, minus the segment from $(0, 0)$ to $(0, a)$. As it stands, Theorem 10.9 applies to continuous functions f whose support lies in D_0. To remove this restriction, proceed as in Example 10.4.

12. Let I^k be the unit cube and Q^k be the standard simplex in $\mathbb{R}^k$; i.e.,

$$I^k = \{\mathbf{u} = (u_1, \dots, u_k) \,\vdots\, 0 \le u_i \le 1, \forall i\}, \text{ and}$$
$$Q^k = \{\mathbf{x} = (x_1, \dots, x_k) \,\vdots\, x_i \ge 0, \textstyle\sum x_i \le 1.\}$$

Define $\mathbf{x} = T(\mathbf{u})$ by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1-u_1)u_2 \\ &\vdots \\ x_k &= (1-u_1)\dots(1-u_{k-1})u_k. \end{aligned}$$

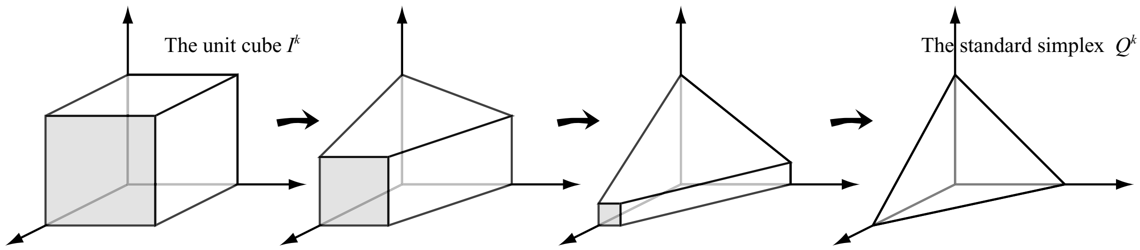

FIGURE 1. The transform T, "in slow motion".

◇ Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1-u_i).$$

This computation is similar to (28)–(30), in Partitions of Unity, p.251. By induction, we begin with the (given) basis step $x_1 = u_1$, then assume

$$\sum_{i=1}^j x_i = 1 - \prod_{i=1}^j (1-u_i), \qquad \text{for some } j < k.$$

Using this assumption immediately, we can write

$$\begin{aligned} \textstyle\sum_{i=1}^{j+1} x_i &= \textstyle\sum_{i=1}^{j} x_i + x_{j+1} \\ &= \left(1 - \textstyle\prod_{i=1}^j (1-u_i)\right) + \left((1-u_1)\dots(1-u_{j-1})u_j\right) \\ &= 1 - (1-u_1)\dots(1-u_j) + (1-u_1)\dots(1-u_j)u_{j+1} \\ &= 1 - \left((1-u_1)\dots(1-u_j) - (1-u_1)\dots(1-u_j)u_{j+1}\right) \\ &= 1 - (1-u_1)\dots(1-u_j)(1-u_{j+1}) \\ &= 1 - \textstyle\prod_{i=1}^{j+1}(1-u_i). \end{aligned}$$

◇ Show that T maps I^k onto Q^k, that T is 1-1 in the interior of I^k, and that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for $i = 1, 2, \ldots, k$.

Note that T is a continuous map (actually, smooth) because each of its component functions is a polynomial.

First, we show that $T(\partial I^k) = \partial Q^k$, so that the boundary of I^k is mapped onto the boundary of Q^k. Injectivity and surjectivity for the interior will be given by the existence of the inverse function.

Let $\mathbf{u} \in \partial I^k$. Then for at least one u_j, either $u_j = 0$ or $u_j = 1$. $u_j = 0$ implies $x_j = 0$ so that $\mathbf{x} \in E_j \subseteq \partial Q^k$. Meanwhile, $u_j = 1$ implies $\sum x_i = 1 - \prod(1 - u_i) = 1 - 0 = 1$, which puts $\mathbf{x} \in E_0 \subseteq \partial Q^k$. Therefore, $T(\partial I^k) \subseteq \partial Q^k$.

Let $\mathbf{x} \in \partial Q^k$. Then either $x_j = 0$ for some $1 \leq j \leq k$ (because $x \in E_j$) or else $\sum x_j = 1$ (because $x \in E_0$), or both.

case (i) Let $x_j = 0$ so $\mathbf{x} = (x_1, \ldots, 0, \ldots, x_k)$, and take $\mathbf{x}_\varepsilon = (x_1, \ldots, \varepsilon, \ldots, x_k)$, so $\mathbf{x}_\varepsilon \in (Q^k)^\circ$. We have

$$\begin{aligned} T^{-1}(\mathbf{x}_\varepsilon) &= \left(x_1, \tfrac{x_2}{1-x_1}, \tfrac{x_3}{1-x_1-x_2}, \ldots, \tfrac{x_j}{1-x_1-\cdots-x_{j-1}}, \tfrac{x_{j+1}}{1-x_1-\cdots-x_j}, \ldots, \tfrac{x_k}{1-x_1-\cdots-x_{k-1}}\right) \\ &= \left(x_1, \tfrac{x_2}{1-x_1}, \tfrac{x_3}{1-x_1-x_2}, \ldots, \tfrac{\varepsilon}{1-x_1-\cdots-x_{j-1}}, \tfrac{x_{j+1}}{1-x_1-\cdots-x_{j-1}-\varepsilon}, \ldots, \tfrac{x_k}{1-x_1-\cdots-x_{k-1}}\right) \\ &\xrightarrow{\varepsilon \to 0} \left(x_1, \tfrac{x_2}{1-x_1}, \tfrac{x_3}{1-x_1-x_2}, \ldots, 0, \tfrac{x_{j+1}}{1-x_1-\cdots-x_{j-1}}, \ldots, \tfrac{x_k}{1-x_1-\cdots-x_{k-1}}\right) \end{aligned}$$

where the last step uses the continuity guaranteed by the Inverse Function Theorem. Now check that for

$$\mathbf{u} = \left(x_1, \tfrac{x_2}{1-x_1}, \tfrac{x_3}{1-x_1-x_2}, \ldots, 0, \tfrac{x_{j+1}}{1-x_1-\cdots-x_{j-1}}, \ldots, \tfrac{x_k}{1-x_1-\cdots-x_{k-1}}\right),$$

we do in fact have $T(\mathbf{u}) = \mathbf{x}$:

$$\begin{aligned} &T\left(x_1, \tfrac{x_2}{1-x_1}, \tfrac{x_3}{1-x_1-x_2}, \ldots, 0, \tfrac{x_{j+1}}{1-x_1-\cdots-x_{j-1}}, \ldots, \tfrac{x_k}{1-x_1-\cdots-x_{k-1}}\right) \\ &= \Big(x_1, (1-x_1)\tfrac{x_2}{1-x_1}, (1-x_1)(1-\tfrac{x_2}{1-x_1})\tfrac{x_3}{1-x_1-x_2}, \\ &\qquad \ldots, (1-x_1)(1-\tfrac{x_2}{1-x_1})\ldots(1-\tfrac{x_{j-2}}{1-x_1-\cdots-x_{j-3}})\tfrac{x_{j-1}}{1-x_1-\cdots-x_{j-2}}, \\ &\qquad\quad (1-x_1)(1-\tfrac{x_2}{1-x_1})\ldots(1-\tfrac{x_{j-1}}{1-x_1-\cdots-x_{j-2}})\tfrac{x_j}{1-x_1-\cdots-x_{j-1}}, \\ &\qquad\quad (1-x_1)(1-\tfrac{x_2}{1-x_1})\ldots(1-\tfrac{x_j}{1-x_1-\cdots-x_{j-1}})\tfrac{x_{j+1}}{1-x_1-\cdots-x_j}, \\ &\qquad \ldots, (1-x_1)(1-\tfrac{x_2}{1-x_1})\ldots(1-\tfrac{x_{k-1}}{1-x_1-\cdots-x_{k-2}})\tfrac{x_k}{1-x_1-\cdots-x_{k-1}}\Big) \\ &= (x_1, x_2, x_3, \ldots x_{j-1}, 0, (1-x_1-x_2-\cdots-x_{j-1})\tfrac{x_{j+1}}{1-x_1-\cdots-x_j}, \ldots, x_k) \\ &= (x_1, x_2, x_3, \ldots x_{j-1}, 0, x_{j+1}, \ldots, x_k) \end{aligned}$$

case (ii) $\sum_{i=1}^k x_i = 1$.

Then by the previous part, $1 - \prod_{i=1}^k (1-u_i) = 1$, so $\prod_{i=1}^k (1-u_i) = 0$. This can only happen if one of the factors is 0, i.e., if one of the u_i is 1. This puts $u \in \partial I^k$.

For the interior, we have an inverse.

$$\begin{aligned}
S \circ T(\mathbf{u}) &= S(u_1, (1-u_1)u_2, \dots, (1-u_1)(1-u_2)\dots u_k) \\
&= \left(u_1, \frac{(1-u_1)u_2}{1-u_1}, \dots, \frac{(1-u_1)(1-u_2)\dots(1-u_{k-1})u_k}{1-u_1-(1-u_1)u_2-\dots-(1-u_1)(1-u_2)\dots u_{k-1}} \right) \\
&= (u_1, u_2, \dots, u_k) \\
&= \mathbf{u}.
\end{aligned}$$

where the third equality follows by successive distributions against the last factor:

$$\begin{aligned}
&[(1-u_1)(1-u_2)\dots(1-u_{k-2})]\,(1-u_{k-1}) \\
&\qquad = (1-u_1)(1-u_2)\dots(1-u_{k-2}) - (1-u_1)(1-u_2)\dots(1-u_{k-2})u_{k-1}.
\end{aligned}$$

Injectivity may also be shown directly as follows: let $\mathbf{u}$ and $\mathbf{v}$ be distinct points of $(I^k)^\circ$, with $T(\mathbf{u}) = \mathbf{x}$ and $T(\mathbf{v}) = \mathbf{y}$. We want to show $\mathbf{x} \neq \mathbf{y}$. From the previous part, we have $T^{-1}(\partial Q^k) \subseteq \partial I^k$, so $\mathbf{x}, \mathbf{y} \in (Q^k)^\circ$. Let the j^{th} coordinate be the first one in which $\mathbf{u}$ differs from $\mathbf{v}$, so that $u_i = v_i$ for $i < j$, but $u_j \neq v_j$. Then

$$\begin{aligned}
x_j &= (1-u_1)\dots(1-u_{j-1})u_j \\
&= (1-v_1)\dots(1-v_{j-1})u_j \\
&\neq (1-v_1)\dots(1-v_{j-1})v_j = y_j.
\end{aligned}$$

So $\mathbf{x} \neq \mathbf{y}$.

◇ Show that

$$J_T(\mathbf{u}) = (1-u_1)^{k-1}(1-u_2)^{k-2}\dots(1-u_{k-1}),$$

and

$$J_S(\mathbf{x}) = \left[(1-x_1)(1-x_1-x_2)\dots(1-x_1-\dots-x_{k-1})\right]^{-1}.$$

$$\begin{aligned}
J_T(\mathbf{u}) &= \begin{vmatrix}
1 & 0 & 0 & \dots & 0 \\
-u_2 & 1-u_1 & 0 & \dots & 0 \\
-(1-u_2)u_3 & -(1-u_1)u_3 & (1-u_1)(1-u_2) & \dots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
-(1-u_2)\dots u_k & \dots & \dots & \dots & (1-u_1)\dots(1-u_{k-1})
\end{vmatrix} \\
&= 1 \cdot (1-u_1) \cdot (1-u_1)(1-u_2) \cdot \dots \cdot (1-u_1)(1-u_2)\dots(1-u_{k-1}) \\
&= (1-u_1)^{k-1}(1-u_2)^{k-2}\dots(1-u_{k-1}).
\end{aligned}$$

$$J_S(\mathbf{u}) = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{x_2}{(1-x_1)^{-2}} & \frac{1}{1-x_1} & 0 & \dots & 0 \\ \frac{x_3}{(1-x_1-x_2)^{-2}} & \frac{x_3}{(1-x_1-x_2)^{-2}} & \frac{1}{1-x_1-x_2} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{x_k}{(1-x_1-\dots-x_{k-1})^{-2}} & \frac{x_k}{(1-x_1-\dots-x_{k-1})^{-2}} & \frac{x_k}{(1-x_1-\dots-x_{k-1})^{-2}} & \dots & \frac{1}{1-x_1-\dots-x_{k-1}} \end{vmatrix}$$
$$= 1 \cdot \frac{1}{1-x_1} \cdot \frac{1}{1-x_1-x_2} \cdot \dots \cdot \frac{1}{1-x_1-\dots-x_{k-1}}$$
$$= [(1-x_1)(1-x_1-x_2)\dots(1-x_1-\dots-x_{k-1})]^{-1}$$

13. Let $r_1, \dots r_k$ be nonnegative integers, and prove that

$$\int_{Q_k} x_1^{r_1} \dots x_k^{r_k} dx = \frac{r_1! \dots r_k!}{(k + r_1 + \dots + r_k)!}.$$

Hint: Use Exercise 12, Theorems 10.9 and 8.20. Note that the special case $r_1 = \dots = r_k = 0$ shows that the volume of Q^k is $1/k!$.

Theorem 8.20 gives that for $\alpha, \beta > 0$ we have

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Theorem 10.9 says when T is a 1-1 mapping of an open set $E \subseteq \mathbb{R}^k$ into $\mathbb{R}^k$ with invertible Jacobian, then

$$\int_{\mathbb{R}^k} f(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^k} f(T(\mathbf{x}))|J_T(\mathbf{x})|\, d\mathbf{x}$$

for any $f \in \mathcal{C}(E)$. Since we always have $\int_{cl(E)} f = \int_{int(E)} f$, we can apply this theorem to the T given in the previous exercise:

$$\begin{aligned}
\int_{Q^k} x_1^{r_1} \dots x_k^{r_k} d\mathbf{x} &= \int_{I^k} T_1(\mathbf{u})^{r_1} T_2(\mathbf{u})^{r_2} \dots T_k(\mathbf{u})^{r_k} |J_T(\mathbf{u})|, d\mathbf{u} \\
&= \int_{I^k} u_1^{r_1} \left[(1-u_1)u_2\right]^{r_2} \dots \left[(1-u_1)\dots(1-u_{k-1})u_k\right]^{r_k} \\
&\qquad \cdot \left[(1-u_1)^k (1-u_2)^{k-1} \dots (1-u_{k-1})\right]\, d\mathbf{u} \\
&= \int_{I^k} u_1^{r_1} u_2^{r_2} \dots u_k^{r_k} (1-u_1)^{r_2+\dots+r_k+k-1} (1-u_2)^{r_3+\dots+r_k+k-2} \\
&\qquad \dots (1-u_{k-2})^{r_{k-1}+r_k+2} (1-u_{k-1})^{r_k+1}\, d\mathbf{u} \\
&= \left(\int_I u_1^{r_1}(1-u_1)^{r_2+\dots+r_k+k-1}\, dx_1\right) \left(\int_I u_2^{r_2}(1-u_2)^{r_3+\dots+r_k+k-2}\, dx_1\right) \\
&\qquad \dots \left(\int_I u_{k-1}^{r_{k-1}}(1-u_{k-1})^{r_k+1}\, dx_k\right) \left(\int_I u_k^{r_k}\, dx_k\right).
\end{aligned}$$

Now we can use Theorem 8.20 to evaluate each of these, using the respective values

$$\alpha_j = 1 + r_j, \beta = r_{j+1} + \dots + r_k + k - j + 1,$$

so that, except for the final factor

$$\int_I u_k^{r_k}\, dx_k = \left[\frac{u_k^{r_k+1}}{r_k+1}\right]_0^1 = \frac{1}{r_k+1},$$

the above product of integrals becomes a product of gamma functions:

$$= \tfrac{\Gamma(1+r_1)\Gamma(r_2+\dots r_k+k)}{\Gamma(1+r_1+r_2+\dots+r_k+k)} \cdot \tfrac{\Gamma(1+r_2)\Gamma(r_3+\dots r_k+k-1)}{\Gamma(r_2+r_3+\dots+r_k+k)} \cdot \tfrac{\Gamma(1+r_3)\Gamma(r_4+\dots r_k+k-2)}{\Gamma(r_3+r_4+\dots+r_k+k-1)} \cdots \tfrac{\Gamma(1+r_{k-1})\Gamma(r_k+2)}{\Gamma(1+2+r_{k-1}+r_k)} \cdot \tfrac{1}{r_k+1}$$

then making liberal use of $\Gamma(1 + r_j) = r_j!$ from Theorem 8.18(b), we have

$$= \tfrac{r_1! \dots r_{k-1}!}{(r_1+\dots r_k+k)!} \cdot \tfrac{\Gamma(r_2+\dots+r_k+k)}{\Gamma(r_2+\dots+r_k+k)} \cdot \tfrac{\Gamma(r_3+\dots+r_k+k-1)}{\Gamma(r_3+\dots+r_k+k-1)} \cdots \tfrac{\Gamma(r_k+2)}{r_k+1}.$$

Now making use of $x\Gamma(x) = \Gamma(x+1)$ from Theorem 8.19(a), to simplify the final factor, we have

$$\tfrac{\Gamma(r_k+2)}{r_k+1} = \tfrac{\Gamma((r_k+1)+1)}{r_k+1} = \tfrac{(r_k+1)\Gamma(r_k+1)}{r_k+1} = \Gamma(r_k+1) = r_k!,$$

so that the final formula becomes

$$\int_{Q^k} x_1^{r_1} \dots x_k^{r_k} d\mathbf{x} = \tfrac{r_1! \dots r_k!}{(r_1 + \dots r_k + k)!}.$$

Note: if we take $r_j = 0, \forall j$, then this formula becomes

$$\int_{Q^k} 1 d\mathbf{x} = \frac{0! \dots 0!}{(0 + \dots 0 + k)!} = \frac{1}{k!},$$

showing that the volume of Q^k is $\frac{1}{k!}$.

15. If ω and λ are k- and m-forms, respectively, prove that $\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega$.

Write the given forms as

$$\omega = \sum a_{i_1,\dots,i_k}(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum\nolimits_I a_I dx_I$$

and

$$\lambda = \sum b_{j_1,\dots,j_k}(x)\, dx_{j_1} \wedge \dots \wedge dx_{j_m} = \sum\nolimits_J a_J dx_J.$$

Then we have

$$\omega \wedge \lambda = \sum_{I,J} a_I(x)\, b_J(x)\, dx_I \wedge dx_J \quad \text{and} \quad \lambda \wedge \omega = \sum_{I,J} a_I(x)\, b_J(x)\, dx_J \wedge dx_I,$$

so it suffices to show that for every summand,

$$dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I.$$

Making excessive use of anticommutativity (Eq. (46) on p.256),

$$\begin{aligned} dx_I \wedge dx_J &= dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_m} \\ &= (-1)^k dx_{j_1} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_m} \\ &= (-1)^{2k} dx_{j_1} \wedge dx_{j_2} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_3} \wedge \dots \wedge dx_{j_m} \\ &\vdots \\ &= (-1)^{mk} dx_{j_1} \wedge \dots \wedge dx_{j_m} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \\ &= (-1)^{km} dx_J \wedge dx_I. \end{aligned}$$

16. If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_k]$ is an oriented affine k-simplex, prove that $\partial^2\sigma = 0$, directly from the definition of the boundary operator ∂. Deduce from this that $\partial^2\Psi = 0$ for every chain Ψ.

Denote $\sigma_i = [\mathbf{p}_0, \ldots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_k]$ and for $i < j$, let σ_{ij} be the $(k-2)$-simplex obtained by deleting $\mathbf{p}_i$ and $\mathbf{p}_j$ from σ.

Now we use Eq. (85) to compute

$$\partial\sigma = \sum_{j=0}^{k} (-1)^j \sigma_j.$$

Now we apply (85) again to the above result and get

$$\begin{aligned}\partial^2\sigma = \partial(\partial\sigma) = \partial\left(\sum_{j=0}^{k}(-1)^j\sigma_j\right) &= \sum_{j=0}^{k}(-1)^j\partial\sigma_j \\ &= \sum_{j=0}^{k}(-1)^j\sum_{i=0}^{k-1}(-1)^i\sigma_{ij} \\ &= \sum_{j=0}^{k}\sum_{i=0}^{k-1}(-1)^{i+j}\sigma_{ij}\end{aligned}$$

Lemma. For $i < j$, $\sigma_{ij} = \sigma_{j-1,i}$.

Proof. These both correspond to removing the same points $\mathbf{p}_i$, $\mathbf{p}_j$, just in different orders (because $\mathbf{p}_j$ is the $(j-1)^{\text{th}}$ vertex of σ_i).

$\sigma = [\mathbf{p}_0, ..., \mathbf{p}_i, ..., \mathbf{p}_j, ..., \mathbf{p}_k] \;\mapsto\; \sigma_j = [\mathbf{p}_0, ..., \mathbf{p}_i, ..., \mathbf{p}_k] \;\mapsto \sigma_{ij} = [\mathbf{p}_0, ..., \mathbf{p}_k]$

$\sigma = [\mathbf{p}_0, ..., \mathbf{p}_i, ..., \mathbf{p}_j, ..., \mathbf{p}_k] \;\mapsto\; \sigma_i = [\mathbf{p}_0, ..., \mathbf{p}_j, ..., \mathbf{p}_k] \;\mapsto \sigma_{j-1,i} = [\mathbf{p}_0, ..., \mathbf{p}_k]$.

□

Thus we split the previous sum along the "diagonal":

$$\partial^2\sigma = \underbrace{\sum_{i \geq j = 0}^{k-1} (-1)^{i+j}\sigma_{ij}}_{A} + \underbrace{\sum_{i<j=1}^{k} (-1)^{i+j}\sigma_{ij}}_{B}.$$

Continuing on with B for a bit, we have

$$\begin{aligned}B &= \sum_{i<j=1}^{k} (-1)^{i+j}\sigma_{j-1,i} && \text{by lemma} \\ &= \sum_{j=1}^{k}\sum_{i=0}^{j-1}(-1)^{i+j}\sigma_{j-1,i} && \text{rewriting the sum} \\ &= \sum_{i=1}^{k}\sum_{j=0}^{i-1}(-1)^{i+j}\sigma_{i-1,j} && \text{swap dummies } i \leftrightarrow j\end{aligned}$$

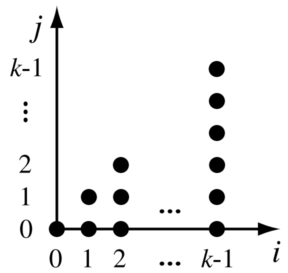

FIGURE 2. Reindexing the double sum over i, j.

$$= \sum_{i=0}^{k-1} \sum_{j=0}^{i} (-1)^{i+j+1} \sigma_{i,j} \qquad \text{reindex } i \mapsto i+1$$

$$= (-1) \sum_{j=1}^{k-1} \sum_{i=j}^{k-1} (-1)^{i+j} \sigma_{i,j} \qquad \text{see Figure 2}$$

$$= -A.$$

Whence

$$\partial^2 \sigma = A + B = A - A = 0.$$

In order to see how this works with the actual faces, define $F_k^i : Q^{k-1} \mapsto Q^k$ for $i = 0, \dots, k-1$ to be the affine map $F_k^i = (\mathbf{e}_0, \dots, \hat{\mathbf{e}}_i, \dots \mathbf{e}_k)$, where $\hat{\mathbf{e}}_i$ means omit $\mathbf{e}_i$, i.e.,

$$F_k^i(\mathbf{e_j}) = \begin{cases} \mathbf{e}_j, & j < i \\ \mathbf{e}_{j+1}, & j \geq i. \end{cases}$$

Then define the i^{th} face of σ to be $\sigma_i = \sigma \circ F_k^i$. Then

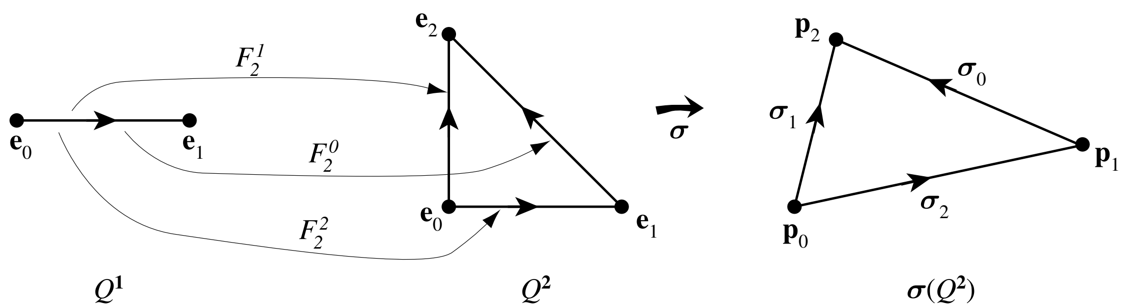

FIGURE 3. The mappings F_k^i and σ, for $k = 2$.

$$\begin{aligned} \partial^2 \sigma = \partial(\partial \sigma) &= \partial(\sigma_0 - \sigma_1 + \sigma_2) \\ &= \partial(\sigma_0) - \partial(\sigma_1) + \partial(\sigma_2) \\ &= (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}_2 - \mathbf{p}_0) + (\mathbf{p}_1 - \mathbf{p}_0) \\ &= 0 \end{aligned}$$

17. Put $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

◇ Explain why it is reasonable to call J^2 the positively oriented unit square in $\mathbb{R}^2$. Show that ∂J^2 is the sum of 4 oriented affine 1-simplices. Find these.

τ_1 has $\mathbf{p}_0 = \mathbf{0}, \mathbf{p}_1 = \mathbf{e}_1, \mathbf{p}_2 = \mathbf{e}_1 + \mathbf{p}_2$, with arrows pointing toward higher index.
τ_2 has $\mathbf{p}_0 = \mathbf{0}, \mathbf{p}_1 = \mathbf{e}_1, \mathbf{p}_2 = \mathbf{e}_1 + \mathbf{p}_2$, with arrows pointing toward lower index.

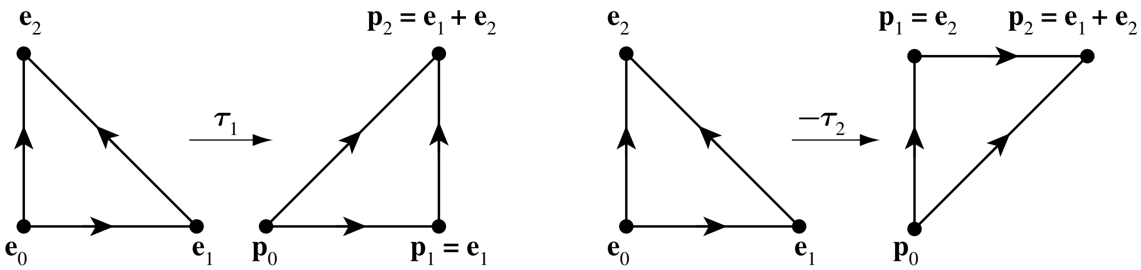

FIGURE 4. τ_1 and $-\tau_2$.

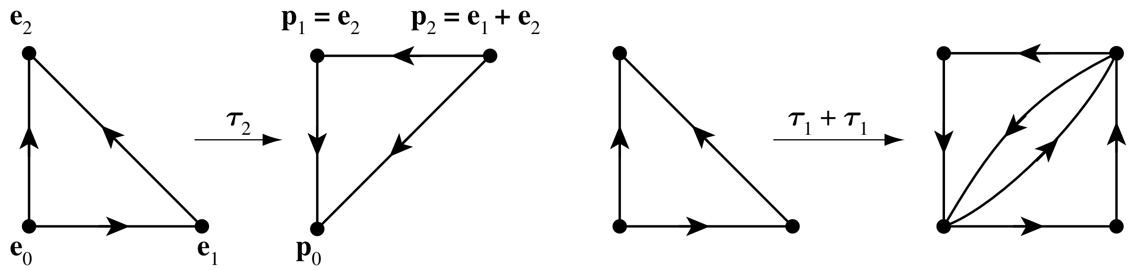

FIGURE 5. τ_2 and $\tau_1 + \tau_2$.

So

$$\begin{aligned}
\partial(\tau_1 + \tau_2) &= \partial[\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - \partial[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] \\
&= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1] - [\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_2] \\
&= [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}].
\end{aligned}$$

So the image of $\tau_1 + \tau_2$ is the unit square, and it is oriented counterclockwise, i.e., positively.

◇ What is $\partial(\tau_1 - \tau_2)$?

$$\begin{aligned}
\partial(\tau_1 - \tau_2) &= \partial[\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + \partial[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] \\
&= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_2] \\
&= [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_2] - 2[\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1].
\end{aligned}$$

20. State conditions under which the formula

$$\int_{\Phi} f\, d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts.
Hint: $d(f\omega) = (df) \wedge \omega + f\, d\omega$.

By the definition of d, we have

$$d(f\omega) = d(f \wedge \omega) = (df \wedge \omega) + (-1)^0 (f \wedge d\omega) = (df) \wedge \omega + f\, d\omega.$$

Integrating both sides yields

$$\int_{\Phi} d(f\omega) = \int_{\Phi} (df) \wedge \omega + \int_{\Phi} f\, d\omega.$$

If the conditions of Stokes' Theorem are met, i.e., if Φ is a k-chain of class $\mathcal{C}''$ in an open set $V \subseteq \mathbb{R}^n$ and ω is a $(k-1)$-chain of class $\mathcal{C}'$ in V, then we may apply it to the left side and get

$$\int_{\partial\Phi} f\omega = \int_{\Phi} (df) \wedge \omega + \int_{\Phi} f\, d\omega,$$

which is equivalent to the given formula.

21. Consider the 1-form

$$\eta = \frac{x\,dy - y\,dx}{x^2 + y^2} \quad \text{in } \mathbb{R}^2 \backslash \{\mathbf{0}\}.$$

(a) Show $\int_{\gamma} \eta = 2\pi \neq 0$ but $d\eta = 0$.

Put $gg(t) = (r\cos t, r\sin t)$ as in the example. Then

$$x^2 + y^2 \mapsto r^2, dx \mapsto -r\sin t\, dt, \text{ and} dy \mapsto r\cos t\, dt$$

and the integral becomes

$$\begin{aligned}
\int_{\gamma} \eta &= \int_0^{2\pi} -\frac{r\sin t}{r^2}(-r\sin t)dt + \int_0^{2\pi} \frac{r\cos t}{r^2}(r\cos t)dt \\
&= \int_0^{2\pi} \sin^2 t + \cos^2 t dt \\
&= \int_0^{2\pi} dt \\
&= 2\pi \neq 0.
\end{aligned}$$

However, using $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_1} dx_1$, we compute

$$\begin{aligned}
d\eta &= d\left(\frac{x\,dy}{x^2 + y^2} - \frac{y\,dx}{x^2 + y^2}\right) \\
&= d\left(\frac{x\,dy}{x^2 + y^2}\right) - d\left(\frac{y\,dx}{x^2 + y^2}\right)
\end{aligned}$$

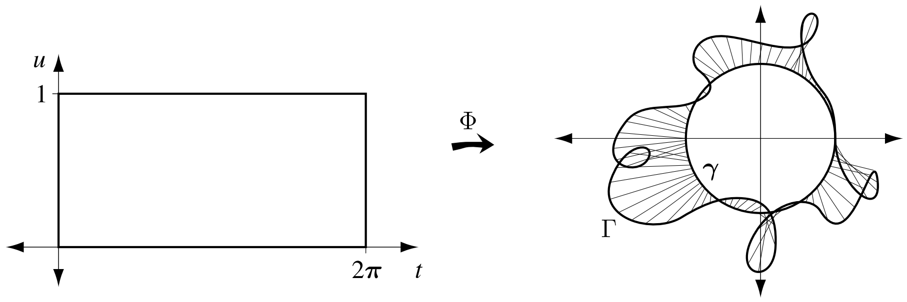

FIGURE 6. The homotopy from Γ to γ.

$$
\begin{aligned}
&= d\left(\frac{x}{x^2+y^2}\right) \wedge dy + \frac{x}{x^2+y^2} \wedge d(dy) \\
&\qquad - d\left(\frac{y}{x^2+y^2}\right) \wedge dx - \frac{y}{x^2+y^2} \wedge d(dx) \\
&= \left[\frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2}dx + \frac{(x^2+y^2)\cdot 0 - x(2y)}{(x^2+y^2)^2}dy\right] \wedge dy \\
&\qquad - \left[\frac{(x^2+y^2)\cdot 0 - y(2x)}{(x^2+y^2)^2}dx + \frac{(x^2+y^2)\cdot 1 - y(2y)}{(x^2+y^2)^2}dy\right] \wedge dx \\
&= \tfrac{1}{(x^2+y^2)^2}\Big((y^2-x^2)dx \wedge dy - 2xy\, dy \wedge dy \\
&\qquad + 2xy\, dx \wedge dx - (x^2-y^2)dy \wedge dx\Big) \\
&= \tfrac{1}{(x^2+y^2)^2}\Big((y^2-x^2)dx \wedge dy + (x^2-y^2)dx \wedge dy\Big) \\
&= \tfrac{1}{(x^2+y^2)^2}(y^2-x^2+x^2-y^2)dx \wedge dy \\
&= 0\, dx \wedge dy = 0.
\end{aligned}
$$

(b) With γ as in (a), let Γ be a $\mathcal{C}''$ curve in $\mathbb{R}^2\backslash\{\mathbf{0}\}$, with $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain $\mathbf{0}$ for any $t \in [0, 2\pi]$. Prove that $\int_\Gamma \eta = 2\pi$.

For $0 \leq t \leq 2\pi$ and $0 \leq u \leq 1$, define the straight-line homotopy between γ and Γ:

$$\Phi(t,u) = (1-u)\Gamma(t) + u\gamma(t), \quad \Phi(t,0) = \Gamma(t), \quad \Phi(t,1) = \gamma(t).$$

Then Φ is a 2-surface in $\mathbb{R}^2\backslash\{\mathbf{0}\}$ whose parameter domain is the rectangle $R = [0, 2\pi] \times [0, 1]$ and whose image does not contain $\mathbf{0}$. Now define $s(x) : I \to \mathbb{R}^2$ by $s(x) = (1-x)\gamma(0) + x\Gamma(x)$, so $s(I)$ is the segment connecting the base point of γ to the base point of Γ. Also, denote the boundary of R by

$$\partial R = R_1 + R_2 + R_3 + R_4,$$

where $R_1 = [\mathbf{0}, 2\pi\mathbf{e}_1]$, $R_2 = [2\pi\mathbf{e}_1, 2\pi\mathbf{e}_1 + \mathbf{e}_2]$, $R_3 = [2\pi\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2]$, $R_4 = [\mathbf{e}_2, \mathbf{0}]$.

$$\begin{aligned}\partial\Phi &= \Phi(\partial R)\\ &= \Phi(R_1 + R_2 + R_3 + R_4)\\ &= \Phi(R_1) + \Phi(R_2) + \Phi(R_3) + \Phi(R_4)\\ &= \Gamma + s + \overline{\gamma} + \overline{s}\\ &= \Gamma + s - \gamma - s\\ &= \Gamma - \gamma,\end{aligned}$$

where $\overline{\gamma}$ denotes the reverse of the path γ (i.e., traversed backwards). Then

$$\begin{aligned}\int_\Gamma \eta - \int_\gamma \eta = \int_{\Gamma-\gamma} \eta &= \int_{\partial\Phi} \eta && \partial\Phi = \Gamma - \gamma\\ &= \int_\Phi d\eta && \text{by Stokes}\\ &= \int_\Phi 0 && \text{by (a)}\\ &= 0\end{aligned}$$

shows that

$$\int_\Gamma \eta = \int_\gamma \eta.$$

(c) Take $\Gamma(t) = (a\cos t, b\sin t)$ where $a, b > 0$ are fixed. Use (b) to show

$$\int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t} dt = 2\pi.$$

We have

$$x(t) = \Gamma_1(t) = a\cos t \quad \text{and} \quad y(t) = \Gamma_2(t) = b\sin t$$

and

$$dx = \tfrac{\partial x}{\partial t} dt = -a\sin t\, dt \quad \text{and} \quad dy = \tfrac{\partial y}{\partial t} dt = b\cos t\, dt,$$

so

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = \frac{ab\sin^2 t + ab\cos^2 t}{a^2\cos^2 t + b^2\sin^2 t} = \frac{ab}{a^2\cos^2 t + b^2\sin^2 t},$$

and the integral becomes

$$\begin{aligned}2\pi = \int_\Gamma \eta &= \int_0^{2\pi} \frac{ab\cos^2 t\,dt + ab\sin^2 t\,dt}{a^2\cos^2 t + b^2\sin^2 t}\\ &= \int_0^{2\pi} \frac{ab}{a^2\cos^2 t + b^2\sin^2 t}\, dt.\end{aligned}$$

(d) Show that

$$\eta = d\left(\arctan \tfrac{y}{x}\right)$$

in any convex open set in which $x \neq 0$ and that

$$\eta = d\left(-\arctan \tfrac{x}{y}\right)$$

in any convex open set in which $y \neq 0$. Why does this justify the notation $\eta = d\theta$, even though η isn't exact in $\mathbb{R}^2 \backslash \{\mathbf{0}\}$?

$\theta(x,y) = \arctan\left(\frac{y}{x}\right)$ is well-defined on

$$D^+ = \{(x,y) \vdots x > 0\}.$$

On D^+ we have

$$\begin{aligned} d\theta &= d\left(\arctan \tfrac{y}{x}\right) \\ &= \tfrac{\partial\theta}{\partial x}dx + \tfrac{\partial\theta}{\partial y}dy \\ &= \frac{\frac{\partial}{\partial x}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}dx + \frac{\frac{\partial}{\partial y}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}dy \\ &= \frac{-yx^{-2}\,dx}{1+\left(\frac{y}{x}\right)^2} + \frac{\frac{1}{x}\,dy}{1+\left(\frac{y}{x}\right)^2} \\ &= \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}. \end{aligned}$$

So $d\theta = \eta$ on D^+ (and similarly on $D^- = \{(x,y) \vdots x < 0\}$), which is everywhere θ is defined. If we take

$$B^+ = \{(x,y) \vdots y > 0\},$$

then

$$d\left(-\arctan \tfrac{x}{y}\right) = -\frac{1/y\,dx}{1+(x/y)^2} + \frac{(x/y)^2\,dy}{1+(x/y)^2} = -\frac{y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2},$$

and similarly on $B^- = \{(x,y) \vdots y < 0\}$.

We say $\eta = d\theta$ because θ gives $\arg(x,y)$; if we let $\gamma(t) = (\cos t, \sin t)$ as above, then this becomes

$$\arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{\sin t}{\cos t}\right) = t,$$

where the inverse is defined, i.e., for $0 \leq t \leq 2\pi$.

Suppose η were exact: $\exists\lambda$, a 0-form on $\mathbb{R}^2_0$ such that $d\lambda = \eta$. Then

$$\begin{aligned} & d\theta - d\lambda = d(\theta - \lambda) = 0 \quad \implies \quad \theta - \lambda = c \\ \implies \quad & \theta = \lambda + c \\ \implies \quad & \theta = \lambda \end{aligned}$$

but λ is continuous and θ isn't. Alternatively, if η were exact,

$$\int_\gamma \eta = \int_\gamma d\lambda = \int_{\partial\gamma} \lambda = \int_\varnothing \lambda = 0,$$

contradicting (b). ↯

(e) Show (b) can be derived from (d).

Taking $\eta = d\theta$,

$$\int_\Gamma \eta = \int_\Gamma d\theta = \Big[\theta\Big]_0^{2\pi} = 2\pi - 0 = 2\pi.$$

(f) Γ is a closed $\mathcal{C}'$ curve in $\mathbb{R}_0^2$ implies that

$$\frac{1}{2\pi}\int_\Gamma \eta = Ind(\Gamma) := \frac{1}{2\pi i}\int_a^b \frac{\Gamma'(t)}{\Gamma(t)}\,dt.$$

We use the following basic facts from complex analysis:

$$\log z = \log|z| + i\arg z, \quad \text{so} \quad \arg z = -i(\log z - \log|z|).$$

First, fix $|\Gamma| = c$, so Γ is one or more circles around the origin in whatever direction, but of constant radius. Then

$$\begin{aligned}\int_\Gamma \eta = \int_\Gamma d\theta &= \int_a^b d(\theta(\Gamma(t))) \\ &= -i\int_a^b d(\log\Gamma(t) - \log|\Gamma(t)|) \\ &= -i\int_a^b d(\log\Gamma(t)) \\ &= -i\int_a^b \frac{\Gamma'(t)}{\Gamma(t)}\,dt.\end{aligned}$$

Now use (b) for arbitrary Γ.

24. Let $\omega = \sum a_i(\mathbf{x})\,dx_i$ be a 1-form of class $\mathcal{C}''$ in a convex open set $E \subseteq \mathbb{R}^n$. Assume $d\omega = 0$ and prove that ω is exact in E.

◇ Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega, \qquad \mathbf{x} \in E.$$

Apply Stokes' theorem to the 2-form $d\omega = 0$ on the affine-oriented 2-simplex $[\mathbf{p},\mathbf{x},\mathbf{y}]$ in E.

By Stokes' Thm,

$$\begin{aligned}
0 = \int_{[\mathbf{p},\mathbf{x},\mathbf{y}]} d\omega &= \int_{\partial[\mathbf{p},\mathbf{x},\mathbf{y}]} \omega && \text{Stokes' Thm}\\
&= \int_{[\mathbf{x},\mathbf{y}]-[\mathbf{p},\mathbf{y}]+[\mathbf{p},\mathbf{x}]} \omega && \text{def of } \partial\\
&= \int_{[\mathbf{x},\mathbf{y}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{p},\mathbf{x}]} \omega && \text{Rem 10.26, p.267}\\
&= \int_{[\mathbf{x},\mathbf{y}]} \omega - f(\mathbf{y}) + f(\mathbf{x}) && \text{def of } f\\
f(\mathbf{y}) - f(\mathbf{x}) &= \int_{[\mathbf{x},\mathbf{y}]} \omega.
\end{aligned}$$

◇ Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y})\,dt$$

for $\mathbf{x}, \mathbf{y} \in E$. Hence $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$.

Note that $\gamma = [\mathbf{x},\mathbf{y}]$ is the straight-line path from $\mathbf{x}$ to $\mathbf{y}$, i.e., $\gamma(t) = (1-t)\mathbf{x}+t\mathbf{y}$. Then the right-hand integral from the last line of the previous derivation becomes

$$\begin{aligned}
f(\mathbf{y}) - f(\mathbf{x}) &= \int_{[\mathbf{x},\mathbf{y}]} \omega\\
&= \int_{[\mathbf{x},\mathbf{y}]} \sum_{i=1}^n a_i(\mathbf{u})\,du_i\\
&= \int_0^1 \sum_{i=1}^n a_i\left((1-t)\mathbf{x} + t\mathbf{y}\right) \tfrac{\partial}{\partial t} \Big((1-t)x_i + ty_i\Big)\,dt\\
&= \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i\left((1-t)\mathbf{x} + t\mathbf{y}\right) dt.
\end{aligned}$$

Putting $\mathbf{y} = \mathbf{x} + s\mathbf{e}_i$, we plug this into the formula for the i^{th} partial derivative. Note that all terms other than $j = i$ are constant with respect to x_i, and hence vanish. This leaves

$$\begin{aligned}
(D_i f)(\mathbf{x}) &= \lim_{s\to 0} \frac{f(\mathbf{x}+s\mathbf{e}_i) - f(\mathbf{x})}{s} \\
&= \lim_{s\to 0} \tfrac{(x_i+s)-x_i}{s} \int_0^1 a_i\left((1-t)\mathbf{x} + t(\mathbf{x}+s\mathbf{e}_i)\right) dt \\
&= \lim_{s\to 0} \tfrac{1}{s}\cdot s \int_0^1 a_i\left(\mathbf{x}+ts\mathbf{e}_i\right)\right) dt \\
&= \lim_{s\to 0} \int_0^1 a_i\left(\mathbf{x}+ts\mathbf{e}_i\right)\right) dt \\
&= \int_0^1 \lim_{s\to 0} a_i\left(\mathbf{x}+ts\mathbf{e}_i\right)\right) dt \\
&= \int_0^1 a_i\left(\mathbf{x}\right) dt \\
&= a_i(\mathbf{x}) \int_0^1 dt \\
&= a_i(\mathbf{x}).
\end{aligned}$$

We can pass the limit through the integral using uniform convergence.

27. Let E be an open 3-cell in $\mathbb{R}^3$, with edges parallel to the coordinate axes. Suppose $(a,b,c)\in E$, $f_i \in \mathcal{C}'(E)$ for $i=1,2,3$

$$\omega = f_1 dy\wedge dz + f_2 dz\wedge dx + f_3 dx\wedge dy,$$

and assume that $d\omega = 0$ in E. Define

$$\lambda = g_1 dx + g_2 dy$$

where

$$\begin{aligned}
g_1(x,y,z) &= \int_c^z f_2(x,y,s)\,ds - \int_b^y f_3(x,t,c)\,dt \\
g_2(x,y,z) &= -\int_c^z f_1(x,y,s)\,ds,
\end{aligned}$$

for $x,y,z)\in E$.

◇ Prove that $d\lambda = \omega$ in E.

We compute the exterior derivative:

$$\begin{aligned}
d\lambda &= d(g_1 dx + g_2 dy) \\
&= dg_1\wedge dx + g_1 \wedge d(dx) + dg_2\wedge dy + g_2\wedge d(dy) \\
&= d\left[\int_c^z f_2(x,y,s)\,ds - \int_b^y f_3(x,t,c)\,dt\right]\wedge dx + d\left[-\int_c^z f_1(x,y,s)\,ds\right]\wedge dy
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{\partial}{\partial y}\left(\int_c^z f_2(x,y,s)\,ds\right) dy\wedge dx + \tfrac{\partial}{\partial z}\left(\int_c^z f_2(x,y,s)\,ds\right) dz\wedge dx \\
&\qquad - \tfrac{\partial}{\partial y}\left(\int_b^y f_3(x,t,c)\,dt\right) dy\wedge dx - \tfrac{\partial}{\partial z}\left(\int_b^y f_3(x,t,c)\,dt\right) dz\wedge dx \\
&\qquad - \tfrac{\partial}{\partial x}\left(\int_c^z f_1(x,y,s)\,ds\right) dx\wedge dy - \tfrac{\partial}{\partial z}\left(\int_c^z f_1(x,y,s)\,ds\right) dz\wedge dy \\
&= \tfrac{\partial}{\partial z}\left(\int_c^z f_1(x,y,s)\,ds\right) dy\wedge dz + \tfrac{\partial}{\partial z}\left(\int_c^z f_2(x,y,s)\,ds - \int_b^y f_3(x,t,c)\,dt\right) dz\wedge dx \\
&\qquad \tfrac{\partial}{\partial y}\left(\int_b^y f_3(x,t,c)\,dt - \int_c^z f_2(x,y,s)\,ds\right) dx\wedge dy \\
&\qquad - \tfrac{\partial}{\partial x}\left(\int_c^z f_1(x,y,s)\,ds\right) dx\wedge dy
\end{aligned}
$$

Note that the third integral above is constant with respect to z, and hence vanishes.

$$
\begin{aligned}
&= f_1(x,y,z)dy\wedge dz + f_2(x,y,z)dz\wedge dx + f_3(x,y,c)dx\wedge dy \\
&\qquad - \left[\int_c^z \left(\tfrac{\partial}{\partial y}f_2(x,y,s) + \tfrac{\partial}{\partial x}f_1(x,y,s)\right) ds\right] dx\wedge dy \\
&= f_1(x,y,z)dy\wedge dz + f_2(x,y,z)dz\wedge dx \\
&\qquad + \left[f_3(x,y,c) - \int_c^z \left(\tfrac{\partial}{\partial y}f_2(x,y,s) + \tfrac{\partial}{\partial x}f_1(x,y,s)\right) ds\right] dx\wedge dy. \qquad (10.1)
\end{aligned}
$$

Since ω is closed,

$$
\begin{aligned}
0 = d\omega &= d(f_1dy\wedge dz) + d(f_2dz\wedge dx) + d(f_3dx\wedge dy) \\
&= df_1\wedge dy\wedge dz + f_1\wedge d(dy\wedge dz) + df_2\wedge dz\wedge dx + f_2\wedge d(dz\wedge dx) \\
&\qquad + df_3\wedge dx\wedge dy + f_3\wedge d(dx\wedge dy) \\
&= \tfrac{\partial}{\partial x}f_1dx\wedge dy\wedge dz + \tfrac{\partial}{\partial y}f_2dy\wedge dz\wedge dx + \tfrac{\partial}{\partial z}f_3dz\wedge dx\wedge dy \\
&= \left(\tfrac{\partial}{\partial x}f_1 + \tfrac{\partial}{\partial y}f_2 + \tfrac{\partial}{\partial z}f_3\right) dx\wedge dy\wedge dz
\end{aligned}
$$

$$
\implies \tfrac{\partial}{\partial z}f_3 = -\left(\tfrac{\partial}{\partial x}f_1 + \tfrac{\partial}{\partial y}f_2\right),
$$

so the last line of the previous derivation, (10.1), becomes

$$
\begin{aligned}
&\left[f_3(x,y,c) + \int_c^z \left(\tfrac{\partial}{\partial z}f_3\right) ds\right] dx\wedge dy \\
&= \left[f_3(x,y,c) + \Big(f_3(x,y,z) - f_3(x,y,c)\Big)\right] dx\wedge dy \\
&= f_3(x,y,z)\,dx\wedge dy
\end{aligned}
$$

and we finally obtain

$$
\begin{aligned}
d\lambda &= f_1(x,y,z)dy\wedge dz + f_2(x,y,z)dz\wedge dx + f_3(x,y,z)dx\wedge dy \\
&= \omega.
\end{aligned}
$$

◇ Evaluate these integrals when $\omega = \zeta$ and thus find the form $\lambda = -(z/r)\eta$, where

$$\eta = \tfrac{x\,dy - y\,dx}{x^2+y^2}.$$

Now the form is

$$\omega = \zeta = \tfrac{1}{r^3}\left(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy\right),$$

so

$$f_1 = \frac{x}{r^3}, \quad f_2 = \frac{y}{r^3}, \quad f_3 = \frac{z}{r^3}.$$

The integrals are

$$\int_c^z f_2(x,y,s)\,ds = \int_c^z y(x^2+y^2+s^2)^{-3/2}\,ds = \tfrac{y}{x^2+y^2}\left(\tfrac{z}{r} - \tfrac{c}{\sqrt{x^2+y^2+c^2}}\right),$$

$$\int_b^y f_3(x,t,c)\,dt = \int_b^y c(x^2+t^2+c^2)^{-3/2}\,dt = \tfrac{c}{x^2+c^2}\left(\tfrac{y}{\sqrt{x^2+y^2+c^2}} - \tfrac{b}{\sqrt{x^2+b^2+c^2}}\right),$$

$$\int_c^z f_1(x,y,s)\,ds = \int_c^z x(x^2+y^2+s^2)^{-3/2}\,ds = \tfrac{x}{x^2+y^2}\left(\tfrac{z}{r} - \tfrac{c}{\sqrt{x^2+y^2+c^2}}\right).$$

Thus $\lambda = g_1\,dx + g_2\,dy$ is

$$\begin{aligned}
\lambda &= \tfrac{y}{x^2+y^2}\left(\tfrac{z}{r} - \tfrac{c}{\sqrt{x^2+y^2+c^2}}\right)dx - \tfrac{c}{x^2+c^2}\left(\tfrac{y}{\sqrt{x^2+y^2+c^2}} - \tfrac{b}{\sqrt{x^2+b^2+c^2}}\right)dx \\
&\qquad - \tfrac{x}{x^2+y^2}\left(\tfrac{z}{r} - \tfrac{c}{\sqrt{x^2+y^2+c^2}}\right)dy \\
&= \left(-\tfrac{z}{r}\right)\left(-\tfrac{y\,dx}{x^2+y^2} + \tfrac{x\,dy}{x^2+y^2}\right) \\
&\qquad + \left[\tfrac{c}{x^2+c^2}\left(\tfrac{b}{\sqrt{x^2+b^2+c^2}} - \tfrac{y}{\sqrt{x^2+y^2+c^2}}\right) - \tfrac{yc}{(x^2+y^2)\sqrt{x^2+y^2+c^2}}\right]dx \\
&\qquad + \tfrac{xc}{(x^2+y^2)\sqrt{x^2+y^2+c^2}}dy \\
&= \left(-\tfrac{z}{r}\right)\eta,
\end{aligned}$$

due to some magic cancellations at the end.

28. Fix $b > a > 0$ and define

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

for $a \le r \le b, 0 \le \theta \le 2\pi$. The range of Φ is thus an annulus in $\mathbb{R}^2$. Put $\omega = x^3 dy$, and show

$$\int_\Phi d\omega = \int_{\partial\Phi} \omega$$

by computing each integral separately.

Starting with the left-hand side,

$$d\omega = \tfrac{\partial}{\partial x} x^3 dx \wedge dy + \tfrac{\partial}{\partial y} x^3 dy \wedge dy = 3x^2 dx \wedge dy + 0 = 3x^2 dx \wedge dy,$$

so

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$

whence

$$3x^2 dx \wedge dy = 3r^2(\cos^2\theta)\, r\, dr\, d\theta,$$

and thus

$$\begin{aligned}
\int_\Phi d\omega &= \int_0^{2\pi}\int_a^b 3r^3\cos^2\theta\, dr\, d\theta \\
&= \left(\int_0^{2\pi}\cos^2\theta\, d\theta\right)\left(\int_a^b 3r^3\, dr\right) \\
&= \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \left[\tfrac{3}{4}r^4\right]_a^b \\
&= \left(\pi + \tfrac{1}{4}(\sin 4\pi - \sin 0)\right) \cdot \tfrac{3}{4}\left(b^4 - a^4\right) \\
&= \tfrac{3\pi}{4}(b^4 - a^4).
\end{aligned}$$

Meanwhile, on the right-hand side, note that

$$\partial\Phi = \Gamma - \gamma,$$

as in Example 10.32 or Exercise 21(b). Thus,

$$\begin{aligned}
\int_\Phi \omega = \int_{\Gamma-\gamma} \omega = \int_\Gamma x^3\, dy - \int_\gamma x^3\, dy \\
&= \int_0^{2\pi} b^3\cos^3\theta(b\cos\theta)\, d\theta - \int_0^{2\pi} a^3\cos^3\theta(a\cos\theta)\, d\theta \\
&= b^4\int_0^{2\pi}\cos^4\theta\, d\theta - a^4\int_0^{2\pi}\cos^4\theta\, d\theta \\
&= \left(b^4 - a^4\right)\int_0^{2\pi}\cos^4\theta\, d\theta \\
&= \left(b^4 - a^4\right)\cdot\frac{3\pi}{4}.
\end{aligned}$$

11. Lebesgue Theory

1. If $f \geq 0$ and $\int_E f\,d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E.

First define

$$E_n := \{f > \tfrac{1}{n}\} \qquad \text{and} \qquad A := \bigcup E_n = \{f \neq 0\}.$$

We need to show that $\mu A = 0$, so suppose not. Then

$$\begin{aligned} \mu A > 0 &\implies \sum E_n > 0 \\ &\implies \mu E_n > 0 \quad \text{for some } n. \end{aligned}$$

But then we'd have

$$\int_E f\,d\mu \geq \int_{E_n} f\,d\mu > \tfrac{1}{n} \cdot \mu(E_n) > 0. \qquad ↯$$

2. If $\int_A f\,d\mu = 0$ for every measurable subset A of a measurable set E, then $f(x) =_{\text{ae}} 0$ on E.

First, suppose $f \geq 0$. Then $f =_{\text{ae}} 0$ on every $A \subseteq E$ by the previous exercise. In particular, $E \subseteq E$, so $f =_{\text{ae}} 0$ on E.

Now let $f \not\geq 0$. Then $f = f^+ - f^-$, and each of f^+, f^- are 0 on E by the above remark, so $f =_{\text{ae}} 0$ on E, too.

3. If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.

Define $f(x) = \overline{\lim} f_n(x)$, so f is measurable by (11.17) and $|f_n(x) - f(x)|$ is measurable by (11.16),(11.18). Then we can write the given set as

$$\begin{aligned} &\{x \,\vdots\, \forall \varepsilon > 0, \exists N \text{ s.t. } n \geq N \implies |f_n(x) - f(x)| < \varepsilon\} \\ &\quad = \{x \,\vdots\, \forall k \in \mathbb{N}, \exists N \text{ s.t. } n \geq N \implies |f_n(x) - f(x)| < 1/k\} \\ &\quad = \bigcap_{k=1}^{\infty} \{x \,\vdots\, \exists N \text{ s.t. } n \geq N \implies |f_n(x) - f(x)| < 1/k\} \\ &\quad = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{x \,\vdots\, |f_n(x) - f(x)| < 1/k\} \end{aligned}$$

4. If $f \in \mathcal{L}(\mu)$ on E and g is bounded and measurable on E, then $fg \in \mathcal{L}(\mu)$ on E.

Since g is bounded, $\exists M$ such that $|g| \leq M$. Then

$$\left| \int_E fg\,d\mu \right| \leq M \int_E |f|\,d\mu < \infty$$

by (11.23(d)) and (11.26).