SOLUTIONS TO SELECTED PROBLEMS FROM RUDIN

DAVID SEAL

Abstract. I became bored sometime this August 2006, and decided to review some of my analysis, by reading my old analysis book[1] from Junior year. While I'm at it, I decided to type up some solutions to a few problems that I scratched out solutions to on notepaper in order to pick up some LaTeXpractice along the way. This paper has solutions to some of the problems I was able to solve, indeed many of the problems in this book were too challenging to solve in a weekend. All of these problems were selected from *Principles of Mathematical Analysis*[1] by Walter Rudin.

Contents

Ok, now the real work. But, first a definition. A metric space is called *separable* if it contains a countable dense subset. Problem #21 on page 45 asks you to prove that $\mathbb{R}^k$ is separable. The second definition on this page is given in the following problem.

1. The Real and Complex Number System

OK fine, I didn't write up any of these. I just wanted the numbers to line up.

2. Basic Topology

Problem 1 (pg. 45, #23)**.** *A collection* $\{V_\alpha\}$ *of open subsets of* X *is said to be a* base *for* X *if the following is true: For every* $x \in X$ *and every open set* $G \subset X$ *such that* $x \in G$*, we have* $x \in V_\alpha \subset G$ *for some*

α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a countable *base.* Hint: *Take all neighborhoods with rational radius and center in some countable dense subset of X.*

The hint tells us exactly what to do. The set

$$V := \left\{B_r(x) : \ r \in \mathbb{Q}^+, \ x \in C\right\} \tag{1}$$

where C is the countable dense subset gives the desired result.

Solution. Set V as in (1). This set is countable since both $\mathbb{Q}$ and C are countable. Suppose $G \subset X$ is open. Let $x \in G$ and find an $\epsilon > 0$ such that $B_{2\epsilon}(x) \subset G$. Since C is dense, there exists a $p \in C$, where $d(p, x) < \epsilon$. In addition, the density of the rationals on the real line provides an $r \in \mathbb{Q}$ where $\epsilon > r > d(p, x) > 0$. Therefore, $x \in B_r(p)$. Furthermore, if $y \in B_r(p)$, we have that

$$d(y, x) \leq d(y, p) + d(p, x) < 2r < 2\epsilon,$$

so that $B_r(p) \subset G$, the desired result. □

This problem set continues in the next one. Rudin[1] sure does like to give important definitions and theorems in the problem sets!

Problem 2 (pg. 45, #24)**.** *Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. (A hint is given in the book. I won't type it here, but I will follow it).*

The best course of action is to follow the hint.

Solution. Fix $1/n > 0$. Start with any $x_1 \in X$. Having chosen $x_1, x_2, \ldots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq 1/n$ for $i = 1, \ldots, j$. Assume this course of action is infinite, and we'll show a contradiction from this assumption. The set defined as

$$X_{1/n} := \{x_k\}_{k=1}^{\infty}$$

is infinite because each x_k is distinct; it therefore has a limit point due to the hypothesis. Let's say $x_N \in \{x_k\}_{k=1}^{\infty}$ is the limit point. However, x_N cannot be a limit point, for

$$B_{\frac{1}{2n}}(x) \cap \left(\{x_k\}_{k=1}^{\infty} \setminus \{x_N\}\right) = \emptyset.$$

because this set has the property that each $x_i \in X_{1/n}$ is at least $1/n$ away from every other point in the set. Therefore, the process of choosing the x_i's is a finite one for any positive $1/n$; we also have the nice property that $\bigcup_k B_{\frac{1}{n}}(x_k) \supset X$ for every positive $1/n$, (otherwise we would have been able to choose an additional x_i). The set

$X^* := \cup_{n=1}^{\infty} X_{\frac{1}{n}}$ yields a countable dense subset. Therefore, X is separable (see pbm. #22) which shows that it has a countable base (see pbm #23). □

Gee, that was fun. I see it takes a lot longer to type these problems up in LaTeXrather than by hand, but I think it'll be a useful tool for me to learn anyway. OK, on to the next problem.

3. Numerical Sequences and Series

Problem 3 (pg.78, #7)**.** *Prove that the convergence of* $\sum a_n$ *implies the convergence of*

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n > 0$*.*

I'll define a sequence

$$b_n := \max\left\{1/n^2, a_n\right\},$$

and claim that this sequence converges.

Proof of claim: Let $\epsilon > 0$. From the Cauchy criterion of convergence, (both $\sum \frac{1}{n^2}$ and $\sum a_n$ converge absolutely), there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned} \sum_{k=N}^{N+j} \frac{1}{n^2} &< \epsilon/2 \qquad \text{and} \\ \sum_{k=N}^{N+j} a_n &< \epsilon/2 \end{aligned}$$

for any $j \in \mathbb{N}$. From the definition of b_n, the following inequalities are fairly imminent. We will replace any missing terms with the remaining terms in the convergent sequences:

$$\sum_{k=N}^{N+j} b_k \leq \sum_{k=N}^{N+j} 1/n^2 + \sum_{k=N}^{N+j} a_k < \epsilon/2 + \epsilon/2 = \epsilon.$$

By the cauchy criterion for convergence, $\sum b_n$ must converge. □

With that bit out of the way, we can provide the solution.

Solution. Starting with $b_n \geq 1/n^2$ and $b_n \geq a_n$, we have that $b_n^2 \geq \frac{a_n}{n^2}$ after multiplying these inequalities together. After taking square roots, we have that $b_n \geq \frac{\sqrt{a_n}}{n}$ and the comparison test implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$. □

GWEN, HERE IS MY PROOF:

Theorem 1. *If $0 < a \leq b$ for some $a, b \in \mathbb{R}$, then $\sqrt{a} \leq \sqrt{b}$.*

Proof. Suppose not. That is, suppose there exist an $0 < a < b$ for which $\sqrt{a} \nleq \sqrt{b}$. I.e.

$$\sqrt{a} > \sqrt{b}.$$

If we multiply the previous inequality together by itself, we obtain

$$\sqrt{a}\sqrt{a} > \sqrt{b}\sqrt{b},$$

In other words,

$$a > b.$$

This is a contradiction. □

Problem 4 (pg. 79, #12)**.** *Suppose $a_n > 0$ and $\sum a_n$ converges. Put*

$$r_n := \sum_{m=n}^{\infty} a_m.$$

(1) *Prove that*

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(2) *Prove that*

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Solution (Part (1))*.* Rather than follow the books use of an $m < n$, I will choose to fix an $m, j \in \mathbb{N}$, and prove this with a $m < m + j$. I believe this is much more explicit. Hence, the equation we seek to prove can be reformulated as

$$\frac{a_m}{r_m} + \cdots + \frac{a_{m+j}}{r_{m+j}} > 1 - \frac{r_{m+j}}{r_m}. \tag{2}$$

Since $a_n > 0$ for every n, we must have that r_n is a decreasing sequence: $r_m > r_{m+i}$ for any $i \in \mathbb{N}$. Hence, for any $i \in \mathbb{N}$,

$$\frac{r_m}{r_{m+i}} > 1. \tag{3}$$

Applying equation (3) a few times, in addition to the fact $a_{m+j} > 0$, we have that

$$\begin{aligned} r_m - r_{m+j} &= a_m + \cdots + a_{m+j-1} \\ &< a_m + \cdots + a_{m+j} \\ &< a_m + a_{m+1}\left(\frac{r_m}{r_{m+1}}\right) + \cdots + a_{m+j}\left(\frac{r_m}{r_{m+j}}\right). \end{aligned}$$

Whence,

$$1 - \frac{r_{m+j}}{r_m} < \frac{a_m}{r_m} + \frac{a_{m+1}}{r_{m+1}} + \cdots + \frac{a_{m+j}}{r_{m+j}};$$

which is the desired result for half the problem.

In order to show $\sum \frac{a_n}{r_n}$ diverges, it is enough show that it fails the cauchy criterion for convergence. Fix $\epsilon = 1/2$ and let $N \in \mathbb{N}$. There exists a $j \in \mathbb{N}$ such that $r_{N+j} \leq \frac{r_N}{2}$, whence $1 - \frac{r_{N+j}}{r_N} \geq 1/2$. If we apply the inequality just shown, we have

$$\sum_{k=N}^{N+j} \frac{a_k}{r_k} > 1 - \frac{r_{N+j}}{r_N} \geq 1/2.$$

This shows that $\sum \frac{a_n}{r_n}$ will never pass the cauchy criterion for convergence, which means it must be divergent. □

Solution (Part (2)). We will start by proving the inequality asked for. The following inequalities each proceed from the previous one:

$$\begin{aligned} 0 &< \left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)^2 \ ; \\ 0 &< r_n - 2\sqrt{r_{n+1} r_n} + r_{n+1} \ ; \\ -r_{n+1} &< r_n - 2\sqrt{r_{n+1} r_n} \ ; \\ r_n - r_{n+1} &< 2\left(r_n - \sqrt{r_{n+1} r_n}\right); \\ a_n &< 2\left(r_n - \sqrt{r_{n+1} r_n}\right). \end{aligned}$$

The final inequality is a restatement of the previous one since $a_n = r_n - r_{n+1}$. The desired result comes after dividing by $\sqrt{r_n}$;

$$\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right). \tag{4}$$

To prove the second part for this question; namely the convergence of $\sum \frac{a_n}{\sqrt{r_n}}$, we might ask: what if we add up a bunch of these $\frac{a_n}{\sqrt{r_n}}$'s?

$$\begin{aligned}\frac{a_n}{\sqrt{r_n}} &+ \frac{a_{n+1}}{\sqrt{r_{n+1}}} + \cdots + \frac{a_{n+j}}{\sqrt{r_{n+j}}} \\ &< 2\left((\sqrt{r_n} - \sqrt{r_{n+1}}) + \cdots + (\sqrt{r_{n+j}} - \sqrt{r_{n+j+1}})\right) \\ &= 2\left(\sqrt{r_n} - \sqrt{r_{n+j+1}}\right) \\ &< 2\sqrt{r_n}.\end{aligned}$$

The sequence $s_n := \sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}}$ is bounded above by $\sqrt{r_1}$, and monotonically increasing. Therefore the sum, $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{r_n}}$ converges. □

The next problem is a multi-series problem.

Problem 5 (pg. 82, #24)**.** *Let X be a metric space.*

(1) *Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X* equivalent *if*
$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$
Prove this is an equivalence relation.

(2) *Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define*
$$\Delta(P, Q) := \lim_{n\to\infty} d(p_n, q_n);$$
by excercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

(3) *Prove that the resulting metric space X^* is complete.*
I FIND THIS QUESTION DIFFICULT, AND HAVEN'T SOLVED IT YET; BUT I HAVE WRITTEN SOME IDEAS DOWN

(4) *for each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that*
$$\Delta(P_p, Q_q) = d(p, q)$$
for all $p, q \in X$. In other words, the mapping $\varphi : X \to X^*$ defined by $p \mapsto P_p$ is an isometry.

Solution (Part (1)). There are three items we need for an equivalence relation.

(1) $\{p_n\} \sim \{p_n\}$ since
$$\lim_{n\to\infty} d(p_n, p_n) = \lim_{n\to\infty} 0 = 0.$$

(2) Suppose $\{p_n\} \sim \{q_n\}$. Then $\{q_n\} \sim \{p_n\}$ because $d(p_n, q_n) = d(q_n, p_n)$, so the limit still goes to 0.
(3) Suppose $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{x_n\}$ for some $\{x_n\}$. Then $d(p_n, x_n) \leq d(p_n, q_n) + d(q_n, x_n)$. After taking limits, the right hand side goes to zero, so we have that $\{p_n\} \sim \{x_n\}$. □

Solution (Part (2)). Suppose both $\{p_n\}$, $\{p_n{}'\} \in P$ and that $\{q_n\}$, $\{q_n{}'\} \in Q$. Set the following values:

$$a := \lim_{n\to\infty} d(p_n, q_n);$$
$$a' := \lim_{n\to\infty} d(p'_n, q'_n).$$

These values exist (from problem #23), and we desire to show that $a = a'$ in order for $\Delta(P, Q)$ to be well defined.

Two applications of the triangle inequality give

$$0 \leq d(p_n, q_n) \leq d(p_n, p'_n) + d(p'_n, q'_n) + d(q'_n, q_n).$$

Sending $n \to \infty$, we obtain

$$0 \leq a \leq a'. \tag{5}$$

Applying the triangle inequality again:

$$0 \leq d(p'_n, q'_n) \leq d(p'_n, p_n) + d(p_n, q_n) + d(q_n, q'_n);$$

so sending $n \to \infty$, we obtain

$$0 \leq a' \leq a. \tag{6}$$

Together, (5) and (6) show $a = a'$. □

Solution (Part (3)). In order to show that X^* is complete, we must show that given a cauchy sequence $\{P_n\} \in X^*$, (under the Δ metric), there is a set $P \in X^*$ such that $\lim_{n\to\infty} P_n = P$.

Suppose $\{P_n\} \in X^*$ is cauchy. For every $\epsilon > 0$ there exists a corresponding $N \in \mathbb{N}$ such that for every $j \in \mathbb{N}$, and every $n \geq N$

$$\Delta(P_n, P_{n+j}) < \epsilon.$$

Refering to the definition for Δ, if $\{p_k^{(n)}\} \in P_n$ and $\{p_k^{(n+j)}\} \in P_{n+j}$,

$$\lim_{k\to\infty} d\left(p_k^{(n)}, p_k^{(n+j)}\right) = \Delta(P_n, P_{n+j}) < \epsilon.$$

Let's start by defining P to be the set which would have the desired property for which $P_n \mapsto P$. Define:

$$P := \Big\{\{p_k\}_{k=1}^{\infty} : \forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ \exists \{q_k^{(n)}\} \in P_n, \lim_{k\to\infty} d\left(p_k, q_k^{(n)}\right) < \epsilon\Big\}. \tag{7}$$

The first task is to show this set is non-empty. To start, there exists an $N_1 < N_2$ such that every $n_1 \geq N_1$, $n_2 \geq N_2$,

$$\begin{aligned} \Delta(P_{n_1}, P_{n_1+j}) &< 2^{-1}; \\ \Delta(P_{n_2}, P_{n_2+j}) &< 2^{-2}. \end{aligned} \tag{8}$$

In order to fill in the whole sequence, take any $p_1, p_2, \ldots, p_{N_1-1} \in X$. According to (8), for $N_1 \leq j_1 \leq N_2$ there are sequences $\{p_k^{(j_1)}\}$ such that

$$\lim_{k\to\infty} d(p_k^{(j_1)}, p_k^{(N_1)}) < 2^{-1}.$$

For these values, $N_1 \leq k \leq N_2$, set $p_k = p_k^{(N_1)}$. We proceed by induction. Let $i \in \mathbb{N}$, and suppose p_k has been chosen for each $k = 1, 2, \ldots, N_i$. There exists an N_{i+1}, $N_i < N_{i+1}$ such that

$$\Delta(P_{N_i}, P_{k_i}) < 2^{-(i+1)} \qquad \text{for} N_i \leq k_i; \tag{9}$$

THIS ISN'T QUITE RIGHT. I'M INDEXING ON THE WRONG VARIABLE... □

The final part (4) will not be proven. We're taking the limit of a constant, so $\Delta(P_p, Q_q) = d(p, q)$ is obvious. This is pretty cool nonetheless! OK - these problems are taking SUPER long to type up. Maybe I should start defining keyboard macros? Ok, I have a few defined, but a lot of the keys are already being used...

4. Continuity

Problem 6 (pg. 101 #20)**.** *If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by*

$$p_E(x) := \inf_d (x, z).$$

1. *Prove that $p_E(x) = 0$ iff $x \in E$.*
2. *Prove that p_E is a uniformly continuous function on X, by showing that*

   $$|p_E(x) - p_E(y)| \leq d(x, y)$$

   for all $x, y \in X$.

The book gives a hint. We'll solve this problem by following it.

Solution (Part (1)). First, a proof of $\Leftarrow$.

Proof ($\Leftarrow$). Suppose $x \in \bar{E}$. For all $r > 0$, there exists a

$$y \in (B_r(x) \setminus \{x\}) \cap E \neq \emptyset.$$

If $p_E(x) > 0$, there exists a y, $d(x, y) < p_E(x)/2$, so $p_E(x)$ is *not* a lower bound of $\{d(x, z) : \ z \in E\}$. □

Now, a proof of ($\Rightarrow$).

Proof ($\Rightarrow$). If $x \notin \bar{E}$, then there exists an $r > 0$, $B_r(x) \subset \bar{E}^c$. (Since $\bar{E}^c$ is open). This implies that

$$p_E(x) = \inf_{z \in E} \geq r > 0.$$

Therefore, by the contrapositive, $p_E(x) = 0$ implies $x \in \bar{E}$. □

Solution (Part (2))*.* Let $x, y \in X$, and take any $\epsilon > 0$. There exists a $z \in E$ such that

$$d(y, z) \leq p_E(y) + \epsilon. \tag{10}$$

(Use the *greatest* lower bound property). Since $z \in E$, we have, after one application of the triangle inequality, as well as an application of equation (10):

$$\begin{aligned} p_E(x) &\leq d(x, z) \leq d(x, y) + d(y, z) \\ &\leq d(x, y) + p_E(y) + \epsilon. \end{aligned}$$

Sending $\epsilon \downarrow 0$, we obtain that

$$p_E(x) - p_E(y) \leq d(x, y).$$

Since $d(x, y) = d(y, x)$, we can exchange the roles of x, y in all the previous lines in order to obtain the desired result. Namely, that

$$|p_E(x) - p_E(y)| \leq d(x, y).$$

□

Problem 7 (pg. 101, #21)**.** *Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$.*

Show that the conclusion may fail for two disjoint closed sets.

Solution. By problem #20, we know $p_F : K \to \mathbb{R}$ is uniformly continuous. Since K compact, we may apply the Max/Min theorem. There exists an $x_0 \in K$, $p_F(x_0) \geq p_F(x)$ for every $x \in K$. Whence, if we set $\delta = p_F(x_0)/2$ we have for $x \in K$, $y \in F$,

$$d(x, y) \geq p_F(x) \geq p_F(x_0) > \delta.$$

□

The conclusion fails if we consider subsets of $\mathbb{R}^2$:

$$\begin{aligned} K &:= \{(x, 0) : \ x \in \mathbb{R}\}; \\ F &:= \left\{(x, 1/(1 + x^2) : \ x \in \mathbb{R}\right\}; \end{aligned}$$

both of which are closed sets, and fail to meet the criterion displayed in the problem.

5. Differentiation

Problem 8 (pg. 114, #4)**.** *If*

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1},$$

where $C_0, C_1, \ldots, C_n$ *are real constants, prove that the equation*

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 *and* 1*.*

Solution. Consider the function, defined by

$$F(x) := \int_0^x f(t) \text{ dt},$$

where $f(x) := C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n$. Apply the mean value theorem to F on the interval $[0, 1]$.

Problem 9. *Suppose* f' *is continuous on* $[a, b]$ *and* $\epsilon > 0$*. Prove that there exists a* $\delta > 0$ *such that*

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta$*,* $a \leq x \leq b$*,* $a \leq t \leq b$*.*

Solution. Let $\epsilon > 0$. As f' is a continuous function on a compact set, f' is uniformly continuous. There exists a $\delta > 0$, for every $x, y \in [a, b]$, $|f'(x) - f'(y)| < \epsilon$ if $|x - y| < \delta$. Using this δ, apply the mean value theorem to the interval $[\min\{t, x\}, \max\{t, x\}]$. □

References

[1] Rudin, Walter "Principles of Mathematical Analysis," McGraw Hill, 1990.