Royden, Real Analysis 3rd ed.
Chapter 5

Problem 5-3 **a.** Obvious. But it should be

$$D_+ f(c) \le D^+ f(c) \le 0 \le D_- f(c) \le D^- f(c).$$

b. At the end point a we can only have $D^+ f(a)$ and $D_+ f(a)$. If f has a local maximum at a, we have

$$D_+ f(a) \le D^+ f(a) \le 0.$$

■

Problem 5-4 We may assume $D^+ f \ge \varepsilon > 0$. Otherwise apply the result to $f(x) + \varepsilon x$ and then let $\varepsilon \downarrow 0$.

Suppose the contrary that there exist $c < d$ such that $f(c) > f(d)$. Consider $\max_{[a,d]} f$, which must be attained at some point in $[a, d)$. By assumption there exist, for any $x \in [a, b)$, arbitrarily small $\delta_x > 0$ such that

$$f(x + \delta_x) - f(x) \ge \frac{\varepsilon \delta_x}{2}$$

which implies no point in $[a, d)$ can be a local maximum. A contradiction. ■

Problem 5-6 If $g(\gamma + h) \ne g(\gamma)$, then

$$\frac{f \circ g(\gamma + h) - f \circ g(\gamma)}{h} = \frac{f(g(\gamma + h)) - f(g(\gamma))}{g(\gamma + h) - g(\gamma)} \cdot \frac{g(\gamma + h) - g(\gamma)}{h} \quad (1)$$

a. If $g'(\gamma) > 0$, then $g(\gamma + h) - g(\gamma) \approx h \cdot g'(\gamma) > 0$ for h small and positive. Since the second term on the right-hand side of (1) converges to $g'(\gamma)$, we get by taking $\limsup$ that

$$D^+ f \circ g(\gamma) = D^+ f(g(x)) \cdot g'(\gamma).$$

b. Similar to part **a**, except that $g(\gamma + h) < g(\gamma)$ and the second term is negative. The fact that we need is the following

$$\overline{\lim}\, a_n \cdot b_n = \underline{\lim}\, a_n \cdot \lim b_n, \text{ if } \lim b_n < 0.$$

c. If $g(\gamma + h) = g(\gamma)$ then the left hand side of (1) is 0. We need only to consider the case $g(\gamma + h) \ne g(\gamma)$. Since $g'(\gamma) = 0$,

$$\frac{g(\gamma + h) - g(\gamma)}{h} = o(1) \text{ as } h \to 0.$$

The first term on the right-hand side is bounded by $\max(|D^+f(g(\gamma))|, |D_+f(g(\gamma))|, |D^-f(g(\gamma))|, |D_-f(g(\gamma))|)$. When multiplying together, the limit is 0. ■

Problem 5-7 **a.** By Theorem 5 we may assume f is a monotone increasing function. Then it is clear (why ?) that

$$\lim_{x\uparrow c} f(x) = f(c-) \text{ and } \lim_{x\downarrow c} f(x) = f(c+).$$

Moreover $f(c-) \leq f(c) \leq f(c+)$ and $f(x)$ is continuous at $x = c$ if and only if $f(c-) = f(c+)$. Hence the set of discontinuous points D is $\{x : f(c-) < f(c+)\}$ and may be written as

$$D = \cup_{n=1}^{\infty} D_n, \text{ when } D_n = \left\{x \in [a,b] : f(c+) - f(c-) > \frac{1}{n}\right\}.$$

Enough to show each D_n is countable. Let $a < x_1 < x_2 < x_3 < \cdots < x_m < b$ belong to D_n. With $x_0 = a$ and $x_{m+1} = b$,

$$\begin{aligned} f(b) - f(a) &= (f(b) - f(x_m + \varepsilon)) + (f(x_m + \varepsilon) - f(x_m - \varepsilon)) + \cdots \\ &\quad + (f(x_1 + \varepsilon) - f(x_1 - \varepsilon)) + f(x_1 - \varepsilon) + f(a) \\ &= [f(b) - f(x_m + \varepsilon)] + \sum_{k=1}^{m} (f(x_k + \varepsilon) - f(x_k - \varepsilon)) \\ &\quad + [f(x_1 - \varepsilon) + f(a)]. \end{aligned}$$

By letting ε small enough and using the monotonicity of f,

$$f(b) - f(a) \geq \sum_{k=1}^{m} (f(x_k + \varepsilon) - f(x_k - \varepsilon)) \geq \sum_{k=1}^{m} (f(x_k+) - f(x_k-)) \geq \frac{m}{n}.$$

Hence $m \leq (f(b) - f(a)) \cdot n$. That is $|D_n| \leq (f(b) - f(a)) \cdot n + 2$ and is finite.

b. Let $Q = \{a_1, a_2, \ldots\}$. For each a_n define $f_n(x) = \chi_{(a_n,1]}(x)$ so that f_n is increasing and discontinuous at $x = a_n$. (When $a_n = 1$, we set $f_n(x) = \chi_{\{1\}}(x)$ for this purpose.) The following function is what we want:

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

Since $\sum_{n=1}^{\infty} 2^{-n} < \infty$, the Weierstrass M Theorem tells us first that f is continuous at any irrational points and then

$$\sum_{n\neq m}^{\infty} 2^{-n} f_n(x) = f(x) - 2^{-m} f_m(x)$$

is continuous at $x = a_m$. Since f_m is discontinuous at $x = a_m$, so does f. ■

Problem 5-10 **a.** No. Let $x_n = ((n + 1/2)\pi)^{-1/2}$ for $n \geq 0$. Then $x_n \downarrow 0$ and $f(x_n) = (-1)^n 2/[(2n+1)\pi]$. Hence

$$\sum_{n=1}^{N} |f(x_n) - f(x_{n+1})| \geq \frac{4}{\pi}\sum_{n=1}^{N}\frac{1}{2n+1} \to \infty.$$

b. Yes. It is based on the following facts:
I. $h \in BV$ if h' is bounded on $[-1, 1]$.
II. For $x \neq 0$,

$$|g'(x)| = \left|2x\sin\frac{1}{x} - \cos\frac{1}{x}\right| \leq 2x + 1 \leq 3$$

and $g'(0) = 0$ as shown below

$$\left|\frac{g(h) - g(0)}{h}\right| = \left|h\sin\frac{1}{h}\right| \leq |h| \to 0 \quad \text{as } h \to 0.$$

■

Problem 5-11 By Lemma 4,

$$f(x) - f(a) = P_a^x - N_a^x. \tag{1}$$

Since both P and N are monotone increasing, Theorem 3 implies

$$\int_a^b (P_a^x)' + (N_a^x)'\,dx \leq P_a^b + N_a^b = T_a^b. \tag{2}$$

By (1), $|f'(x)| = \left|(P_a^x)' - (N_a^x)'\right| \leq (P_a^x)' + (N_a^x)'$. The conclusion follows from (2). ■

Problem 5-13 Enough to show

$$P_a^b(f) = \int_a^b (f')^+ \text{ and } N_a^b(f) = \int_a^b (f')^-. \tag{1}$$

By summing together, we will get

$$T_a^b(f) = P_a^b(f) + N_a^b(f) = \int_a^b (f')^+ + (f')^- = \int_a^b |f'|.$$

In the following we will show the first equation in (1) as the second one can be obtained similarly. For any partition $\{x_0 = a,\ x_1, \ldots,\ x_N = b\}$,

$$\begin{aligned}\sum_{j=1}^{N} \left(f\left(x_j\right) - f\left(x_{j-1}\right)\right)^+ &= \sum_{j=1}^{N} \left(\int_{x_{j-1}}^{x_j} f'(t)\, dt\right)^+ \\ &\leq \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \left(f'(t)\right)^+ dt = \int_a^b \left(f'\right)^+.\end{aligned}$$

(You should verify the inequality above). Hence

$$P_a^b(f) \leq \int_a^b \left(f'\right)^+ \tag{2}$$

To show the other direction, let $E = \{x \in [a, b] : f'(x) > 0\}$. Then

$$\int_a^b \left(f'\right)^+ = \int_E f'. \tag{3}$$

By using Proposition 3-15 and Proposition 4-14, we can find, for any $\varepsilon > 0$, a set $U = \cup_{i=1}^m (a_i, b_i)$ such that $U \Delta E$ is small and then

$$\left| \int_E f' - \int_U f' \right| < \varepsilon.$$

For any partition $\mathbf{P}$ with $\{a_i, b_i; 1 \leq i \leq m\} \subseteq \mathbf{P}$,

$$\begin{aligned}\sum_{j=1}^{N} \left(f\left(x_j\right) - f\left(x_{j-1}\right)\right)^+ &\geq \sum_{j=1}^{N} \left(f\left(b_j\right) - f\left(a_j\right)\right)^+ \\ &= \sum_{j=1}^{N} \int_{(a_j, b_j)} f' = \int_U f' \geq \int_E f' - \varepsilon.\end{aligned}$$

In view of (3), we get by $\varepsilon \downarrow 0$ that

$$P_a^b(f) \geq \int \left(f'\right)^+$$

and then (1) holds by combining with (2). ■

Problem 5-14 **a.** It is easier to use Theorem 14. Here we use the definition only. By definition we have, for any $\varepsilon > 0$, some $\delta > 0$ such that

$$\sum_{j=1}^{n} \left| h\left(x_j'\right) - h\left(x_j\right) \right| < \varepsilon \quad \text{if} \quad \sum_{j=1}^{n} \left| x_j' - x_j \right| < \delta$$

where $h = f$ or g. Hence

$$\begin{aligned}\sum_{j=1}^{n} \left|(f \pm g)\left(x_j'\right) - (f \pm g)\left(x_j\right)\right| &\leq \sum_{j=1}^{n} \left|f\left(x_j'\right) - f\left(x_j\right)\right| + \sum_{j=1}^{n} \left|g\left(x_j'\right) - g\left(x_j\right)\right| \\ &< 2\varepsilon\end{aligned}$$

if $\sum_{j=1}^{n} \left|x_j' - x_j\right| < \delta$.

b. Let $(\sup|f(x)|) \vee (\sup|g(x)|) = M$. Then

$$\begin{aligned}\left|(f \cdot g)\left(x_j'\right) - (f \cdot g)\left(x_j\right)\right| &= \left|f\left(x_j'\right)\left(g\left(x_j'\right) - g\left(x_j\right)\right) + \left(f\left(x_j'\right) - f\left(x_j\right)\right) g\left(x_j\right)\right| \\ &\leq M\left(\left|f\left(x_j'\right) - f\left(x_j\right)\right| + \left|g\left(x_j'\right) - g\left(x_j\right)\right|\right).\end{aligned}$$

What remains is routine.

c. Let $\inf|f(x)| = N > 0$ because f is continuous and $f' \neq 0$ on $[a, b]$. Then

$$\left|\frac{1}{f\left(x_j'\right)} - \frac{1}{f\left(x_j\right)}\right| = \frac{\left|f\left(x_j'\right) - f\left(x_j\right)\right|}{\left|f\left(x_j'\right) f\left(x_j\right)\right|} \leq N^{-2}\left|f\left(x_j'\right) - f\left(x_j\right)\right|.$$

What remains is routine. ■

Problem 5-15 Let $f_m(x) = \sum_{n=1}^{N \wedge m} b_n / 2^n$. It has been shown in Problem 2-48 that

$$f_m \uparrow f \text{ and } f' = 0 \text{ a.e.}$$

Since f_m is increasing,

$$f_m(x) \leq f_m(y) \text{ for } x \leq y.$$

Now let $m \to \infty$ and we get the monotonicity of f. If f would be absolutely continuous, we have from Theorem 14 that

$$1 = f(1) = f(0) + \int_0^1 f'(t)\, dt = 0 + 0, \text{ a contradiction.}$$

■

Problem 5-16 **a.** By Theorem 3, f' is integrable. Write

$$f(x) = \left[\int_a^x f'(t)\, dt\right] + \left(f(x) - \int_a^x f'(t)\, dt\right)$$

and use Theorem 14. The second term above is a singular function.

b. Let $c = b$ in the proof of Lemma 13. We then have

$$y_0 = a \leq x_1 < y_1 \leq x_2 < y_2 \leq \cdots < y_n \leq b = x_{n+1}$$

and

$$\sum_{k=0}^{n}(x_{k+1}-y_k)<\delta,\ \sum_{k=1}^{n}|f(y_k)-f(x_k)|\le\eta(b-a). \tag{1}$$

Since f is $\nearrow$,

$$f(b)-f(a)=\sum_{k=1}^{n}(f(y_k)-f(x_k))+\sum_{k=0}^{n}(f(x_{k+1})-f(y_k)).$$

By (1),

$$\sum_{k=0}^{n}(f(x_{k+1})-f(y_k))>f(b)-f(a)-\eta(b-a)$$

and the intervals $\{(y_k,x_{k+1}):0\le k\le n\}$ are what we want.

c. Let $J=\cup_{k=1}^{n}(x_k,y_k)$. Then by using Theorem 3 and (1)

$$\int_J f'=\sum_{k=1}^{n}\int_{x_k}^{y_k}f'\le\sum_{k=1}^{n}(f(y_k)-f(x_k))<\eta(b-a).$$

Since f' is integrable by Theorem 3, we get from (1) and Proposition 4-14 that

$$\int_{[a,b]\setminus J}f'=\int_{\cup_{k=1}^{n}(y_k,x_{k+1})}f'<\varepsilon$$

if we let δ in (1) satisfy the requirement in Theorem 4-14.

Combining together,

$$0\le\int_a^b f'\le\eta(b-a)+\varepsilon$$

which implies $\int_a^b f'=0$ by letting $\eta,\,\varepsilon\downarrow 0$. Since $f'\ge 0$, we have $f'=0$ a.e..

d. Let $\varepsilon,\,\delta>0$ be given

$$f(b)-f(a)=\sum_{n=1}^{\infty}(f_n(b)-f_n(a))<\infty,$$

there is an N such that

$$\alpha=\sum_{n=N+1}^{\infty}(f_n(b)-f_n(a))<\frac{\varepsilon}{2}. \tag{3}$$

Let $F_N(x)=\sum_{n=1}^{N}f_n(x)$. Obviously F_N is singular. By part **b**, there exist a finite collection $\{(c_k,d_k)\}$ of nonoverlapping intervals such that

$$\sum(d_k-c_k)<\delta \text{ and } \sum F_N(d_k)-F_N(c_k)>F_N(b)-F_N(a)-\frac{\varepsilon}{2}. \tag{4}$$

Combining (3) and (4) ,

$$\begin{aligned}\sum f(d_k)-f(c_k) &\geq \sum (F_N(d_k)-F_N(c_k)) > F_N(b)-F_N(a)-\frac{\varepsilon}{2}\\ &= f(b)-f(a)-\alpha-\frac{\varepsilon}{2} > f(b)-f(a)-\varepsilon.\end{aligned}$$

The conclusion follows from part **c**.

e. Let $g(x)$ be 1 on $[1,\infty)$, 0 on $(-\infty,0]$ and the Cantor ternary function on $[0,1]$. Then g is continuous, $g'=0$ a.e.. Let $Q\cap[0,1]=\{a_1,a_2,\ldots\}$. Define

$$g_{m,n}(x)=g\left(\frac{x-a_m}{a_n-a_m}\right) \text{ when } a_m<a_n$$

and then

$$g(x)=\sum_{\substack{m,n=1\\ a_m<a_n}}^{\infty} g_{m,n}(x)\diagup 2^{m+n}.$$

Note that

$$\sum_{m,n=1}^{\infty}\frac{1}{2^{m+n}}=\sum_{k=2}^{\infty}\sum_{m+n=k}\frac{1}{2^{m+n}}=\sum_{k=2}^{\infty}\frac{k-1}{2^k}<\infty.$$

Since $g_{m,n}(a_m)=0<1=g_{m,n}(a_n)$ and Q is dense, g is strictly increasing on $[0,1]$. Moreover, g is continuous by Weierstrass M Theorem and singular by part **d.** ■

Problem 5-17 **a.** Roughly speaking, that g is absolutely continuous implies

$$\text{if } \sum_{i=1}^{n}|x_i'-x_i|=o(1) \text{ then } \sum_{i=1}^{n}|g(x_i')-g(x_i)|=o(1).$$

Using the absolutely continuity of F, we then have

$$\sum_{i=1}^{n}|F(g(x_i'))-F(g(x_i))|=o(1).$$

That is $F\circ g$ is absolutely continuous.

b. Fix $\varepsilon>0$. For any $x\in E$, we can find arbitrarily small interval $[x,x+h)$ such that

$$|g(y)-g(x)|<\varepsilon(y-x) \text{ for } y\in[x,x+h). \tag{1}$$

Using Vitali's Lemma, there are a finite number of disjoint intervals $\{I_j=[x_j,x_j+h_j):1\leq j\leq N\}$ in $[a,b]$ such that

$$m^*\left(E\diagdown\cup_{j=1}^{N}I_j\right)<\delta$$

where δ is the number corresponding to ε in the definition of the absolute continuity of f.

By definition of outer measure, we can find open intervals $J_k = (c_k, d_k)$ such that

$$\cup_{k=1}^{\infty} J_k \supseteq E \diagdown \cup_{j=1}^{N} I_j \text{ with } \sum_{k=1}^{\infty} |J_k| = \sum_{k=1}^{\infty} (d_k - c_k) < \delta. \tag{2}$$

Since $E \subseteq \left(\cup_{j=1}^{N} I_j\right) \cup \left(\cup_{k=1}^{\infty} J_k\right)$,

$$g(E) \subseteq \left(\cup_{j=1}^{N} g(I_j)\right) \cup \left(\cup_{k=1}^{\infty} g(J_k)\right). \tag{3}$$

By (1), $g(I_j) \subseteq (g(x_j) - \varepsilon h_j, g(x_j) + \varepsilon h_j)$ and then

$$m^* \left(\cup_{j=1}^{N} g(I_j)\right) \leq \sum_{j=1}^{N} 2\varepsilon h_j \leq 2\varepsilon (b-a). \tag{4}$$

Since g is continuous, $g(J_k)$ is an interval, which may be taken as an open interval as this won't change its measure. Also $g(J_k)$ can be approximated by $\cup_{l=0}^{n_k} (g(x_{k,l}), g(x_{k,l+1}))$, where $\mathbf{P}_k = \{x_{k,0} = c_k, x_{k,1}, \ldots, x_{k,n_k}, x_{k,n_k+1} = d_k\}$ is a partition of J_k. By (2),

$$\sum_{k=1}^{L} \sum_{l=0}^{n_k} |x_{k,l} - x_{k,l+1}| < \delta \quad \text{for any } L.$$

It follows from the absolute continuity of g that

$$\sum_{k=1}^{L} \sum_{l=0}^{n_k} |g(x_{k,l}) - g(x_{k,l+1})| < \varepsilon. \tag{5}$$

We may assume $g(J_k)$ is an open interval as its measure is unchanged by deleting the endpoints. Now

$$g\left(\cup_{k=1}^{\infty} J_k\right) = \cup_{k=1}^{\infty} g(J_k)$$

is an open set as can be written as a countable union of open intervals W_k by Proposition 2-8. Each W is the union of some $g(J_k)$'s. For brevity we may assume $g(J_k)$ are mutually disjoint.

By varying $\mathbf{P}_k$ of J_k we get from (5) that

$$m\left(\cup_{k=1}^{L} g(J_k)\right) < \varepsilon \quad \text{for any } L \geq 1.$$

Letting $L \to \infty$ and then $\varepsilon \downarrow 0$, we have

$$m\left(g\left(\cup_{k=1}^{\infty} J_k\right)\right) = m\left(\cup_{k=1}^{\infty} g(J_k)\right) = 0. \tag{6}$$

Combining with (3) and (4), we finally get $m^*(g(E)) = 0$. That is the same to say $m(g(E)) = 0$. ■

Problem 5-18 It is contained in the solution to Problem 5-17. See from the line next to (4) there. ■

Problem 5-21 **a.** By Proposition 2-8, $O = \cup_{i=1}^{\infty} I_i$, where $I_i = (\alpha_i, \beta_i)$ are disjoint. Then $g^{-1}(O) = \cup_{i=1}^{\infty} g^{-1}(I_i)$. Since g is continuous and monotone increasing, $g^{-1}(I_i)$ is an open interval, say, (u_i, v_i) with $g(u_i) = \alpha_i$ and $g(v_i) = \beta_i$.

In order to show

$$mO = \int_{g^{-1}(O)} g'(x)\, dx, \tag{1}$$

it suffices to show by Proposition 3-13 and Proposition 4-12 that

$$\beta_i - \alpha_i = mI_i = \int_{g^{-1}(I_i)} g'(x)\, dx = \int_{u_i}^{v_i} g'(x)\, dx.$$

By Corollary 15, the RHS is $g(v_i) - g(u_i) = \beta_i - \alpha_i$. We are done.

b & c. Let $\mathfrak{g}$ be the family of all measurable set $E \subseteq [a, b]$ such that

$$mE = \int_{g^{-1}(E)\cap H} g'(x)\, dx. \tag{2}$$

Part **a** shows that (2) holds for any open set E because

$$\int_F g'(x)\, dx = \int_{F\cap H} g'(x)\, dx \quad \text{for any measurable set } F.$$

The same argument used in proving (1) shows that $\mathfrak{g}$ is closed under taking complement and countable unions. In view of Theorem 3-10, it remains to show that $E \in \mathfrak{g}$ if $mE = 0$. By definition there exists open set $O_n \supseteq E$ with $mO_n < 1/n$. Then $g^{-1}(E) \subseteq g^{-1}(O_n)$, which is open by the continuity of g and

$$\frac{1}{n} > mO_n = \int_{g^{-1}(O_n)\cap H} g'(x)\, dx. \tag{3}$$

Let $\underline{\lim}\, g^{-1}(O_n) = F$, which is $\cup_{n=1}^{\infty} \cap_{k\geq n} g^{-1}(O_k)$ and measurable. Note that $F \supseteq g^{-1}(E)$. Applying Fatou's Lemma to (3),

$$0 \geq \int_{F\cap H} g'(x)\, dx. \tag{4}$$

Since $g'(x) > 0$ on H, it follows that $m(F \cap H) = 0$. Because $g^{-1}(E) \subseteq F$, $m^*(g^{-1}(E) \cap H) = 0$ and thus $g^{-1}(E) \cap H$ is measurable by Lemma 3-6. Moreover (4) shows

$$mE = 0 = \int_{g^{-1}(E)\cap H} g'(x)\, dx.$$

This completes the proof.

d. Rewrite (2) as

$$\int_c^d \chi_E(x)\,dx = mE = \int_a^b \chi_E(g(x)) \cdot g'(x)\,dx. \tag{5}$$

If $f \geq 0$ on $[c,d]$, we can find simple function f_n, e.g.,

$$f_n(x) = \sum_{k=1}^{2^{2n}} k{\cdot}2^{-n}\chi_{E_{n,k}}(x), \text{ where } E_{n,k} = \left\{x : k \cdot 2^{-n} < f(x) \leq (k+1) \cdot 2^{-n}\right\},$$

such that $f_n \uparrow f$ everywhere. By (5),

$$\int_c^d f_n(x)\,dx = \int_a^b f_n(g(x)) \cdot g'(x)\,dx.$$

Since $\lim\limits_{n\to\infty} f_n(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$ for each $x \in [a,b]$, $f \circ g \cdot g'$ is measurable and by the Monotone Convergence Theorem

$$\int_c^d f(x)\,dx = \int_a^b f(g(x)) \cdot g'(x)\,dx.$$

■

Problem 5-23 **a.** Choose $a < c < b$ and consider the supporting line through $(c, \varphi(c))$. We have

$$\varphi(x) \geq m(x-c) + \varphi(c) \quad \text{for } a < x < b. \tag{1}$$

It also holds for $x = a$ as φ is continuous at $x = a$ by assumption. Hence

$$\varphi(x) \geq -|m|(b-a) + \varphi(c).$$

b. By Proposition 17 and Corollary 15,

$$\varphi(x) - \varphi(c) = \int_c^x D^+f(t)\,dt \quad \text{for } x \in (a,b). \tag{2}$$

If D^+f never changes sign, then $D^+f \geq 0$ or $D^+f \leq 0$. In either case (2) shows φ is $\nearrow$ or $\searrow$ on (a,b). Hence $\varphi(a+)$ and $\varphi(b-)$ exist, where $\pm\infty$ is allowed. Otherwise there exists a point $c \in (a,b)$ such that $D^+f(x) \geq 0$ on (c,b) and $D^+f(x) \leq 0$ on (a,c) because $D^+f \uparrow$ by Proposition 17.

It is then clear from (2) that φ is $\searrow$ on $(a,c]$ and $\nearrow$ on $[c,b)$ and the same conclusion follows as above.

If a (or b) is finite, (1) implies by letting $x \downarrow a$ (or $x \uparrow b$) that $\varphi(a+)$ (or $\varphi(b-)$) cannot be $-\infty$.

c. The formula holds for $t \in [0,1]$ and $x, y \in (a,b)$. If $a \in I$ then we just let $x \downarrow a$ and use the continuity of φ. Repeat the same argument if $b \in I$ as well. ■

Problem 5-24 ($\Leftarrow$) If $\varphi''(x) \geq 0$ for each $x \in (a,b)$, then φ' exists of course and by Mean-Value Theorem

$$\varphi'(y) - \varphi'(x) = (y-x)\varphi''(\xi) \geq 0 \quad \text{for } y > x.$$

This is $\varphi' \uparrow$ on (a,b) and then φ is convex by Proposition 18.

($\Rightarrow$) By Proposition $\varphi' \uparrow$. Since $\varphi''(x)$ exists, we obviously have $\varphi'' \geq 0$ on (a,b). ■

Problem 5-25 **a.** Compute φ'' and use Corollary 19.

b. When $p > 1$, $\varphi''(t) > 0$ on $(0,\infty)$. Now apply the following result:

If $\varphi'' > 0$ on (a,b), then φ is strictly convex on $[a,b]$.

Proof.

By Problem 5–23**c**, we need only to show that

$$\varphi(\lambda x + (1-\lambda)y) < \lambda\varphi(x) + (1-\lambda)\varphi(y) \text{ for } x, y \in (a,b) \text{ and } 0 < \lambda < 1.$$

Assume $x < y$. The formula above is equivalent to

$$\frac{\varphi(\lambda x + (1-\lambda)y) - \varphi(x)}{(1-\lambda)(y-x)} < \frac{\varphi(y) - \varphi(\lambda x + (1-\lambda)y)}{\lambda(y-x)}.$$

By the Mean-Value Theorem, the $LHS = \varphi'(\xi_1)$ and the $RHS = \varphi'(\xi_2)$, where

$$x < \xi_1 < \lambda x + (1-\lambda)y < \xi_2 < y.$$

By the Mean-Value Theorem again,

$$RHS - LHS = \varphi'(\xi_2) - \varphi'(\xi_1) = (\xi_2 - \xi_1)\varphi''(z) > 0,$$

where $\xi_1 < z < \xi_2$. It is done. ■

Problem 5-26 It is clear from the proof of Jensen Inequality that the equality holds in Proposition 20 if and only if for a.e. t in $(0,1)$

$$\varphi(f(t)) = m(f(t) - \alpha) + \varphi(\alpha). \tag{1}$$

Here $y = m(x-\alpha) + \varphi(\alpha)$ is a supporting line at $(\alpha, \varphi(\alpha))$. When $\varphi(x) = e^x$, $\varphi''(x) > 0$ and thus φ is strictly convex. It is easy to show that for any $y \neq \alpha$

$$e^y > e^\alpha(y-\alpha) + e^\alpha. \tag{2}$$

Combining (1) and (2), it follows that the equality holds in Corollary 21 if and only if $f(t) = \alpha$ a.e.. That is $f(t)$ is a constant a.e..

Remark. Simple algebraic operation shows that (2) is equivalent to

$$e^{(y-\alpha)} > 1 + (y - \alpha) \text{ or } e^x > 1 + x \quad \text{for } x \neq 0.$$

You may use calculus like

$$e^x - 1 = e^x - e^0 = \int_0^x e^t dt$$

or use

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{when } x > 0.$$

■

Problem 5-27 Let $S_n = \sum_{j=1}^n \alpha_j$. Then $S_0 = 0 < S_1 < S_2 < \cdots < 1$. Define f on $(0,1)$ such that $f\big|_{(S_{n-1},S_n)} = \log \xi_n$ for $n > 0$. It is easy to see

$$\exp\left(\int_0^1 f(t)\,dt\right) = \exp\left(\sum_{n=1}^{\infty} \alpha_n \log \xi_n\right) = \prod_{n=1}^{\infty} \xi_n^{\alpha_n}$$

and

$$\int_0^1 \exp(f(t))\,dt = \sum_{n=1}^{\infty} \alpha_n \exp(\log \xi_n) = \sum_{n=1}^{\infty} \alpha_n \xi_n.$$

The conclusion follows from Corollary 21 if f is integrable. For general f, we may define f_N with $f_N = f$ on $(0, S_n)$ and is 1 on $(S_n, 1)$. Corollary 21 shows

$$\prod_{k=1}^{n} \xi_k^{\alpha_k} = \sum_{k=1}^{n} \alpha_k \xi_k.$$

Now let $n \to \infty$. ■

Problem 5-28 $f(t) = -\log t$ is a convex function. Now apply the Jensen Inequality. ■

Problem 2-48 Let $f_m(x) = \sum_{n=1}^{N \wedge m} b_n / 2^n$ for $m = 1, 2, \ldots$. Draw the figures of f_1 and f_2. Then you should see how to get f_3, $f_4, \ldots$ and so on. Each f_m is a continuous function and

$$|f_m(x) - f(x)| \leq \sum_{n=m+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^m} \to 0 \quad \text{as } m \to \infty.$$

That is $f_m \to f$ uniformly. Hence f is also a continuous function. What remains is clear from the figure of f. ■