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On certain subclasses of prestarlike functions [1]

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Abstract

In the present investigation, we introduce and study interesting properties of a new unified class of pre-starlike functions with negative coefficients in the open unit disk Δ. These properties include growth and distortion, radii of convexity, radii of starlikeness and radii of close to convexity.

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1 Introduction and Motivations

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

that are analytic in the open unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{S}^*(\alpha)$, we mean the class of analytic functions that satisfy the analytic condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in \Delta)$$

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for $0 \leq \alpha < 1$. In particular, $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^*$, the well-known standard class of starlike functions. For functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or Convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \Delta. \tag{2}$$

A function $f \in \mathcal{S}$ is said to be convex of order $\alpha, (0 \leq \alpha < 1)$ if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \Delta).$$

This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K} = \mathcal{K}(0)$, is the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha).$$

Let the function

$$\mathcal{S}_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha)}}, \quad (z \in \Delta,\, 0 \leq \alpha < 1) \tag{3}$$

which is the extremal function for the class $\mathcal{S}^*(\alpha)$. We also note that $\mathcal{S}_\alpha(z)$ can be written in the form

$$\mathcal{S}_\alpha(z) = z + \sum_{n=2}^{\infty} |C_n(\alpha)| z^n, \tag{4}$$

where

$$C_n(\alpha) = \frac{\prod_{j=2}^{n}(j - 2\alpha)}{(n-1)!} \quad (n \in \mathbb{N} \setminus \{1\}, \mathbb{N} := \{1, 2, 3, \cdots\}). \tag{5}$$

We note that $C_n(\alpha)$ is decreasing in α and satisfies

$$\lim_{n \to \infty} C_n(\alpha) = \begin{cases} \infty & \text{if } \alpha < \frac{1}{2} \\ 1 & \text{if } \alpha = \frac{1}{2} \\ 0 & \text{if } \alpha > \frac{1}{2} \end{cases}. \tag{6}$$

Let $\mathcal{R}[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$ denote the class of pre-starlike functions satisfying the following condition

$$\left| \frac{\dfrac{(h*\Phi)(z)}{(h*\Psi)(z)} - 1}{2\gamma(B-A)\left(\dfrac{(h*\Phi)(z)}{(h*\Psi)(z)} - \zeta\right) - B\left(\dfrac{(h*\Phi)(z)}{(h*\Psi)(z)} - 1\right)} \right| < \beta, \tag{7}$$

where

$$h(z) = f * \mathcal{S}_\alpha(z),\ 0 < \beta \leq 1,\ 0 \leq \zeta < 1,$$

$$\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n,$$

and

$$\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$$

analytic in Δ, with $\lambda_n \geq 0, \mu_n \geq 0$, $\lambda_n \geq \mu_n$, for $n = 2, 3, 4, \cdots$, and

$$\frac{B}{2(B-A)} < \gamma \leq \begin{cases} \frac{B}{2(B-A)\zeta}, & \text{if } \zeta \neq 0 \\ 1 & \text{if } \zeta \neq 0 \end{cases}$$

for fixed $-1 \leq A \leq B \leq 1$ and $0 \leq B \leq 1$.

We note that a function f is so called pre-starlike of order α function $(0 \leq \alpha < 1)$ if and only if $f * \mathcal{S}_\alpha$ is a starlike function of order α, which was introduced by Ruscheweyh [5]. Many subclasses of the pre-starlike function were studied by Silverman and Silvia [6], (see also [9]), Owa and Ahuja [4], Maslina Darus [2] and also by Uralegaddi and Sarangi [10]. Our results generalize the results of Maslina Darus [2] and also some other known results.

Let T denote the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0. \tag{8}$$

Let us write

$$\mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B] = \mathcal{R}[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B] \cap T,$$

where T is the class all functions with negative coefficients and of the form (8) that are analytic and univalent in Δ.

In this paper, we make a systematic investigation of the newly defined class $\mathcal{R}[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$. It is assumed throughout this paper that $\Phi(z)$, and $\Psi(z)$ satisfy the conditions stated in (7) and that $(h*\Psi)(z) \neq 0$ for $z \in \Delta$.

2 Coefficient inequalities

Our main tool in this paper is the following result, which can be easily proven, and the details are omitted.

Theorem 1 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, *be in the class* $\mathcal{A}$. *If, for some* A *and* B *with* $-1 \leq A < B \leq 1$,

$$\sum_{n=2}^{\infty} \sigma(\Phi, \Psi, n) C_n(\alpha)|a_n| \leq 2\beta\gamma(1-\zeta)(B-A) \tag{9}$$

where

$$\sigma(\Phi, \Psi, n) = (\lambda_n - \mu_n)(1 - B\beta) + 2\gamma\beta(B-A)(\lambda_n - \zeta\mu_n) \tag{10}$$

then $f \in \mathcal{R}[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$. *The result is sharp.*

Proof. Suppose the condition (9) holds for all admissible values of A and B. In view of (7), it is enough to prove that

$$\left| \frac{\dfrac{(h*\Phi)(z)}{(h*\Psi)(z)} - 1}{2\gamma(B-A)\left(\dfrac{(h*\Phi)(z)}{(h*\Psi)(z)} - \zeta\right) - B\left(\dfrac{(h*\Phi)(z)}{(h*\Psi)(z)} - 1\right)} \right| < \beta, \tag{11}$$

For $|z| = r$, $0 \leq r < 1$, we have
$|(h*\Phi)(z) - (h*\Psi)(z)|$

$$-\beta \left| 2\gamma(B-A)\left\{(h*\Phi)(z) - \zeta(h*\Psi)(z)\right\} - B\left\{(h*\Phi)(z) - (h*\Psi)(z)\right\}\right|$$

$$\leq \sum_{n=2}^{\infty} \left\{(\lambda_n - \mu_n)(1 - B\beta) + 2\beta\gamma(B-A)(\lambda_n - \zeta\mu_n)\right\} C_n(\alpha)|a_n|r^n -$$

$$-2\beta\gamma(1-\zeta)(B-A)r$$

which is clearly lesser than zero as $r \to 1$ in view of (9). Thus, (11) is satisfied and hence, $f \in \mathcal{R}[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$.
For the choices of

$$\Psi(z) = (1-\lambda)\frac{z}{1-z} + \lambda\frac{z}{(1-z)^2}$$

and

$$\Phi(z) = (1-\lambda)\frac{z}{(1-z)^2} + \lambda\frac{z+z^2}{(1-z)^3},$$

we get the following result of Maslina Darus [2]

Corollary 1 *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, be in the class $\mathcal{A}$. If, for some A and B with $-1 \leq A < B \leq 1$,*

$$\sum_{n=2}^{\infty} \sigma(\Phi, \Psi, n) C_n(\alpha) |a_n| \leq 2\beta\gamma(1-\zeta)(B-A) \tag{12}$$

where

$$\sigma(\Phi, \Psi, n) = (1 + (n-1)\lambda)\left\{n - 1 + 2\beta\gamma(n-\zeta)(B-A) - B\beta(n-1)\right\} \tag{13}$$

then $f \in \mathcal{R}[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$.

Theorem 2 *Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, be in the class T. Then $f \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$ if and only if (9) is satisfied.*

Proof. In view of Theorem 1, it is sufficient to show the "only if" part. Thus, let $f \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$. Then,

$$|\omega(z)| = \left| \frac{\frac{(h*\Phi)(z)}{(h*\Psi)(z)} - 1}{2\gamma(B-A)\left(\frac{(h*\Phi)(z)}{(h*\Psi)(z)} - \zeta\right) - B\left(\frac{(h*\Phi)(z)}{(h*\Psi)(z)} - 1\right)} \right|. \tag{14}$$

Using the power series expansion for f, Φ and Ψ, we get,

$$|\omega(z)| = \tag{15}$$

$$\left| \frac{\sum_{n=2}^{\infty} (\lambda_n - \mu_n) C_n(\alpha) a_n z^n}{\sum_{n=2}^{\infty} \left[2\gamma(B-A)(\lambda_n - \zeta\mu_n) - B(\lambda_n - \mu_n)\right] a_n C_n(\alpha) z^n - 2\gamma(B-A)(1-\zeta)|z|} \right|$$

$$< \beta$$

and hence,

(16)

$$\Re\left(\frac{\sum_{n=2}^{\infty} (\lambda_n - \mu_n) C_n(\alpha) a_n z^n}{\sum_{n=2}^{\infty} \left[2\beta\gamma(B-A)(\lambda_n - \zeta\mu_n) C_n(\alpha) a_n z^n - \beta B(\lambda_n - \mu_n) C_n(\alpha) a_n z^n\right] - 2\gamma(B-A)(1-\zeta)z} \right)$$

is less than β for all $z \in \Delta$. We consider real values of z and take $z = r$ with $0 < r < 1$. Then, for $r = 0$, the denominator of (16) is positive and so is for all $r, 0 \leq r < 1$. Then (16) gives,

$$\sum_{n=2}^{\infty} \sigma(\Phi, \Psi, n) C_n(\alpha) |a_n| r^{n-1} \leq 2\beta\gamma(1-\zeta)(B-A) \tag{17}$$

where $\sigma(\Phi, \Psi, n)$ is as defined in (10). Letting $r \to 1^-$, we get (9).
For the choices of

$$\Psi(z) = (1-\lambda)\frac{z}{1-z} + \lambda\frac{z}{(1-z)^2}$$

and

$$\Phi(z) = (1-\lambda)\frac{z}{(1-z)^2} + \lambda\frac{z+z^2}{(1-z)^3},$$

we get the following result of Maslina Darus [2]

Corollary 2 *Let* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$*, be in the class* T*. If, for some* A *and* B *with* $-1 \leq A < B \leq 1$*,*

$$\sum_{n=2}^{\infty} \sigma(\Phi, \Psi, n) C_n(\alpha) a_n \leq 2\beta\gamma(1-\zeta)(B-A) \tag{18}$$

where

$$\sigma(\Phi, \Psi, n) = (1 + (n-1)\lambda)\{n - 1 + 2\beta\gamma(n-\zeta)(B-A) - B\beta(n-1)\} \tag{19}$$

then $f \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$.

Indeed, since $f \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$ from (9), we have

$$\sum_{n=2}^{\infty} \sigma(\Phi, \Psi, n) C_n(\alpha) |a_n| \leq 2\beta\gamma(1-\zeta)(B-A),$$

where $\sigma(\Phi, \Psi, n)$ is as defined in (10). Hence for all $n \geqq 2$, we have

$$a_n \leq \frac{2\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi, \Psi, n) C_n(\alpha)}.$$

whenever $-1 \leq A < B \leq 1$. Hence we state this important observation as a separate theorem.

Theorem 3 *If* $f \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$, *then*

$$a_n \leq \frac{2\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi, \Psi, n)C_n(\alpha)}, \ n \geq 2, \tag{20}$$

where $-1 \leq A < B \leq 1$. *Equality in (20) holds for the function*

$$f(z) = z - \frac{\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi, \Psi, n)(1-\alpha)} z^2. \tag{21}$$

3 Covering theorem and Distortion Bounds

Theorem 4 *If* $f \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$ *then* $f \in T^*(\xi)$, *the class of starlike functions of order* ξ, $(0 \leq \xi < 1)$, *with*

$$\xi = 1 - \frac{2\beta\gamma(1-\zeta)(B-A)}{(1+\beta)\sigma(\Phi, \Psi, 2)C_2(\alpha) - 2\beta\gamma(1-\zeta)(B-A)}.$$

This result is sharp with the extremal function being

$$f(z) = z - \frac{\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi, \Psi, 2)} z^2. \tag{22}$$

Proof. It is sufficient to show that (9) implies $\sum_{n=2}^{\infty}(n-\xi)a_n \leq 1-\xi$ [6], that is,

$$\frac{n-\xi}{1-\xi} \leq \frac{\sigma(\Phi, \Psi, n)C_n(\alpha)}{2\beta\gamma(1-\zeta)(B-A)}, \ n \geqq 2. \tag{23}$$

Since, for $n \geqq 2$, (23) is equivalent to

$$\xi \ \leq \ 1 - \frac{2(n-1)\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi, \Psi, n)C_n(\alpha) - 2\beta\gamma(n-1)(1-\zeta)(B-A)} = \Phi_1(n),$$

and $\Phi_1(n) \leq \Phi_(2)$, (23) holds true for any $n \geq 2$, and for $-1 \leq B < A \leq 1$. This completes the proof of the Theorem 4.

Theorem 5 *Let* $\sigma(\Phi, \Psi, n)$ *be as defined in (10). Then, for* $f \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$ *with* $z = re^{i\theta} \in \Delta$, *we have*

$$r - \frac{\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi, \Psi, 2)} r^2 \leq |f(z)| \leq r + \frac{\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi, \Psi, 2)} r^2. \tag{24}$$

Proof. Observing that $C_n(\alpha)$ defined by (5) is increasing for $0 \le \alpha \le \frac{1}{2}$, we find from Theorem 2 that,

$$\sum_{n=2}^{\infty} a_n \le \frac{2\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi,\Psi,2)C_2(\alpha)}. \tag{25}$$

Using (8) and (25), we readily have for $z \in \Delta$,

$$|f(z)| \ge |z| - |z|^2 \sum_{n=2}^{\infty} a_n \ge |z| - \frac{2\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi,\Psi,2)C_2(\alpha)}|z|^2, \tag{26}$$

$$|f(z)| \le |z| + |z|^2 \sum_{n=2}^{\infty} a_n \le |z| + \frac{2\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi,\Psi,2)C_2(\alpha)}|z|^2, \tag{27}$$

and noting that $C_2(\alpha) = 2(1-\alpha)$, we get the assertion (24) of Theorem 5.

Theorem 6 *If $f \in \mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$, then for $|z| = r < 1$*

$$1 - \frac{2\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi,\Psi,2)}r \le |f'(z)| \le 1 + \frac{2\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi,\Psi,2)}r. \tag{28}$$

Proof. Using (8), we readily have for $z \in \Delta$,

$$|f'(z)| \ge 1 - |z| \sum_{n=2}^{\infty} a_n \ge |z| - \frac{4\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi,\Psi,2)C_2(\alpha)}|z|, \tag{29}$$

$$|f'(z)| \le |z| + |z|^2 \sum_{n=2}^{\infty} a_n \le |z| + \frac{4\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi,\Psi,2)C_2(\alpha)}|z|, \tag{30}$$

and noting that $C_2(\alpha) = 2(1-\alpha)$, we get the assertion (28) of Theorem 6.

4 Radii of close-to-convexity, starlikeness and convexity

Theorem 7 *Let the function f be in the class $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$. Then $f(z)$ is close-to-convex of order ρ, $0 \le \rho < 1$ in $|z| < r_1(\beta,\gamma,A,B,\rho)$, where*

$$r_1(\beta,\gamma,A,B,\rho)=\inf_n\left[\frac{(1-\rho)\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2n\beta\gamma(1-\zeta)(B-A)}\right]^{\frac{1}{n-1}}, \qquad n\geqq 2,$$

with $\sigma(\Phi,\Psi,n)$ be defined as in (10). This result is sharp for the function $f(z)$ given by (22).

Proof. Since

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,$$

we get

$$f'(z)=1-\sum_{n=2}^{\infty}na_nz^{n-1}.$$

It is sufficient to show that $|f'(z)-1|\leq 1-\rho$, $0\leq\rho<1$, for $|z|<r_1(\beta,\gamma,A,B,\rho)$, or equivalently

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right)a_n|z|^{n-1}\leq 1. \tag{31}$$

By Theorem 1, (31) will be true if

$$\left(\frac{n}{1-\rho}\right)|z|^{n-1}\leq\frac{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2\beta\gamma(1-\zeta)(B-A)}$$

or, if

$$|z|\leq\left[\frac{(1-\rho)\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2n\beta\gamma(1-\zeta)(B-A)}\right]^{\frac{1}{n-1}}, \quad n\geqq 2. \tag{32}$$

The theorem follows easily from (32).

Theorem 8 *Let the function $f\in T$ be in the class $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$ Then $f(z)$ is starlike of order ρ, $0\leq\rho<1$ in $|z|<r_2(\beta,\gamma,A,B,\rho)$, where*

$$r_2(\beta,\gamma,A,B,\rho)=\inf_n\left[\frac{(1-\rho)\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2(n-\rho)\beta\gamma(1-\zeta)(B-A)}\right]^{\frac{1}{n-1}}, \qquad n\geq 2,$$

with $\sigma(\Phi,\Psi,n)$ be defined as in (10). This result is sharp for the function $f(z)$ given by (22).

Proof. Since,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,$$

we get

$$f'(z) = 1 - \sum_{n=2}^{\infty} n a_n z^{n-1}.$$

It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \text{ or equivalently} \tag{33}$$

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{n=2}^{\infty} (n-1) a_n z^n}{z - \sum_{n=2}^{\infty} a_n z^n} \right| \leq 1 - \rho \text{ or}$$

$$\sum_{n=2}^{\infty} \left(\frac{n-\rho}{1-\rho} \right) a_n |z|^{n-1} \leq 1,$$

for $0 \leq \rho < 1$, and $|z| < r_2(\beta, \gamma, A, B, \rho)$. By Theorem 1, (33) will be true if

$$\left(\frac{n-\rho}{1-\rho} \right) |z|^{n-1} \leq \frac{\mathcal{C}_n(\alpha) \sigma(\Phi, \Psi, n)}{2\beta\gamma(1-\zeta)(B-A)}$$

or, if

$$|z| \leq \left[\frac{(1-\rho)\mathcal{C}_n(\alpha)\sigma(\Phi, \Psi, n)}{2(n-\rho)\beta\gamma(1-\zeta)(B-A)} \right]^{\frac{1}{n-1}}, \quad n \geqq 2. \tag{34}$$

The theorem follows easily from (32).

Theorem 9 *Let the function $f \in T$ be in the class $\mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$. Then $f(z)$ is convex of order ρ, $0 \leq \rho < 1$ in $|z| < r_3(\beta, \gamma, A, B, \rho)$ where*

$$r_3(\beta, \gamma, A, B, \rho) = \inf_n \left[\frac{(1-\rho)\mathcal{C}_n(\alpha)\sigma(\Phi, \Psi, n)}{2n(n-\rho)\beta\gamma(1-\zeta)(B-A)} \right]^{\frac{1}{n-1}}, \qquad n \geqq 2,$$

with $\sigma(\Phi, \Psi, n)$ be defined as in (10). This result is sharp for the function $f(z)$ given by (22).

Proof. It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\rho \qquad \text{or equivalently} \tag{35}$$

$$\left|\frac{-\sum_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} na_n z^{n-1}}\right| \leq 1-\rho \qquad \text{or equivalently}$$

$$\sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right) a_n|z|^{n-1} \leq 1,$$

for $0 \leq \rho < 1$ and $|z| < r_3(\beta,\gamma,A,B,\rho)$. By Theorem 1, (35) will be true if

$$\left(\frac{n(n-\rho)}{1-\rho}\right)|z|^{n-1} \leq \frac{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2\beta\gamma(1-\zeta)(B-A)}$$

or, if

$$|z| \leq \left[\frac{(1-\rho)\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2n(n-\rho)\beta\gamma(1-\zeta)(B-A)}\right]^{\frac{1}{n-1}}, \quad n \geqq 2. \tag{36}$$

The theorem follows easily from (36).

5 Extreme points of the class $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$

Theorem 10 *Let $f_1(z)=z$ and*

$$f_n(z) = z - \frac{2\beta\gamma(1-\zeta)(B-A)}{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}z^n, \qquad n \geqq 2 \tag{37}$$

and with $\sigma(\Phi,\Psi,n)$ be defined as in (10). Then $f \in T$ belong to the class $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$, if and only if it can be represented in the form

$$f(z) = \sum_{n=1}^{\infty}\xi_n f_n(z), \qquad \xi_n \geqq 0, \qquad \sum_{n=1}^{\infty}\xi_n = 1. \tag{38}$$

Proof. Suppose $f(z)$ can be written as in (38). Then

$$f(z) = z - \sum_{n=2}^{\infty}\xi_n\left\{\frac{2\beta\gamma(1-\zeta)(B-A)}{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}\right\}z^n.$$

Now,

$$\sum_{n=2}^{\infty} \frac{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2\beta\gamma(1-\zeta)(B-A)} \frac{2\beta\gamma(1-\zeta)(B-A)}{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}\xi_n = \sum_{n=2}^{\infty}\xi_n = 1-\xi_1 \leq 1.$$

Thus $f \in \mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$. Conversely, let $f \in \mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$. Then, by using (20),

$$a_n \leq \frac{2\beta\gamma(1-\zeta)(B-A)}{\sigma(\Phi,\Psi,n)\mathcal{C}_n(\alpha)}.$$

Setting,

$$\xi_n = \frac{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}{2\beta\gamma(1-\zeta)(B-A)}a_n, \qquad n \geqq 2,$$

and $\xi_1 = 1 - \sum_{n=2}^{\infty}\xi_n$, we have $f(z) = \sum_{n=1}^{\infty}\xi_n f_n(z)$, with $f_n(z)$ is as given in (37).

Corollary 3 *The extreme points of* $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$, *are the functions* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{2\beta\gamma(1-\zeta)(B-A)}{\mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)}z^n, \quad n \geqq 2.$$

As in earlier theorems, we can deduce known results for various other classes and we omit details.

Theorem 11 *The class* $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$ *is closed under linear combination.*

Proof. Let $f, g \in \mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$. Let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0,$$

and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n,\ b_n \geq 0.$$

For η such that $0 \leq \eta \leq 1$, it is sufficient to show that the function h, defined by $h(z) = (1-\eta)f(z) + \eta g(z)$, $z \in \Delta$, belongs to $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$.

Since,

$$h(z) = z - \sum_{n=2}^{\infty} [(1-\eta)a_n + \eta b_n]\, z^n,$$

applying Theorem 2, we get

$$\begin{aligned}\sum_{n=2}^{\infty} \mathcal{C}_n(\alpha) \quad & \sigma(\Phi,\Psi,n)[(1-\eta)a_n + \eta b_n] \\ = \quad & (1-\eta)\sum_{n=2}^{\infty} \mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)a_n + \eta\sum_{n=2}^{\infty} \mathcal{C}_n(\alpha)\sigma(\Phi,\Psi,n)b_n \\ \leq \quad & 2(1-\eta)\beta\gamma(1-\zeta)(B-A) + 2\eta\beta\gamma(1-\zeta)(B-A) \\ = \quad & 2\beta\gamma(1-\zeta)(B-A).\end{aligned}$$

This implies that $h \in \mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$.

6 Integral Means Inequalities

Lemma 1 *[3] If the functions f and g are analytic in Δ with $g \prec f$, then for $\kappa > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta. \tag{39}$$

In [6], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [7] and settled in [8], that

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\kappa} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\kappa} d\theta,$$

for all $f \in T$, $\kappa > 0$ and $0 < r < 1$. In [8], he also proved his conjecture for the subclasses $T^*(\beta)$ and $\mathcal{C}(\beta)$ of T.

In this section, we obtain integral means inequalities for the functions in the family $\mathcal{R}_T[\zeta,\alpha,\beta,\gamma,\Phi,\Psi,A,B]$. By taking appropriate choices of the parameters Φ,Ψ,A,B we obtain the integral means inequalities for several known as well as new subclasses.

Applying Lemma 1, Theorem 1 and Theorem 10, we prove the following result.

Theorem 12 *Suppose $f(z) \in \mathcal{R}_T[\zeta, \alpha, \beta, \gamma, \Phi, \Psi, A, B]$ and $f_2(z)$ is defined by*

$$f_2(z) = z - \frac{\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi, \Psi, 2)} z^2,$$

with

$$\sigma(\Phi, \Psi, 2) = (\lambda_2 - \mu_2)(1 - B\beta) + 2\gamma\beta(B-A)(\lambda_2 - \zeta\mu_2)$$

Then for $z = re^{i\theta}$, $0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\kappa \, d\theta \le \int_0^{2\pi} |f_2(z)|^\kappa \, d\theta. \tag{40}$$

Proof. For

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,$$

(40) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\kappa d\theta \le \int_0^{2\pi} \left| 1 - \frac{\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi, \Psi, 2)} z \right|^\kappa d\theta.$$

By Lemma 1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi, \Psi, 2)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{\beta\gamma(1-\zeta)(B-A)}{(1-\alpha)\sigma(\Phi, \Psi, 2)} w(z), \tag{41}$$

and using (9), we obtain

$$\begin{aligned} |w(z)| &= \left| \sum_{n=2}^{\infty} \frac{\sigma(\Phi, \Psi, n) C_n(\alpha)}{2\beta\gamma(1-\zeta)(B-A)} |a_n| z^{n-1} \right| \\ &\le |z| \sum_{n=2}^{\infty} \frac{\sigma(\Phi, \Psi, n) C_n(\alpha)}{2\beta\gamma(1-\zeta)(B-A)} |a_n| \\ &\le |z|, \end{aligned}$$

where $\sigma(\Phi, \Psi, n)$ is as defined in (10). This completes the proof Theorem 12.

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