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Some Radius Results for Univalent Functions

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This paper deals with the geometrically motivated properties uniform starlikeness, uniform convexity and starlikeness with respect to symmetrical points. We give sharp radius results for these properties in the class $\mathcal{S}$ of normalized univalent functions. Corresponding results in the class $\mathcal{S}^*$ of starlike functions are also discussed.

1. INTRODUCTION

Let $\mathcal{S}$ be the class of functions, analytic and univalent in the unit disk U, normalized by $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the usual classes of starlike and convex functions in $\mathcal{S}$. Further, we introduce the sets

$$UST = \left\{ f \in \mathcal{S} \mid \operatorname{Re} \frac{f(z) - f(\zeta)}{(z - \zeta) f'(z)} \geq 0, (z, \zeta) \in U \times U \right\}, \tag{1.1}$$

$$SSP = \left\{ f \in \mathcal{S} \mid \operatorname{Re} \frac{f(z) - f(-z)}{z f'(z)} \geq 0, z \in U \right\} \tag{1.2}$$

and

$$UCV = \left\{ f \in \mathcal{S} \mid \left| \frac{z f''(z)}{f'(z)} \right| \leq \operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\}, z \in U \right\}. \tag{1.3}$$

The class SSP, called functions starlike with respect to symmetrical points,

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was introduced by Sakaguchi [8], whereas the classes UST and UCV, called uniformly starlike and uniformly convex functions, were introduced by Goodman [2]. We see that $UST \subset SSP$ because (1.2) is equivalent to taking $\zeta = -z$ in (1.1), and clearly also $UST \subset \mathcal{S}^*$ (choose $\zeta = 0$). As for the class SSP we see from (1.2) that if $f \in SSP$, then $\operatorname{Re}(z(f'(z) + f'(-z)))/(f(z) - f(-z)) \geq 0$, so the function $(f(z) - f(-z))/2 \in \mathcal{S}^*$. This implies, by (1.2), that $SSP \subset \mathcal{C}$, the class of close-to-convex functions. Since a convex set can be described as being starlike with respect to all its points, we have that $\mathcal{K} \subset SSP$, and it is also easily seen from (1.3) that $UCV \subset \mathcal{K}$. Geometrically the property uniform starlikeness (convexity) of a function f means that the image of every circular arc contained in U, with center ζ also in U, is starlike with respect to $f(\zeta)$ (convex). The geometric interpretation of SSP is that for every ζ with $|\zeta| = r < 1$ the angular velocity of $f(z)$ about the point $f(-\zeta)$ is positive at $z = \zeta$ as z traverses the circle $|z| = r$ in the positive direction. The characterization (1.3) of UCV was found by Ma and Minda [4] and the author [7] independently.

2. Main Results

Let $\mathcal{F}$ and $\mathcal{G}$ be two families of normalized analytic functions. If for every $f \in \mathcal{F}$, $f(rz)/r \in \mathcal{G}$ for $r \leq r_0$, and r_0 is the largest number for which this holds, then we say that r_0 is the $\mathcal{G}$ radius (or the radius of the property connected to $\mathcal{G}$) in $\mathcal{F}$. Results of this type are numerous in the literature of univalent functions. The best known are probably the radius of starlikeness in $\mathcal{S}$, which is $\tanh(\pi/4) = 0.655\ldots$, and the radius of convexity, which is $2 - \sqrt{3} = 0.2679\ldots$ [3, pp. 119–121]. We shall mainly be concerned with computing the radii of the properties (1.1), (1.2), and (1.3) in $\mathcal{S}$. We first give the following result.

THEOREM 1. *The radius of uniform starlikeness in $\mathcal{S}$ is the unique root $r_0 \in (0, 1)$ of the equation $\varphi(r) = \pi/2$, where*

$$\varphi(r) = \tan^{-1} \frac{\sqrt{(2 - 4r^2 + 4r^4)\sqrt{1 + 4r^2 - r^4} - 2 + 9r^4 - 3r^6}}{\sqrt{1 + r^2}(4 - 2r^2 + \sqrt{1 + 4r^2 - r^4})} + \log \frac{1 + \sqrt{1 - 5(1 - r^2)^2/(2(1 + r^2)\sqrt{1 + 4r^2 - r^4} + 3 + 4r^2 + r^4)}}{1 - \sqrt{1 - 5(1 - r^2)^2/(2(1 + r^2)\sqrt{1 + 4r^2 - r^4} + 3 + 4r^2 + r^4)}} \tag{2.1}$$

The number r_0 is equal to 0.3691, *correctly rounded to four decimal places.*

Since $UST \subset SSP$, the SSP radius in $\mathcal{S}$ must be greater than or equal to the number r_0 in Theorem 1. The next result shows that it is indeed larger.

THEOREM 2. *The radius of starlikeness with respect to symmetrical points in* $\mathcal{S}$ *is* $\tanh(\pi/8) = 0.3736\ldots$

The proofs, which we give in the next section, will show that the *UST* and *SSP* radii in $\mathcal{S}^*$ are strictly larger than the ones computed for $\mathcal{S}$. We are not able to compute the exact radii in $\mathcal{S}^*$ so far.

THEOREM 3. *Let* $k(z) = z/(1-z)^2$, *the Koebe function. Then* $k(rz)/r \in SSP$ *if and only if* $r \le 1/\sqrt{7} = 0.3779\ldots$

Putting the information in the above theorems together, we can formulate the following result, which at present is the best we can say about the class $\mathcal{S}^*$.

COROLLARY. *Denote the radius of uniform starlikeness in* $\mathcal{S}^*$ *by* r_0^*. *Then*

$$0.369 < r_0^* \le 1/\sqrt{7}.$$

(In fact, by studying the Koebe function, we can see that r_0^* must be strictly less than $1/\sqrt{7}$.)

Remark. It is easy to verify that for the Koebe function, $\operatorname{Re} zk'(z)/k(z) \ge \frac{1}{2}$ if and only if $|z| \le \frac{1}{3}$. Since r_0 (in Theorem 1) is greater than $\frac{1}{3}$, we see that $k(r_0 z)/r_0 \notin \mathcal{S}^*_{1/2}$, where $\mathcal{S}^*_\alpha$ denotes the class of functions starlike or order α. Therefore $UST \not\subset \mathcal{S}^*_{1/2}$. A nice problem would be to try to find the largest $\alpha \ge 0$ such that $UST \subset \mathcal{S}^*_\alpha$.

The author introduced in [7] the class $\mathcal{S}_p$ of starlike functions given by the property that $f \in UCV \Leftrightarrow zf' \in \mathcal{S}_p$ and in [6] it was established that there is no inclusion between $\mathcal{S}_p$ and *UST*. So far, no natural connection between *UST* and *UCV* has been discovered, apart from the similarities in the geometric characterizations. The $\mathcal{S}_p$ radius in $\mathcal{S}^*$ was found in [7] to be $\frac{1}{3}$. This gives immediately that the radius of uniform convexity in $\mathcal{K}$ is $\frac{1}{3}$. However, the radius of uniform convexity in the entire class $\mathcal{S}$ has so far not been computed, so we include that result here.

THEOREM 4. *The radius of uniform convexity in* $\mathcal{S}$ *and in* $\mathcal{S}^*$ *is* $(4-\sqrt{13})/3 = 0.1314\ldots$

3. PROOFS

The proof of Theorems 1 and 2 relies heavily on the linear invariance property of the set $\mathcal{S}$, i.e., the fact that if $f \in \mathcal{S}$, then the function

$$g(w) = \frac{f((w+z)/(1+\bar{z}w)) - f(z)}{(1-|z|^2)f'(z)} \tag{3.1}$$

is also in $\mathcal{S}$. A simple argument using the minimum principle for harmonic functions [6] shows that $f \in UST$ if and only if for every $r < 1$,

$$\operatorname{Re} \frac{f(z) - f(\zeta)}{(z - \zeta)f'(z)} \geq 0$$

for all z and ζ with $|z| = |\zeta| = r$.

Hence for a given $f \in \mathcal{S}$, we want to find the largest $r = |z|$ such that

$$\operatorname{Re} \frac{f(z) - f(xz)}{z(1 - x)f'(z)} \geq 0$$

for all z, $|z| = r$, and all x, $|x| = 1$. Choosing $w = z(x - 1)/(1 - x|z|^2)$, we see that $(w + z)/(1 + \bar{z}w) = xz$, and (3.1) becomes

$$g\left(\frac{z(x - 1)}{1 - x|z|^2}\right) = \frac{f(xz) - f(z)}{(1 - |z|^2)f'(z)}. \tag{3.2}$$

Defining $F(x, z) = (f(z) - f(xz))/z(1 - x)f'(z)$, we now see that

$$F(x, z) = \frac{1 - |z|^2}{1 - x|z|^2} \frac{g(w)}{w},$$

where $w = z(x - 1)/(1 - x|z|^2)$. The property $\operatorname{Re} F(x, z) \geq 0$ is equivalent to $|\arg F(x, z)| \leq \pi/2$, so our problem can be expressed as finding the largest $r = |z|$ such that for all x, $|x| = 1$,

$$\left| \arg \frac{1}{1 - x|z|^2} + \arg \frac{g(w)}{w} \right| \leq \frac{\pi}{2}.$$

It is well known that for $g \in \mathcal{S}$ and $|w| < 1$,

$$\left| \arg \frac{g(w)}{w} \right| \leq \log \frac{1 + |w|}{1 - |w|} \tag{3.3}$$

[1, p. 95], and this bound is sharp for each $w \in U$. Hence we get

$$|\arg F(x, z)| \leq \left| \arg \frac{1}{1 - x|z|^2} \right| + \log \frac{1 + |w|}{1 - |w|}. \tag{3.4}$$

If one writes $x = e^{i\theta}$, the right hand side of (3.4) turns into the function

$$\Psi(r,\theta) = \tan^{-1}\frac{r^2\sin\theta}{1 - r^2\cos\theta} + \log\frac{1 + r\sqrt{2 - 2\cos\theta}/\sqrt{1 - 2r^2\cos\theta + r^4}}{1 - r\sqrt{2 - 2\cos\theta}/\sqrt{1 - 2r^2\cos\theta + r^4}}, \tag{3.5}$$

where it is understood that $\sin\theta \geq 0$. Note that it is possible to choose x such that $\arg(1/(1 - x|z|^2)) \geq 0$ without affecting the value of $\log((1 + |w|)/(1 - |w|))$. What remains in order to establish Theorem 1 is to show that $\varphi(r)$ in the theorem is equal to the maximum of $\Psi(r, \theta)$ for $\theta \in [0, 2\pi\rangle$. The sharpness of the result follows from the fact that for each $w \in U$, equality holds in (3.3) for some $g \in \mathcal{S}$ and that each $g \in \mathcal{S}$ is the image of some $f \in \mathcal{S}$ by the mapping (3.2). We state the desired property of $\varphi(r)$ as a lemma and defer the proof to the end of the paper.

LEMMA. *Let $\varphi(r)$ be as in* (2.1) *and $\psi(r, \theta)$ be as in* (3.5). *Then*

$$\Psi(r,\theta) \leq \varphi(r),$$

and equality holds for

$$\cos\theta = \frac{1}{5}\left[2r^2 - 2 + \frac{1 - (1 + r^2)\sqrt{1 + 4r^2 - r^4}}{r^2}\right].$$

The equation $\varphi(r) = \pi/2$ has exactly one solution in the interval (0, 1).

Theorem 2 is obtained by taking $\theta = \pi$ in (3.5). This gives

$$\arg\frac{f(z) - f(-z)}{zf'(z)} \leq 2\log\frac{1 + r}{1 - r} = \frac{\pi}{2}$$

for $r = \tanh(\pi/8)$.

To prove Theorem 3, we take $f(z) = k(z) = z/(1 - z)^2$ and we get

$$\frac{k(z) - k(-z)}{zk'(z)} = \frac{2(1 - z)(1 + z^2)}{(1 + z)^3}.$$

With $z = re^{i\varphi}$, a simple computation shows that

$$\operatorname{Re}\frac{k(z) - k(-z)}{zk'(z)} = \frac{2(1 - r^2)}{|1 + re^{i\varphi}|^6}[(1 - 6r^2 + r^4) + 2r(1 + r^2)\cos\varphi + 8r^2\cos^2\varphi]. \tag{3.6}$$

Minimizing the expression in the square bracket on the right hand side of (3.6) with respect to $\cos\varphi$, we find that the minimum is taken for

$$\cos\varphi = -\frac{1+r^2}{8r}$$

if $r \geq 4 - \sqrt{15}$ and for $\cos\varphi = -1$ if $r < 4 - \sqrt{15}$. The case of the small r-values is not interesting, because then the expression in (3.6) is strictly positive. Substituting $\cos\varphi = -(1 + r^2)/8r$ into the square bracket of (3.6) and using the fact that $|1 + re^{i\varphi}| < 2$ we get

$$\operatorname{Re}\frac{k(z)-k(-z)}{zk'(z)} \geq \frac{1-r^2}{32}\cdot\frac{1}{8}(7 - 50r^2 + 7r^4),$$

which is equal to zero for $r^2 = \frac{1}{7}$.

Remark. The linear invariance property that we have used for $\mathcal{S}$ does not hold in $\mathcal{S}^*$, so this method cannot be applied to find the *UST* and *SSP* radii in $\mathcal{S}^*$. In principle, it could have been applied to find the corresponding radii in $\mathcal{C}$, the class of close-to-convex functions, because $\mathcal{C}$ has the linear invariance property. The upper bound on $|\arg g(z)/z|$ for $g \in \mathcal{C}$, $|z| = r$, is strictly less than the corresponding one for $g \in \mathcal{S}$, $|z| = r$, as can be seen from the main result in [5]. Therefore we would necessarily get a larger r for the class $\mathcal{C}$ than for the whole of $\mathcal{S}$, and since $\mathcal{S}^* \subset \mathcal{C}$, it follows that the r-values that we have computed for $\mathcal{S}$ cannot be sharp for $\mathcal{S}^*$.

To prove Theorem 4 we apply the following classical result for $f \in \mathcal{S}$ [1, p. 32].

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}, \qquad |z| = r < 1. \tag{3.7}$$

The result is sharp for the Koebe function. To get the *UCV* radius we compute the maximal $r = |z|$ such that

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \tag{3.8}$$

for every $f \in \mathcal{S}$. The inequality (3.8) represents a parabola with vertex in $x = -\frac{1}{2}$ and axis $(-\frac{1}{2}, \infty)$, and the inequality (3.7) is a circular disk intersecting the real axis in $x = -2r(2 - r)/(1 - r^2)$ and $x = 2r(2 + r)/(1 - r^2)$. A simple computation shows that

$$\frac{-2r(2-r)}{1-r^2} = -\frac{1}{2}$$

for $r = (4 - \sqrt{13})/3$ and that for r less than this value the disk (3.7) is completely inside the parabola (3.8).

4. Proof of the Lemma

Let $\Psi(r,\theta)$ be as in (3.5). After some computation we arrive at

$$\frac{\partial}{\partial\theta}\Psi(r,\theta) = \frac{r^2(\cos\theta - r^2)}{1 - 2r^2\cos\theta + r^4} + \frac{\sqrt{2}r\sin\theta}{\sqrt{1-\cos\theta}\sqrt{1-2r^2\cos\theta + r^4}}.$$

Now, $\partial/\partial\theta\ \Psi(r,\theta) = 0$ is equivalent to

$$\frac{-\sqrt{2}r\sin\theta}{\sqrt{1-\cos\theta}\sqrt{1-2r^2\cos\theta+r^4}} = \frac{r^2(\cos\theta - r^2)}{1-2r^2\cos\theta + r^4}. \tag{4.1}$$

Squaring both sides of (4.1), and putting $\cos\theta = x$, we get after some computation the equation

$$5r^2x^3 - (2 + r^2 + 4r^4)x^2 - (4r^2 - 2r^4 - r^6)x + 2 + 2r^4 - r^6 = 0,$$

which has the solutions

$$x_1 = 1,$$

$$x_2 = \frac{1}{5}\left[2r^2 - 2 + \frac{1 + (1+r^2)\sqrt{1+4r^2-r^4}}{r^2}\right]$$

and

$$x_3 = \frac{1}{5}\left[2r^2 - 2 + \frac{1 - (1+r^2)\sqrt{1+4r^2-r^4}}{r^2}\right].$$

We immediately see that x_1 does not fit in (4.1). A further analysis shows that $x_2 > 1$, $-1 < x_3 < 0$ for $r > 0$ and that $\lim_{r\to 0} x_3 = -1$. The only possible solution is therefore x_3, and indeed we can verify that $\cos\theta = x_3$ and $\sin\theta = \sqrt{1 - x_3^2}$ satisfy (4.1). Substituting these values for $\cos\theta$ and $\sin\theta$ in $\Psi(r,\theta)$, we get $\varphi(r)$ as in (2.1). Since $\Psi(r, 0) = \Psi(r, 2\pi) = 0$ and $\Psi(r, \pi) > 0$, we infer that $\varphi(r)$ is a maximum for $\Psi(r,\theta)$.

We see that $\varphi(0) = 0$ and that $\varphi(r) \to \infty$ as $r \to 1$, so to show that $\varphi(r) = \pi/2$ has exactly one root in $(0, 1)$, we show that $\varphi(r)$ is monotonically increasing. This will conclude the proof of the lemma.

Let

$$f(r) = \tan^{-1} \frac{\sqrt{(2 - 4r^2 + 4r^4)\sqrt{1 + 4r^2 - r^4} - 2 + 9r^4 - 3r^6}}{\sqrt{1 + r^2}(4 - 2r^2 + \sqrt{1 + 4r^2 - r^4})}$$
$$= \tan^{-1} \frac{\sqrt{c(r)}}{d(r)}.$$

We first show that $f(r)$ is increasing by showing that $\sqrt{c(r)}/d(r)$ is increasing. Differentiating we get

$$\left(\frac{\sqrt{c(r)}}{d(r)}\right)' = \frac{c'(r)\, d(r) - 2c(r)\, d'(r)}{2\sqrt{c(r)}\, d(r)^2},$$

and

$$\begin{aligned} &c'(r)\, d(r) - 2c(r)\, d'(r) \\ &\quad = \sqrt{1 + r^2}\Bigg[4r^7 - 24r^5 + 148r^3 - 32r \\ &\qquad + \frac{22r^9 - 176r^7 + 398r^5 - 84r^3 + 32r}{\sqrt{1 + 4r^2 - r^4}}\Bigg] \\ &\qquad - \frac{2}{\sqrt{1 + r^2}}[2r^9 - 10r^7 + 22r^5 + 8r^3 - 6r \\ &\qquad + (-11r^7 + 33r^5 - 20r^3 + 6r)\sqrt{1 + 4r^2 - r^4}]. \end{aligned}$$

Now we see that $c'(r)\, d(r) - 2c(r)\, d'(r) > 0$ is equivalent to

$$\begin{aligned} &(1 + r^2)(22r^9 - 176r^7 + 398r^5 - 84r^3 + 32r) \\ &\quad + 2(11r^7 - 33r^5 + 20r^3 - 6r)(1 + 4r^2 - r^4) \\ &\quad > [2(2r^9 - 10r^7 + 22r^5 + 8r^3 - 6r) \\ &\quad - (1 + r^2)(4r^7 - 24r^5 + 148r^3 - 32r)]\sqrt{1 + 4r^2 - r^4}, \end{aligned}$$

which again is equivalent to

$$1 - 3r^2 + 21r^4 - 3r^6 > (1 - 5r^2 - 4r^4)\sqrt{1 + 4r^2 - r^4}.$$

The left hand side of the above inequality is always positive. With both sides squared it becomes

$$25r^{12} - 150r^{10} + 300r^8 - 250r^6 + 75r^4 = 25r^4(3 - r^2)(1 - r^2)^3 > 0,$$

which is true for $r < 1$. This proves that $f'(r) > 0$.

Next, let

$$\begin{aligned} g(r) &= \log\frac{1 + \sqrt{1 - 5(1 - r^2)^2/(2(1 + r^2)\sqrt{1 + 4r^2 - r^4} + 3 + 4r^2 + r^4)}}{1 - \sqrt{1 - 5(1 - r^2)^2/(2(1 + r^2)\sqrt{1 + 4r^2 - r^4} + 3 + 4r^2 + r^4)}} \\ &= \log\frac{1 + \sqrt{1 - 5(1 - r^2)^2/p(r)}}{1 - \sqrt{1 - 5(1 - r^2)^2/p(r)}}. \end{aligned}$$

To show that $g'(r) > 0$ it is enough to show that $p'(r) > 0$. Computing, we find

$$p'(r) = 4r\left(\frac{3 + 5r^2 - 2r^4}{\sqrt{1 + 4r^2 - r^4}} + 2 + r^2\right).$$

The polynomial $3 + 5r^2 - 2r^4$ is positive for $r \in [0, 1]$, so we see that $p'(r) > 0$ for $r \in (0, 1]$, and the proof of the lemma is complete.

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