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Integrated Partial Sums of Convolutions of Univalent Functions

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Let $\mathcal{S}$ be the set of normalized univalent functions. We show that for $f(z)=\sum_{k=1}^{\infty} a_k z^k$, $g(z)=\sum_{k=1}^{\infty} b_k z^k \in \mathcal{S}$,

$$\operatorname{Re} \sum_{k=1}^{n} \frac{a_k b_k}{k} z^{k-1} > 0, \qquad |z| < \frac{1}{2}$$

if $n \leqslant 3$ or $n \geqslant 6$. For $n=4$ we obtain the corresponding result if g is restricted to the class of close-to-convex functions. The results that we obtain are special cases of a multiplier conjecture for univalent functions as well as an extension of an old result about sections of univalent functions.

1. INTRODUCTION

We shall be concerned with the class $\mathcal{A}_0$ of functions of the form $f(z)=z+\sum_{k=2}^{\infty} a_k z^k$, analytic in the unit disk U. Denote by $\mathcal{S}$, $\mathcal{C}$, $\mathcal{S}^*$ and $\mathcal{K}$ the classes of such functions that are univalent, close-to-convex, starlike, and convex in U, respectively. Furthermore, let

$$\mathcal{D} = \{f \in \mathcal{A}_0 \mid |f''(z)| \leqslant \operatorname{Re} f'(z), z \in U\}$$

and

$$\mathcal{R} = \left\{ f \in \mathcal{A}_0 \,\middle|\, \operatorname{Re} \frac{f(z)}{z} > \frac{1}{2}, z \in U \right\}.$$

For $f(z)=\sum_{k=1}^{\infty} a_k z^k$ and $g(z)=\sum_{k=1}^{\infty} b_k z^k$ in $\mathcal{A}_0$ we define the convolution of f and g as $(f * g)(z)=\sum_{k=1}^{\infty} a_k b_k z^k$. The class $\mathcal{D}$ plays a central role in a recently stated conjecture on convolutions of univalent functions [3].

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Conjecture A. Let $f \in \mathscr{D}$, $g, h \in \overline{\text{co}}(\mathscr{S})$. Then

$$\operatorname{Re} \frac{1}{z}(f * g * h)(z) > 0, \qquad z \in U.$$

The conjecture has been proved in a number of special cases. Most of this work is collected in [3]. The class $\mathscr{D}$ is not well described, but some special members of $\mathscr{D}$ are known. We shall work with the functions

$$f_n(z) = \sum_{k=1}^{n} \frac{z^k}{k 2^{k-1}}, \qquad n = 2, 3, \ldots . \tag{1.1}$$

The proof for $f_n \in \mathscr{D}$ is in [3] as well as the following result supporting Conjecture A.

THEOREM A. *For f_n as in* (1.1), $g \in \overline{\text{co}}(\mathscr{S})$, $h \in \overline{\text{co}}(\mathscr{S}^*)$,

$$\operatorname{Re} \frac{1}{z}(f_n * g * h)(z) > 0, \qquad z \in U. \tag{1.2}$$

Remark. What actually is proved in [3] is Theorem A for $g \in \mathscr{S}$ and $h(z) = z(1-z)^{-2}$. But the linearity of the statement (1.2) implies that it automatically extends to the closed convex hull of the functions involved. Since $\overline{\text{co}}(\mathscr{S}^*)$ is generated by the rotations of the Koebe function [4, p. 52], Theorem A in its present form follows.

In order to continue the work in [3] it would be natural to try to extend Theorem A to cover the case when both g and h are in $\mathscr{S}$. Another, less ambitious, extension would be to take $g \in \mathscr{S}$ and $h \in \mathscr{C}$. (For g, h both from $\mathscr{C}$, the conjecture is proved for arbitrary $f \in \mathscr{D}$ [3].) We shall in the present paper prove (1.2) for $g, h \in \mathscr{S}$ when $n \leqslant 3$ and $n \geqslant 6$. In the case $n = 4$ we obtain only the weaker result, where h is restricted to $\mathscr{C}$. The case $n = 5$ is beyond reach in either case at present.

2. THE MAIN RESULT

We now present the main theorem of the paper.

THEOREM 1. (a) *Let $g(z) = \sum_{k=1}^{\infty} a_k z^k \in \overline{\text{co}}(\mathscr{S})$ and $h(z) = \sum_{k=1}^{\infty} b_k z^k \in \overline{\text{co}}(\mathscr{S})$. Then for $n \leqslant 3$ and $n \geqslant 6$ we have*

$$\operatorname{Re} \sum_{k=1}^{n} \frac{a_k b_k}{k} z^{k-1} > 0, \qquad |z| < \frac{1}{2}. \tag{2.1}$$

The radius $\frac{1}{2}$ is the largest possible.

(b) *If h is restricted to $\overline{\text{co}}(\mathscr{C})$ then* (2.1) *holds also for $n = 4$.*

We mention the following equivalence (Lemma 1.1 in [3]). For $H \in \mathcal{A}_0$

$$\operatorname{Re} \frac{1}{z}(H * h)(z) > 0, \qquad \forall h \in \mathscr{C},\ z \in U, \tag{2.2}$$

if and only if

$$\left| \frac{H(z)}{z} - H'(z) \right| \leqslant \operatorname{Re}\left(\frac{H(z)}{z} + H'(z) \right), \qquad z \in U. \tag{2.3}$$

From (2.3) it follows that $\operatorname{Re}(zH'(z)/H(z)) > 0$, i.e., $H \in \mathscr{S}^*$. Hence we have from Theorem 1(b), and the equivalence above, the following result about the sections of functions in $\mathscr{S}$.

COROLLARY 1. *Let* $g(z) = \sum_{k=1}^{\infty} a_k z^k \in \overline{\text{co}}(\mathscr{S})$. *Then*

$$2 \sum_{k=1}^{n} \frac{a_k}{k} \left(\frac{z}{2} \right)^k \in \mathscr{S}^* \tag{2.4}$$

for $n \leqslant 4$ *and* $n \geqslant 6$.

Kobori [5] proved the same result (for all n) for $g \in \mathscr{S}^*$, so this is a considerable extension of a classical result. For the proof of the theorem we need some results about functions in $\mathscr{S}$.

LEMMA 1. *Let* $g(z) = \sum_{k=1}^{\infty} a_k z^k \in \mathscr{S}$. *Then we have*

$$\left| \log \frac{g(z)}{z} + \log(1 - |z|^2) \right| \leqslant \log \frac{1 + |z|}{1 - |z|}, \qquad z \in U, \tag{2.5}$$

$$|a_k| \leqslant k, \qquad k = 2, 3, \ldots, \tag{2.6}$$

$$\left| a_3 - \frac{3}{4} a_2^2 \right| \leqslant \frac{2}{\sqrt{3}} \sqrt{1 - |a_2/2|^2}, \tag{2.7}$$

$$\left| a_4 - \frac{1}{2} a_2^3 \right| \leqslant \frac{2}{\sqrt{5}} \sqrt{(1 + 15\,|a_2/2|^2)(1 - |a_2/2|^2)}. \tag{2.8}$$

The inequality (2.5) is an old result by Grunsky, and (2.7) follows directly from the area theorem. Both results can be found in textbooks on univalent functions, e.g., [2]. The inequality (2.6) is the former Bieberbach conjecture, now deBranges' theorem [1]. For (2.8) we do not have a reference, so we include a proof. Using the area theorem on the function $[g(1/z^2)]^{-1/2}$ we obtain (compare [2, p. 132])

$$\left| \frac{a_2}{2} \right|^2 + \frac{3}{4} \left| \frac{3}{4} a_2^2 - a_3 \right|^2 + \frac{5}{4} \left| \frac{3}{2} a_2 \left(a_3 - \frac{3}{4} a_2^2 \right) + \frac{1}{2} a_2^3 - a_4 \right|^2 \leqslant 1. \tag{2.9}$$

Introducing $x = |a_2/2|$ and $A = |a_3 - \frac{3}{4}a_2^2|$, (2.9) gives

$$a_4 = \frac{1}{2}a_2^3 + \frac{3}{2}a_2\left(a_3 - \frac{3}{4}a_2^2\right) + \sigma\frac{2}{\sqrt{5}}\sqrt{1 - x^2 - \tfrac{3}{4}A^2}, \qquad (|\sigma| \leqslant 1)$$

$$= \frac{1}{2}a_2^3 + \rho,$$

where

$$|\rho| \leqslant 3Ax + \frac{2}{\sqrt{5}}\sqrt{1 - x^2 - \tfrac{3}{4}A^2}. \tag{2.10}$$

Maximizing the right-hand side of (2.10) with respect to A we obtain

$$|\rho| \leqslant \frac{2}{\sqrt{5}}\sqrt{(1 + 15x^2)(1 - x^2)},$$

and (2.8) follows.

We also state another lemma which plays a central role in the proof.

LEMMA 2. *Let* $g(z) = \sum_{k=1}^{\infty} a_k z^k \in \mathscr{S}$, *and define* $G(z) = \sum_{k=1}^{\infty} (a_k/k)\, z^k$:

(a) *For* $\rho < 0.835596$ *the function* $(1/\rho)\, G(\rho z) \in \mathscr{R}$.

(b) *For* $f \in \mathscr{R}$, $g \in \mathscr{A}_0$, $((f * g)/z)(U) \subset \overline{\mathrm{co}}((g/z)(U))$.

Part (a) is proved in [8]. For (b) we refer to [6, pp. 54–55].

Now we turn to the proof of Theorem 1. Since the class $\mathscr{S}$ is preserved under rotations and dilations it suffices to prove (2.1) for $z = \frac{1}{2}$. The case $n = 2$ is trivial and shows also that the radius $\frac{1}{2}$ cannot be extended. We first consider the case when n is large.

The case $n \geqslant 6$. Let ρ_0 be the number 0.8355 (compare Lemma 2), and let r_0 be such that $r_0\rho_0 = \frac{1}{2}$, i.e., $r_0 = 0.5984\ldots$. Using (2.5) we obtain for $|z| \leqslant r_0$ that

$$\operatorname{Re}\frac{g(z)}{z} \geqslant \min_{|\rho| \leqslant 1} \frac{1}{1 - r_0^2} \operatorname{Re}\left(\frac{1 + r_0}{1 - r_0}\right)^{\rho}$$

$$\geqslant 0.25433\ldots > \frac{1}{4}.$$

The minimum is calculated numerically. Let $H(z) = \sum_{k=1}^{\infty} (b_k/k)\, z^k$. Then we have from part (a) of Lemma 2 that $(1/\rho_0)\, H(\rho_0 z) \in \mathscr{R}$, and from part (b) that

$$\operatorname{Re}\left(\frac{g(z)}{z} * \frac{H(z)}{z}\right) > \frac{1}{4}, \qquad |z| \leqslant r_0\rho_0 = \frac{1}{2},$$

i.e.,

$$\operatorname{Re} \sum_{k=1}^{\infty} \frac{a_k b_k}{k} z^{k-1} > \frac{1}{4}, \qquad |z| \leqslant \frac{1}{2}.$$

From this we obtain

$$\operatorname{Re} \sum_{k=1}^{n} \frac{a_k b_k}{k 2^{k-1}} = \operatorname{Re} \sum_{k=1}^{\infty} \frac{a_k b_k}{k 2^{k-1}} - \operatorname{Re} \sum_{k=n+1}^{\infty} \frac{a_k b_k}{k 2^{k-1}}$$
$$> \frac{1}{4} - \sum_{k=n+1}^{\infty} \frac{k}{2^{k-1}}$$

which, by computing, is seen to be positive for $n \geqslant 6$.

The case $n = 3$. We shall now prove that

$$\operatorname{Re}\left(1 + \frac{a_2 b_2}{4} + \frac{a_3 b_3}{12}\right) \geqslant 0. \tag{2.11}$$

For this case we need the following lemma which is proved in [3].

LEMMA 3. *Let* $0 \leqslant \alpha \leqslant \frac{2}{3}$, $x_0(\alpha) := \frac{1}{2}(1 - \alpha/\sqrt{1+\alpha^2})$, *and* $\lambda(x) = x + \alpha\sqrt{1-x^2}$. *Then for* $x_0(\alpha) \leqslant A \leqslant x \leqslant 1$ *we have*

$$\lambda(x)\,\lambda(A/x) \leqslant \lambda^2(\sqrt{A}).$$

We write

$$\gamma = \operatorname{Re}\left(1 + \frac{a_2 b_2}{4} + \frac{a_3 b_3}{12}\right),$$

and, using (2.7),

$$a_3 = \frac{3}{4} a_2^2 + \rho_1 \frac{2}{\sqrt{3}} \sqrt{1 - |a_2/2|^2}, \qquad |\rho_1| \leqslant 1,$$
$$b_3 = \frac{3}{4} b_2^2 + \rho_2 \frac{2}{\sqrt{3}} \sqrt{1 - |b_2/2|^2}, \qquad |\rho_2| \leqslant 1.$$

Then

$$\gamma \geqslant \operatorname{Re}\left(1 + \frac{a_2 b_2}{4} + \frac{3}{64}(a_2 b_2)^2\right)$$
$$- \frac{\sqrt{3}}{6}\left(\left|\frac{a_2}{2}\right|^2 \sqrt{1 - |b_2/2|^2} + \left|\frac{b_2}{2}\right|^2 \sqrt{1 - |a_2/2|^2}\right)$$
$$- \frac{1}{9}\sqrt{1 - |a_2/2|^2}\,\sqrt{1 - |b_2/2|^2}.$$

Writing

$$|a_2| = 2x, \qquad |b_2| = 2y, \qquad a_2 b_2 = 4Ae^{i\varphi}, \qquad A = xy$$

and applying $|x| \leqslant 1$, $|y| \leqslant 1$, we obtain

$$\begin{aligned}\gamma &\geqslant 1 + A\cos\varphi + \frac{3}{4}A^2\cos(2\varphi) - \frac{\sqrt{3}}{6}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \\ &\quad - \frac{1}{9}\sqrt{1-x^2}\sqrt{1-y^2} \\ &= 1 + A\cos\varphi + \frac{3}{4}A^2\cos(2\varphi) \\ &\quad - \frac{3}{4}\left(x + \frac{2}{3\sqrt{3}}\sqrt{1-x^2}\right)\left(y + \frac{2}{3\sqrt{3}}\sqrt{1-y^2}\right) \\ &\quad + \frac{3}{4}A.\end{aligned}$$

Minimizing with respect to φ and applying Lemma 3 with $\alpha = 2/3\sqrt{3}$ $(<\frac{2}{3})$ we obtain

$$\gamma \geqslant \frac{5}{6} - \frac{3}{4}A^2 - \begin{cases} \frac{3}{4}\left[\left(\sqrt{A} + \frac{2}{3\sqrt{3}}\sqrt{1-A}\right)^2 - A\right], & A \geqslant \frac{1}{2} - \frac{1}{\sqrt{31}}, \\ \frac{\sqrt{3}}{3} + \frac{1}{9}, & A < \frac{1}{2} - \frac{1}{\sqrt{31}}. \end{cases}$$

In the lower case we obtain $\gamma > 0.06 > 0$. In the upper case we obtain

$$\gamma \geqslant \frac{13}{18} + \frac{1}{9}A - \frac{3}{4}A^2 - \frac{1}{\sqrt{3}}\sqrt{A(1-A)}.$$

If this is positive for $\frac{1}{2} - 1/\sqrt{31} \leqslant A \leqslant 1$, the proof of (2.11) is complete. Taking squares we see that this will be so if

$$p(A) := 729A^4 - 216A^3 - 956A^2 - 224A + 676 > 0, \qquad 0 \leqslant A \leqslant 1.$$

Also later we encounter the problem of proving that a certain polynomial (of high degree) is positive in some interval. To prove this we have computed the roots of the polynomial with the computer system MAPLE working with 15 digits precision. If this shows that the polynomial has no roots in the interval in question, we have a proof for the polynomial not changing sign on that interval. Computing the zeroes of p, we find them to be

$$x_{1,2} = -0.828602 \pm 0.531486i, \qquad x_{3,4} = 0.976750 \pm 0.053516$$

(all given digits precise). Since $p(0) > 0$, we conclude that $\gamma > 0$. It should be mentioned that the technique used in this proof is the same as used in [3] for a similar case. The result that we obtain here, however, does not follow from the corresponding result in [3].

The case $n = 4$. In this case we are not able to prove (2.1) for $g, h \in \mathscr{S}$, but only for $g \in \mathscr{S}$, $h \in \mathscr{C}$. Using the equivalence (2.2)$\Leftrightarrow$(2.3) we see that what we need to prove is

$$\left|\frac{a_2}{4} + \frac{a_3}{6} + \frac{3a_4}{32}\right| \leqslant \operatorname{Re}\left(2 + \frac{3a_2}{4} + \frac{a_3}{3} + \frac{5a_4}{32}\right). \tag{2.12}$$

We write $a_2 = 2xe^{i\varphi}$, $0 \leqslant x \leqslant 1$. Then using (2.7) and (2.8) we obtain

$$\begin{aligned} a_3 &= 3x^2 e^{i2\varphi} + \rho \frac{2}{\sqrt{3}} \sqrt{1 - x^2}, && |\rho| \leqslant 1 \\ a_4 &= 4x^3 e^{i3\varphi} + \sigma \frac{2}{\sqrt{5}} \sqrt{(1 + 15x^2)(1 - x^2)}, && |\sigma| \leqslant 1. \end{aligned} \tag{2.13}$$

Substituting (2.13) into (2.12), we see that (2.12) follows from the following lemma.

LEMMA 4. *For* $0 \leqslant x \leqslant 1$ *and* $0 \leqslant \varphi \leqslant 2\pi$ *we have*

$$\begin{aligned} &\frac{x}{2}\left(1 + x^2 + \frac{9}{16}x^4 + 2x\cos\varphi + \frac{3}{2}x^3\cos\varphi + \frac{3}{2}x^2\cos(2\varphi)\right)^{1/2} \\ &\quad \leqslant 2 + \frac{3}{2}x\cos\varphi + x^2\cos(2\varphi) + \frac{5}{8}x^3\cos(3\varphi) - \frac{1}{\sqrt{3}}\sqrt{1 - x^2} \\ &\quad\quad - \frac{1}{2\sqrt{5}}\sqrt{(1 + 15x^2)(1 - x^2)}. \end{aligned} \tag{2.14}$$

3. Proof of Lemma 4

Numerical calculations show that it is not much to "give away" in this inequality, so one has to be very careful. The proof is split in several cases, and we start with the small values of x which turned out to be the easiest case.

(i) $x \leqslant \frac{1}{2}$. Minimizing the right-hand side of (2.14) with respect to φ, we find that it attains its minimum for $\varphi = \pi$; hence (2.14) follows if we can prove

$$\frac{x}{2}\left(1 + x^2 + \frac{9}{16}x^4 + 2x\cos\varphi + \frac{3}{2}x^3\cos\varphi + \frac{3}{2}x^2\cos(2\varphi)\right)^{1/2}$$
$$\leqslant 1 - \frac{3}{2}x + x^2 - \frac{5}{8}x^3. \tag{3.1}$$

The right-hand side of (3.1) is positive for the x-values in question, so by squaring and simplifying we find that (3.1) holds if

$$p(x) = 2x^6 - 13x^5 + 18x^4 - 38x^3 + 32x^2 - 24x + 8 \geqslant 0, \qquad 0 \leqslant x \leqslant \tfrac{1}{2}.$$

This polynomial has the roots (exact)

$$x_1 = \tfrac{1}{2}, \qquad x_2 = 2 + \alpha + \beta \approx 5.4,$$
$$x_{3,4} = 2 - \frac{\alpha+\beta}{2} \pm \frac{(\beta-\alpha)\sqrt{3}}{2}i, \qquad x_{5,6} = \pm\sqrt{2i},$$

where $\alpha = (6 - 2\sqrt{115/3}\sqrt{3})^{1/3}$ and $\beta = (6 + 2\sqrt{115/3}\sqrt{3})^{1/3}$. Since $p(0) > 0$, we are done.

(ii) $\frac{1}{2} < x \leqslant 1$, $\cos\varphi \geqslant -x/2$. Define $y = \cos\varphi$. Then the right-hand side of (2.14) equals to

$$f(x, y) := \frac{5}{2}x^3y^3 + 2x^2y^2 - \left(\frac{15}{8}x^3 - \frac{3}{2}x\right)y + 2 - x^2$$
$$-\left(\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{5}}\sqrt{1+15x^2}\right)\sqrt{1-x^2} \tag{3.2}$$

and the left-hand side is

$$g(x, y) := \frac{x}{2}\left(3x^2y^2 + \left(2x + \frac{3}{2}x^3\right)y + 1 - \frac{1}{2}x^2 + \frac{9}{16}x^4\right)^{1/2}. \tag{3.3}$$

We shall find the following inequalities useful:

$$g(x, y) \leqslant \frac{5}{6}x^2\left(y + \frac{x}{2}\right) + \frac{x}{2}\left(1 - \frac{3}{4}x^2\right), \qquad -\frac{x}{2} \leqslant y \leqslant 1 \tag{3.4}$$

$$\frac{1}{\sqrt{3}}+\frac{1}{2\sqrt{5}}\sqrt{1+15x^2}\leqslant\frac{4}{5}x+\frac{7}{10}, \qquad \frac{1}{2}\leqslant x\leqslant 1 \quad (3.5)$$

$$\sqrt{1-x^2}\leqslant\frac{6}{5}-\frac{x}{2}, \qquad \frac{1}{2}\leqslant x\leqslant 1 \quad (3.6)$$

$$\sqrt{225x^2+100x-116}\leqslant\frac{115\sqrt{3}}{9}x-\frac{38\sqrt{3}}{9}\leqslant\frac{1107}{50}x-\frac{73}{10}, \quad \frac{1}{2}\leqslant x\leqslant 1 \quad (3.7)$$

We find that $\partial^2 g/\partial y^2>0$, $-x/2\leqslant y\leqslant 1$, so $g(x, y)$ is bounded above by the straight line (regarding y as the variable)

$$\begin{aligned} h(x, y)&=\frac{g(x, 1)-g(x, -x/2)}{1+x/2}\left(y+\frac{x}{2}\right)+g\left(x, -\frac{x}{2}\right)\\ &=\frac{x^2(2+3x)}{2(2+x)}\left(y+\frac{x}{2}\right)+\frac{x}{2}\left(1-\frac{3}{4}x^2\right)\\ &\leqslant\frac{5}{6}x^2\left(y+\frac{x}{2}\right)+\frac{x}{2}\left(1-\frac{3}{4}x^2\right). \end{aligned}$$

Hence (3.4) follows; (3.5) and (3.6) are both established by studying the quadratic inequalities we obtain when squaring both sides. The expression in the middle of (3.7) is the Taylor polynomial of degree one around $x=\frac{4}{5}$ of the left-hand side. The left inequality then follows from the concavity properties of the left-hand side, and the right inequality is obtained from rational approximations of $\sqrt{3}$.

We now intend to prove that

$$\begin{aligned} f^*(x, y) &:=\frac{5}{2}x^3y^3+2x^2y^2-\left(\frac{15}{8}x^3-\frac{3}{2}x\right)y+2-x^2\\ &\quad-\left(\frac{4}{5}x+\frac{7}{10}\right)\sqrt{1-x^2}\\ &\geqslant\frac{5}{6}x^2\left(y+\frac{x}{2}\right)+\frac{x}{2}\left(1-\frac{3}{4}x^2\right):=g^*(x, y) \end{aligned}$$

which because of (3.4) and (3.5) will imply $f(x, y)\geqslant g(x, y)$, the desired result.

Define $F(x, y):=f^*(x, y)-g^*(x, y)$. Differentiating with respect to y we find that $(\partial F/\partial y)(x, y)=0$ for

$$y=\frac{-8\pm\sqrt{225x^2+100x-116}}{30x}, \quad x\geqslant x_0:=\frac{-10+\sqrt{286}}{45}\approx 0.5294. \qquad (3.8)$$

Of the two values for y in (3.8), the smallest one is always less than $-x/2$. The other, which we denote by y_0, is greater than $-x/2$ if $x > 0.5606\ldots$. $F(x, y)$ is a third-degree polynomial in y with positive leading coefficient; hence y_0 corresponds to a minimum of $F(x, y)$. It is now necessary to prove

$$F(x, 1) \geqslant 0, \qquad \frac{1}{2} \leqslant x \leqslant 1,$$

$$F\left(x, -\frac{x}{2}\right) \geqslant 0, \qquad \frac{1}{2} \leqslant x \leqslant 1,$$

$$F(x, y_0) \geqslant 0, \qquad 0.5606 \leqslant x \leqslant 1.$$

Computing we find that

$$\begin{aligned} F(x, 1) &= \tfrac{7}{12}x^3 + \tfrac{1}{6}x^2 + x + 2 - (\tfrac{4}{5}x + \tfrac{7}{10})\sqrt{1 - x^2} \\ &\geqslant \tfrac{7}{12}x^3 + \tfrac{1}{6}x^2 + \tfrac{1}{5}x + \tfrac{13}{10} > 0. \\ F(x, -x/2) &= -\tfrac{5}{16}x^6 + \tfrac{23}{16}x^4 + \tfrac{3}{8}x^3 - \tfrac{7}{4}x^2 - x/2 + 2 - (\tfrac{4}{5}x + \tfrac{7}{10})\sqrt{1 - x^2} \\ &> -\tfrac{5}{16}x^6 + \tfrac{23}{16}x^4 + \tfrac{3}{8}x^3 - \tfrac{7}{4}x^2 - x/2 + 2 - (\tfrac{4}{5}x + \tfrac{7}{10})(\tfrac{6}{5} - \tfrac{1}{2}x) \\ &= -\tfrac{5}{16}x^6 + \tfrac{23}{16}x^4 + \tfrac{3}{8}x^3 - \tfrac{27}{20}x^2 - \tfrac{111}{100}x + \tfrac{29}{25}. \end{aligned}$$

From this we obtain the polynomial

$$p(x) = -125x^6 + 575x^4 + 150x^3 - 540x^2 - 444x + 464$$

which has the roots

$$x_1 = 1.968954, \qquad x_2 = -1.890177$$

$$x_{3,4} = 0.841041 \pm 0.174057i, \qquad x_{5,6} = -0.880429 \pm 0.759595i.$$

Since $p(0) > 0$, we have $F(x, -x/2) > 0$. Finally we compute

$$\begin{aligned} F(x, y_0) = {} & (\tfrac{29}{1350} - \tfrac{1}{54}x - \tfrac{1}{24}x^2)\sqrt{225x^2 + 100x - 116} \\ & - (\tfrac{1}{24}x^3 + \tfrac{1}{2}x^2 + \tfrac{5}{18}x - \tfrac{1144}{675}) - (\tfrac{4}{5}x + \tfrac{7}{10})\sqrt{1 - x^2} \end{aligned}$$

which will be positive if

$$\begin{aligned} & -\tfrac{1}{24}x^3 - \tfrac{1}{2}x^2 - \tfrac{5}{18}x + \tfrac{1144}{675} - (\tfrac{4}{5}x + \tfrac{7}{10})\sqrt{1 - x^2} \\ & \geqslant (\tfrac{1}{24}x^2 + \tfrac{1}{54}x - \tfrac{29}{1350})\sqrt{225x^2 + 100x - 116}. \end{aligned}$$

This is, by (3.7), true if

$$\begin{aligned} & -\tfrac{1}{24}x^3 - \tfrac{1}{2}x^2 - \tfrac{5}{18}x + \tfrac{1144}{675} - (\tfrac{4}{5}x + \tfrac{7}{10})\sqrt{1 - x^2} \\ & \geqslant (\tfrac{1}{24}x^2 + \tfrac{1}{54}x - \tfrac{29}{1350})(\tfrac{1107}{50}x - \tfrac{73}{10}) \end{aligned}$$

which again holds if

$$p(x)=67769105625x^6+85165323750x^5+26600097825x^4$$
$$-163971829800x^3-138703141256x^2-6974285760x$$
$$+136719867600\geqslant 0, \qquad \tfrac{1}{2}\leqslant x\leqslant 1.$$

We find the roots of $p(x)=0$ to be

$$x_{1,2}=0.932938\pm 0.0369766i$$
$$x_{3,4}=-0.593805\pm 1.127873i$$
$$x_{5,6}=-0.967482\pm 0.698857i$$

with all given digits precise. Hence, there are no roots in $[\frac{1}{2}, 1]$, and we have proved that $F(x, y_0)>0$, $x\in[\frac{1}{2}, 1]$.

(iii) $\frac{1}{2}<x\leqslant 1$, $\cos\varphi<-\pi/2$. As before set $y=\cos\varphi$, and let $f(x, y)$ and $g(x, y)$ be as in (3.2) and (3.3).

LEMMA 5. *For* $-1\leqslant y<-x/2$ *we have*

$$\text{(a)}\quad f(x, y)\geqslant f(x, -1), \qquad \tfrac{1}{2}<x\leqslant 1$$

$$\text{(b)}\quad g(x, y)\leqslant\begin{cases} g(x, -x/2), & \frac{1}{2}<x\leqslant\frac{2}{3}\\ g(x, -1), & \frac{2}{3}\leqslant x\leqslant 1.\end{cases}$$

Proof. $f(x, y)-f(x, -1)=\frac{5}{2}x^3y^3+2x^2y^2-(\frac{15}{8}x^3-\frac{3}{2}x)\,y+\frac{5}{8}x^3-2x^2+\frac{3}{2}x$ is monotonically increasing (as a function of y) when $x<\frac{2}{15}\sqrt{29}$ and has a local minimum for

$$y_0=\frac{-8+\sqrt{225x^2-116}}{30x}$$

when $x\geqslant\frac{2}{15}\sqrt{19}$. To prove (a), we study

$$f(x, y_0)-f(x, -1)=-\frac{225x^2-116}{5400}\sqrt{225x^2-116}$$
$$+\frac{5}{8}x^3-\frac{3}{2}x^2+\frac{3}{2}x-\frac{206}{675}.$$

This expression is positive if the polynomial

$$p(x)=-20250x^5+51075x^4-52720x^3+30824x^2-9888x+1584>0.$$

TABLE I

	$p(x)$		$q(x)$	
n	$\mathrm{Re}(x_n)$	$\mathrm{Im}(x_n)$	$\mathrm{Re}(x_n)$	$\mathrm{Im}(x_n)$
1/2	1.065931	±0.368367	0.875452	±1.559512
3/4	0.387738	±1.323358	0.843386	±0.0999483
5/6	−0.453669	±1.115972	−0.218838	±1.213894

We find that

$$p(x) = -(10x - 11)(45x^2 - 32x + 12)^2$$

which shows that it has only one real root, $x = \frac{11}{10}$, and that $p(x) > 0$ for $x < \frac{11}{10}$.

To prove (b) we recall from the proof of (3.4) that $\partial^2 g/\partial y^2 > 0$; hence $g(x, y)$ will attain its maximum in the endpoints of the interval $[-1, -x/2]$. $g(x, -x/2) = (x/2)(1 - \frac{3}{4}x^2)$ and $g(x, -1) = (x/2)(1 - x + \frac{3}{4}x^2)$, and it is easily verified that $g(x, -x/2) \geqslant g(x, -1)$ when $\frac{1}{2} < x \leqslant \frac{2}{3}$, and the other way around when $\frac{2}{3} \leqslant x \leqslant 1$. ∎

Using Lemma 5 the following inequalities will give the desired result, $f(x, y) \geqslant g(x, y)$:

$$f(x, -1) \geqslant g\left(x, -\frac{x}{2}\right), \qquad \tfrac{1}{2} < x \leqslant \tfrac{2}{3}, \tag{3.9}$$

$$f(x, -1) \geqslant g(x, -1), \qquad \tfrac{2}{3} \leqslant x \leqslant 1. \tag{3.10}$$

Equation (3.9) holds if the polynomial

$$p(x) = 100x^6 - 200x^5 + 364x^4 - 488x^3 + 485x^2 - 512x + 351 > 0, \qquad \tfrac{1}{2} < x \leqslant \tfrac{2}{3},$$

and (3.10) holds if the polynomial

$$q(x) = 100x^6 - 300x^5 + 689x^4 - 888x^3 + 985x^2 - 912x + 351 > 0, \qquad \tfrac{2}{3} \leqslant x \leqslant 1.$$

Table I gives the roots of $p(x)$ and $q(x)$. This ends the proof of Lemma 4.

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