Ukrainian Mathematical Journal, Vol. 53, No. 2, 2001

ESTIMATES OF THE KOLMOGOROV WIDTHS OF CLASSES OF ANALYTIC FUNCTIONS REPRESENTABLE BY CAUCHY-TYPE INTEGRALS. I

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UDC 517.5

In the Banach space of functions analytic in a Jordan domain $\Omega \subset \mathbb{C}$, we establish order estimates for the Kolmogorov widths of certain classes of functions that can be represented in Ω by Cauchy-type integrals along the rectifiable curve $\Gamma = \partial\Omega$ and can be analytically continued to $\Omega' \supset \Omega$ or to $\mathbb{C}$..

In this paper, we study the approximation properties of classes of functions analytic in a simply connected domain $\Omega \subset \mathbb{C}$ bounded by a finite closed rectifiable Jordan curve Γ.

We establish estimates for the approximation of these classes of functions by finite-dimensional subspaces of certain functional Banach spaces.

As sets of functions to be approximated, we consider the classes $L^{\psi}_{\beta}\mathfrak{N}(\Omega)$ (see [1] and [2]); in the first part of the paper, we consider these classes under certain restrictions on the sequence $\{\psi(k),\ k \in \mathbb{N}\}$ that defines a class for which functions from $L^{\psi}_{\beta}\mathfrak{N}(\Omega)$ are analytically continuable outside Γ to a domain $\Omega' \supset \Omega$ or even to the entire complex plane $\mathbb{C}$.

We briefly explain the scheme of definition of the classes $L^{\psi}_{\beta,p}(\Omega)$. For this purpose, we first denote by $L^{\psi}_{\beta}\mathfrak{N}(T)$ the natural realization of the classes $L^{\psi}_{\beta}\mathfrak{N}$ of complex-valued 2π-periodic locally summable functions of a real variable (ψ,β)-differentiable in the Stepanets sense (see, e.g., [3]) on the circle $T = \{w\colon |w| = 1\}$. Let $\Phi(\cdot)$ and $\Psi(\cdot)$ be functions realizing a conformal homeomorphism between the exterior of the domain $\overline{\Omega} = \Omega \cup \Gamma$ and the exterior of the disk $\overline{D} = \{w\colon |w| \le 1\}$; we assume that the function $\Phi(\cdot)$ satisfies the conditions

$$\lim_{z\to\infty} \Phi(z)/z = \alpha > 0 \quad \text{and} \quad \Phi(\infty) = \infty.$$

We denote by $L(\Gamma)$ the space of functions defined and summable on Γ and define the sets

$$L^{\psi}_{\beta,p}(\Gamma) := \left\{ f \in L(\Gamma)\colon f \circ \Psi \in L^{\psi}_{\beta,p}(T) \right\}$$

and

$$L^{\psi}_{\beta,p}(\Omega) := \left\{ \mathcal{K} f\colon \mathcal{K} f(z) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,\ f \in L^{\psi}_{\beta,p}(\Gamma),\ z \in \Omega \right\}.$$

Further, we set

$$\tilde{L}_q(\Gamma) = \left\{ \varphi\colon \|\varphi\|_{\Gamma,q} := \left(\int_{\Gamma} |\varphi(z)|^q |\Phi'(z)| |dz| \right)^{1/q} < \infty \right\}, \quad 1 < q < \infty,$$

where $\Phi'(\cdot)$ is the derivative of the function $\Phi(\cdot)$ extended by continuity to Γ.

Institute of Mathematics, Ukrainian Academy of Sciences, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 53, No. 2, pp. 229–237, February, 2001. Original article submitted October 12, 1999.

0041–5995/01/5302–0259 $25.00

Functions from the sets $L^{\psi}_{\beta,p}(\Omega)$ are regarded as elements of the space $\tilde{A}_q(\overline{\Omega})$, $1 < q < \infty$, consisting of analytic functions $\varphi(\cdot)$ in Ω and having angular boundary values $\overline{\varphi} \in \tilde{L}_q(\Gamma)$ almost everywhere on Γ. The norm of an element $\varphi(\cdot)$ is identified with the norm of $\overline{\varphi}(\cdot)$. We assume that the curve Γ that bounds the domain Ω and is completely determined by the function $\Phi(\cdot)$ $(\text{or } \Psi(\cdot))$ satisfies the following conditions:

(i) Γ is a regular curve, i.e.,

$$\sup_{z\in\Gamma}\sup_{r>0} \mu(\Theta_z(r))/r < \infty,$$

where $\Theta_z(r) = \{\zeta \in \Gamma : |\zeta - z| \le r\}$ and $\mu(A)$ is the Lebesgue measure of the set A on Γ;

(ii) $|\Phi'(\cdot)|$ satisfies the following analog (for weights on Γ) of the Muckenhoupt A_q-condition (see [4]):

$$\sup_{z\in\Gamma}\sup_{r>0}\left(\frac{1}{\mu(\theta_z(r))}\int\limits_{\theta_z(r)}|\Phi'(\zeta)||d\zeta|\right)\left(\frac{1}{\mu(\theta_z(r))}\int\limits_{\theta_z(r)}|\Phi'(\zeta)|^{-1/(q-1)}|d\zeta|\right)^{q-1} < \infty,$$

where $1 < q < \infty$. The set of closed rectifiable Jordan curves Γ satisfying conditions (i) and (ii) is denoted by RA_q.

For functions $\varphi \in \tilde{A}_q(\overline{\Omega})$, we denote by $\rho_n(\varphi;\cdot)_{\Gamma,q}$ and $E_n(\varphi;\cdot)_{\Gamma,q}$ the following approximation characteristics:

$$\rho_n(\varphi; z) := \left|\varphi(z) - S^F_{n-1}(\varphi; z)\right|, \qquad z \in \Omega,$$

where

$$S^F_n(\varphi; z) = \sum_{k=0}^{n} a_k(\varphi) F_k(z)$$

is the partial Faber sum of order n of the function $\varphi(\cdot)$,

$$a_k(\varphi) = \frac{1}{2\pi i}\int\limits_T \varphi(\Psi(w))w^{-k-1}dw, \qquad k = 0, 1, \dots,$$

and $F_k(\cdot)$ is the Faber polynomial of degree k for the domain Ω (see, e.g., [5, p. 350]), and

$$E_n(\varphi)_{\Gamma,q} := \inf_{p_n\in\mathcal{P}_n} \|\varphi(z) - p_n(z)\|_{\Gamma,q},$$

where

$$\mathcal{P}_n = \left\{p_n : p_n(z) = \sum_{k=0}^{n} c_k z^k,\ c_k \in \mathbb{C}\right\}$$

is the space of algebraic polynomials of degree n.

Correspondingly, we define

$$\mathcal{E}_n(W)_{\Gamma,q} := \sup_{\varphi \in W} \|\rho_n(\varphi;\cdot)\|_{\Gamma,q}$$

and

$$E_n(W)_{\Gamma,q} := \sup_{\varphi \in W} E_n(\varphi)_{\Gamma,q}, \qquad W \subset \tilde{A}_q(\overline{\Omega}).$$

We denote by C_1, $K, \ldots$ absolute positive constants, i.e., constants independent of n.

1. Main Result

We set

$$\mathfrak{M}'_{\infty} = \{\psi \in I_0 \colon (\exists K > 0 \ \ \forall t \ge 1\colon \quad \eta(t) - t \le K)\},$$

where I_0 is the set of positive functions decreasing to zero on $[1; \infty)$, $\eta(t) = \psi^{-1}(\psi(t)/2)$, and $\psi^{-1}(\cdot)$ is the function inverse to $\psi(\cdot)$.

Let

$$\mathfrak{N}_R := \left\{\psi \in \mathfrak{M}'_{\infty} \colon \lim_{k\to\infty} \ln|\psi(k)|^{-1/k} = R > 1\right\}$$

and

$$\mathfrak{N}_{\infty} := \left\{\psi \in \mathfrak{M}'_{\infty} \colon \lim_{k\to\infty} \ln|\psi(k)|^{-1/k} = \infty\right\}.$$

Assuming that the condition $\Gamma \in RA_p$, $1 < p < \infty$, is satisfied and using the properties of the Faber transformation $\mathcal{F}_{\Omega}$ for functions analytic in Ω (see Proposition 1) and conditions for the convergence of series in Faber polynomials $F_k(\cdot)$ determined by the domain Ω (see, e.g., Chaps. VI and VII in [6]), one can show that the following assertions are true:

(i) $L^{\psi}_{\beta,p}(\Omega)$ is the set of functions analytic in the domain $\Omega_R \supset \Omega$ bounded by the curve $\Gamma_R = \{z \colon |\Phi(z)| = R\}$ if $\psi \in \mathfrak{N}_R$, e.g., $\psi(t) = R^{-t}$, $R > 1$;

(ii) $L^{\psi}_{\beta,p}(\Omega)$ is the set of entire functions if $\psi \in \mathfrak{N}_{\infty}$, e.g., $\psi(t) = R^{-t^r}$, R, $r > 1$.

We now define the approximation characteristic to be investigated in this paper.

Let X be a Banach space with the norm $\|\cdot\|_X$, let W be a centrally symmetric set in X, and let L_n be an arbitrary n-dimensional subspace in X.

Definition 1. *The quantity*

$$d_n(W;X) = \inf_{L_n \subset X} \sup_{x \in W} \inf_{y \in L_n} \|x - y\|_X$$

is called the n-dimensional Kolmogorov width of the set W in the space X.

The following statement is true:

Theorem 1. *Let* Ω *be a domain bounded by a curve* $\Gamma \in RA_q$, $q > 1$. *If* $\psi \in \mathfrak{M}'_\infty$ *and* $\beta \in \mathbb{R}$, *then*[1]

$$d_n(L^{\psi}_{\beta,p}(\Omega); \tilde{A}_q(\overline{\Omega})) \asymp \psi(n), \qquad 1 < p, \quad q < \infty. \tag{1}$$

2. One Relation for Functions from the Set $\mathfrak{M}'_\infty$

Lemma 1. *If* $\psi \in \mathfrak{M}'_\infty$, *then*

$$\sum_{l=1}^{n} \frac{1}{\psi(l)} << \frac{1}{\psi(n)}. \tag{2}$$

Proof. By definition, for $\psi \in \mathfrak{M}'_\infty$ there exists a constant $K > 0$ such that $\eta(t) - t \le K$ for any $t \ge 1$ (in what follows, we assume that K is integer). Let $n \ge K$ be arbitrarily fixed and let $n_0 = [n/K] + 1$, where $[x]$ denotes the integer part of a number $x \in \mathbb{R}$ (clearly, in this case, $(n_0 - 1)K \le n < n_0 K$).

Further, since it is obvious that

$$\underbrace{\eta(\eta(\ldots\eta(t)))}_{m \text{ times}} \le mK$$

for any $m \in \mathbb{N}$, we have

$$2^{-m}\psi(t) \ge \psi(t + mK);$$

in particular,

$$\psi(rK) \ge 2^{s-r}\psi(sK), \quad s, r \in \mathbb{N}, \quad s \ge r.$$

Taking into account this inequality, we get

$$\psi(n)\sum_{l=1}^{n}\frac{1}{\psi(l)} < 2K + \psi((n_0-1)K)\sum_{l=K}^{(n_0-1)K}\frac{1}{\psi(l)}$$

$$\le 2K + \psi((n_0-1)K)\sum_{i=1}^{n_0-2}\sum_{l=iK}^{(i+1)K}\frac{1}{\psi(l)}$$

$$< 2K + (K+1)\sum_{i=1}^{n_0-2}\frac{\psi((n_0-1)K)}{\psi((i+1)K)} < 2K + (K+1)\sum_{i=1}^{n_0-2}\left(\frac{1}{2}\right)^{n_0-2-i} \le C,$$

which is equivalent to relation (2).

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[1] The relation $\alpha(n) \asymp \beta(n)$ means that there exist constants $C_1, C_2 > 0$ that do not depend on n and are such that $\forall n \ge 1$ $C_1 \le \alpha(n)/\beta(n) \le C_2$. If $C_1 \le \alpha(n)/\beta(n)$ or $\alpha(n)/\beta(n) \le C_2$, then we write $\alpha(n) \gg \beta(n)$ and $\alpha(n) \ll \beta(n)$, respectively.

3. Faber Transformation and Its Inverse

Consider the linear operators $\mathcal{F}_\Omega$ and $\mathcal{F}^\Omega$ acting according to the equalities

$$\mathcal{F}_\Omega[f](z) = \frac{1}{2\pi i}\int_\Gamma \frac{\bar{f}(\Phi(\zeta))}{\zeta - z} d\zeta, \quad f \in H_p, \quad z \in \Omega,$$

and

$$\mathcal{F}^\Omega[g](w) = \frac{1}{2\pi i}\int_{|\tau|=1} \frac{\bar{g}(\Psi(\tau))}{\tau - w} d\tau, \quad g \in \tilde{A}_p(\overline{\Omega}), \quad w \in D,$$

where

$$H_p := \left\{ \varphi \in \mathrm{Hol}(D) \colon \|\varphi\|_{H_p} = \sup_{0<\rho<1} \left\{ \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \varphi(\rho e^{it}) \right|^p dt \right) \right\}^{1/p} < \infty \right\}, \quad 1 \le p < \infty,$$

is the Hardy space of functions analytic in D [7, p. 388] and $\bar{f}$ and $\bar{g}$ denote the boundary values of the functions $f \in H_p$ and $g \in \tilde{A}_p(\overline{\Omega})$, respectively.

Proposition 1. *Let Ω be a domain bounded by a curve $\Gamma \in RA_p$, $1 < p < \infty$. Then the operators $\mathcal{F}_\Omega$ and $\mathcal{F}^\Omega$ realize an isomorphism between the subspaces $H_p \cap \mathcal{P}_m$ and $\tilde{A}_p(\overline{\Omega}) \cap \mathcal{P}_m$.*

Proof. First, note that

$$\mathcal{F}_\Omega[w^k](z) = F_k(z), \quad k = 0, 1, \ldots . \tag{3}$$

Relation (3) is a simple consequence of the following integral representation of Faber polynomials $F_k(\cdot)$ (see, e.g, [5, p. 358]):

$$F_k(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\Phi^k(\zeta)}{\zeta - z} d\zeta, \quad k = 0, 1, \ldots, \quad z \in \Omega.$$

On the other hand, by virtue of Lemma 3 [8, p. 54], we have

$$\mathcal{F}^\Omega[F_k](w) = w^k, \quad k = 0, 1, \ldots . \tag{4}$$

In view of the fact that the mapping $\mathcal{F}_\Omega \colon \mathcal{P}_m \to \mathcal{P}_m$ is bijective [8, p. 61], relations (3) and (4) imply that the operators $\mathcal{F}_\Omega$ and $\mathcal{F}^\Omega$ defined on the linear space $\mathcal{P}_m$ are inverse to one another.

The continuity of these operators in the normed subspaces indicated in Proposition 1 is a consequence of the boundedness of the Cauchy operator $\mathcal{K} \colon \tilde{L}_q(\Gamma) \to \tilde{A}_q(\overline{\Omega})$, $1 < q < \infty$, defined by the equality

$$\mathcal{K} f(z) = (2\pi i)^{-1} \int_\Gamma f(\zeta)(\zeta - z) d\zeta, \quad z \in \Omega,$$

under the condition $\Gamma \in RA_q$ (see, e.g., [9]). Moreover, for $1 < p < \infty$, if Γ is an arbitrary closed rectifiable Jordan curve, then $\|\mathcal{F}_\Omega\|_{H_p \to \tilde{A}_p(\overline{\Omega})} = K_1 < \infty$.

The statement below establishes certain properties of the operator S_n^F that associates every function $f \in \tilde{A}_p(\overline{\Omega})$ with its partial Faber sum of order n.

Proposition 2. *Let Ω be a domain bounded by a curve $\Gamma \in RA_p$, $1 < p < \infty$. Then S_n^F is a projector of $\tilde{A}_p(\overline{\Omega})$ to $\tilde{A}_p(\overline{\Omega}) \cap \mathcal{P}_m$, $m \le n$, i.e., the following assertions are true:*

(i) $\forall p_m \in \mathcal{P}_m: \ S_n^F(p_m) = p_m$;

(ii) $\left\| S_n^F \right\|_{\tilde{A}_p(\overline{\Omega}) \to \tilde{A}_p(\overline{\Omega})} \le C$, *where C is a positive constant independent of n.*

Proof. Assume that an arbitrary polynomial

$$p_m(z) = \sum_{k=0}^{m} c_k z^k$$

is given. Then

$$p_m(z) = \sum_{k=0}^{m} a_k(p_m) F_k(z),$$

where $F_k(z)$, $k = 0, 1, \ldots$, are the Faber polynomials for the domain Ω and $a_k(p_m)$ are the Faber coefficients of the function $p_m(\cdot)$. Assertion (i) is a direct consequence of the equality (Lemma 3 in [8, p. 54])

$$a_k(F_l) = \begin{cases} 0, & k \ne l, \\ 1, & k = l. \end{cases}$$

To prove assertion (ii), we note that [see (3) and (4)]

$$S_n^F(f; z) = \mathcal{F}_\Omega[Q_n](z),$$

where

$$Q_n(w) = \sum_{k=0}^{n-1} a_k(f) w^k$$

and $a_k(f)$ are the Faber coefficients of the function f. Therefore, according to Proposition 1, we have

$$\left\| S_n^F(f; \cdot) \right\|_{\Gamma, p} \le C_1 \|Q_n\|_{T, p}. \tag{5}$$

Since $Q_n(\cdot)$ is the partial Fourier sum of order $n - 1$ of the function $f \circ \Psi \in L_p(T)$, we conclude that, by virtue of the boundedness of the Fourier operator $S_n: L_p(T) \to L_p(T)$, $1 < p < \infty$,

$$\|Q_n\|_{T,p} \leq C_2 \|f \circ \Psi\|_{T,p} = C_2 \|f\|_{\Gamma,p}. \tag{6}$$

Relations (5) and (6) yield assertion (ii).

4. An Analog of the Nikol'skii Inequality

Nikol'skii [10] established the following relation for the values of the norms of an arbitrary trigonometric polynomial

$$t_n(x) = \sum_{k=-n}^{n} c_k e^{ikx}$$

in the spaces $L_r(0;2\pi)$ and $L_s(0;2\pi)$:

$$\|t_n(x)\|_s \ll n^{1/r-1/s} \|t_n(x)\|_r, \quad 1 \leq r < s \leq \infty.$$

A strengthened version of this relation is the inequality

$$\left\|t_{n_1,n_2}(x)\right\|_s \leq C(n_2 - n_1)^{1/r-1/s} \left\|t_{n_1,n_2}(x)\right\|_r, \quad 1 \leq r < s \leq \infty, \tag{7}$$

where $t_{n_1,n_2}(\cdot)$ is an arbitrary polynomial of the form

$$t_{n_1,n_2}(x) = \sum_{|k| \in [n_1;n_2]} c_k e^{ikx}, \quad 0 \leq n_1 < n_2,$$

and C is a constant independent of n_1 and n_2.

Inequality (7) is used in what follows, although it should be noted that there exists a simple analog of it for algebraic polynomials of the form

$$P_{n_1,n_2}(z) = \sum_{k=n_1}^{n_2} c_k z^k, \quad 0 \leq n_1 \leq n_2, \quad z \in \mathbb{C},$$

the set of which is denoted by $\mathcal{P}_{n_1,n_2}$.

Proposition 3. *Let* Ω *be a domain bounded by a curve* $\Gamma \in RA_r$, $1 < r < s < \infty$, *and let* $P_{n_1,n_2} \in \mathcal{P}_{n_1,n_2}$. *Then*

$$\left\|P_{n_1,n_2}\right\|_{\Gamma,s} \leq C(n_2 - n_1)^{1/r-1/s} \left\|P_{n_1,n_2}\right\|_{\Gamma,r}. \tag{8}$$

To prove (8), it suffices to note that any polynomial $P_{n_1,n_2}(\cdot)$ admits a unique decomposition into a finite sum of Faber polynomials $F_k(\cdot)$ for the domain

$$\Omega: P_{n_1,n_2}(z) = \sum_{k=n_1}^{n_2} a_k F_k(z),$$

and, according to Proposition 1 [in view of relations (3) and (4)], we have

$$C_1 \left\| \tilde{t}_{n_1,n_2}(w) \right\|_{T,\nu} \le \left\| P_{n_1,n_2}(z) \right\|_{\Gamma,\nu} \le C_2 \left\| \tilde{t}_{n_1,n_2}(w) \right\|_{T,\nu}, \quad 1 < \nu < \infty, \tag{9}$$

where C_1 and C_2 are positive constants independent of n_1 and n_2 and

$$\tilde{t}_{n_1,n_2}(w) = \sum_{k=n_1}^{n_2} a_k w^k .$$

Combining (9) and (7), we get (8).

5. Proof of Theorem 1

Lower Bound. The successive application of the theorem on a projector [11, p. 207] (using Proposition 2) and Proposition 1 allows one to easily obtain the main relation for the lower bound of the width:

$$d_n(L^{\psi}_{\beta,p}(\Omega); \tilde{A}_q(\overline{\Omega})) \gg d_n(L^{\psi}_{\beta,p}(D) \cap \mathcal{P}_m; H_q \cap \mathcal{P}_m),$$

or

$$d_n(L^{\psi}_{\beta,p}(\Omega); \tilde{A}_q(\overline{\Omega})) \gg d_n(L^{\psi}_{0,p}(T) \cap \mathcal{P}_m; L_q(T) \cap \mathcal{P}_m), \quad m \ge n. \tag{10}$$

First, we assume that $q = 2$.

Let $\tau = \{\tau_s\}_{s=0}^{n}$ be an arbitrary orthonormal system in $L_2(T) \cap \mathcal{P}_m$. Denoting the Fourier coefficients of a function $f(\cdot)$ with respect to the system τ by $(f;\tau_s)$, i.e.,

$$(f;\tau_s) = (2\pi i) \int_{|w|=1} f(w) \overline{\tau_s(w)} dw,$$

and using the expansions

$$\begin{aligned} w^k &= \sum_{s=0}^{n} (w^k;\tau_s)\, \tau_s(w), \quad k = 0, 1, \ldots, n, \\ \tau_l(w) &= \sum_{k=0}^{n} (\tau_l;w^k)\, w^k, \quad l = 0, 1, \ldots, n, \end{aligned} \tag{11}$$

we get

$$\sum_{k=0}^{n} \left| (w^k;\tau_s) \right|^2 = \sum_{s=0}^{n} \left| (w^k;\tau_s) \right|^2 = 1. \tag{12}$$

We set

$$S_k(f;\tau;w) = \sum_{s=0}^{k} (f;\tau_s)\, \tau_s(w)$$

and $a_s^k = (w^k;\tau_s)$. Then

$$\left\| w^k - S_{n-1}(w^k;\tau;w)\right\|_{T,2}^2 = \left\| w^k - \sum_{s=0}^{n-1} a_s^k \tau_s(w)\right\|_{T,2}^2 = \left|a_n^k\right|^2, \quad k = 0, 1, \ldots, n. \tag{13}$$

Consider the function

$$f_k(w) = \begin{cases} \psi(k) w^k, & k = 1, 2, \ldots, n, \\ \psi(1), & k = 0. \end{cases}$$

It is clear that $f_k \in L_{0,p}^{\psi}(T)$ and

$$\sigma_k := \| f_k - S_{n-1}(f_k;\tau;w)\|_{T,2}^2 = \psi^2(k)\left|a_n^k\right|^2, \quad k = 0, 1, \ldots, n.$$

We show that there exist k, $0 \le k \le n$, and $C > 0$ such that

$$\sigma_k \ge C\psi^2(n) \tag{14}$$

(we assume that $\psi(0) = \psi(1)$).

Assume the contrary. Then

$$1 = \sum_{k=0}^{n} \left|a_n^k\right|^2 \le \sum_{k=0}^{n} \sigma_k / \psi^2(k) \le C\psi^2(n) \sum_{k=0}^{n} 1/\psi^2(k). \tag{15}$$

According to Proposition 1, for some $C_0 > 0$ we have

$$\psi^2(n) \sum_{k=0}^{n} 1/\psi^2(k) \le C_0$$

and, therefore, relation (15) is contradictory for $0 < C < 1/C_0$, which proves (14).

Relation (14) yields

$$d_n\big(L_{0,p}^{\psi}(T) \cap \mathcal{P}_n;\ L_2(T) \cap \mathcal{P}_n\big) \gg \psi(n)$$

for any $0 < p < \infty$. This estimate, together with (10), yields

$$d_n\big(L_{\beta,p}^{\psi}(\Omega); \tilde{A}_q(\overline{\Omega})\big) \gg \psi(n)$$

for $q = 2$; in view of the imbedding $\tilde{A}_2(\overline{\Omega}) \supset \tilde{A}_q(\overline{\Omega})$, $2 \le q < \infty$, this estimate is also true for $1 < p < \infty$ and $2 \le q < \infty$.

The same estimate is also true in the case where $1 < p < \infty$ and $1 < q < 2$. Indeed, let $\varphi = \{\varphi_i\}_{i=1}^n$ be an arbitrary orthonormal system from $\mathcal{P}_n$. Then, setting $\varphi_n^*(w) = w^n$, $w \in D$, taking into account the boundedness of the Fourier operator $S_n\colon L_q(T) \to L_q(T)$ with respect to the system $\{e^{ikt}\}_{k=0}^{\infty}$, and using inequality (7), we get

$$\sup_{f\in L^{\psi}_{0,p}(T)\cap\mathcal{P}_n}\inf_{c_i}\left\| f-\sum_{i=1}^{n}c_i\varphi_i\right\|_{T,q} \gg \sup_{f\in L^{\psi}_{0,p}(T)\cap\mathcal{P}_n}\inf_{c_i}\left\|(S_n-S_{n-1})[f]-\sum_{i=1}^{n}c_i\,(S_n-S_{n-1})[\varphi_i]\right\|_{T,q}$$

$$\gg \sup_{f\in L^{\psi}_{0,p}(T)\cap\mathcal{P}_n}\inf_{c}\left\| f-(c\varphi_n^{*}+(S_{n-1})[f])\right\|_{T,2}$$

$$\gg d_n\big(L^{\psi}_{0,p}(T)\cap\mathcal{P}_n;\, L_2(T)\cap\mathcal{P}_n\big) \gg \psi(n),$$

i.e.,

$$d_n\big(L^{\psi}_{0,p}(T)\cap\mathcal{P}_n;\, L_q(T)\cap\mathcal{P}_n\big) \gg \psi(n)$$

for $1<p<\infty$ and $1<q<2$. Comparing this equality with relation (10), we establish the following estimate under the indicated restrictions on the parameters p and q:

$$d_n\big(L^{\psi}_{\beta,p}(\Omega);\tilde{A}_q(\overline{\Omega})\big) \gg \psi(n).$$

Remark 1. In the case $1<p\le q<\infty$, the lower bound in Theorem 1 can be obtained in a different way. Namely, it suffices to use the estimate $d_n\big(L^{\psi}_{\beta,p}(\Omega);\tilde{A}_p(\overline{\Omega})\big)\asymp\psi(n)$ for $1<p<\infty$ and $\psi\in I_0$, which was established earlier by the author.

Upper Bound. The upper bound in Theorem 1 follows from the estimate from above for the least upper bounds of the best approximations by algebraic polynomials of degree $n-1$ over the set $L^{\psi}_{\beta,p}(\Omega)$ in the space $\tilde{A}_q(\overline{\Omega})$.

The following statement is true:

Theorem 2. *Let* Ω *be a domain bounded by a curve* $\Gamma\in RA_q$, $1<p,\ q<\infty$, *let* $\psi\in\mathfrak{M}'_{\infty}$, *and let* $\beta\in\mathbb{R}$. *Then*

$$E_n\big(L^{\psi}_{\beta,p}(\Omega)\big)_{\Gamma,q} \asymp \psi(n).$$

Proof. One can easily verify that, for $\Gamma\in RA_q$, we have

$$E_n(\mathcal{K}f)_{\Gamma,q} \asymp \|\rho_n(\mathcal{K}f;\cdot)\|_{\Gamma,q} \ll \|\rho_n(F;\cdot)\|_{T,q}, \tag{16}$$

where $f\in\tilde{L}_q(\Gamma)$ and $F=f\circ\Psi$. The upper bound follows from this relation and the estimates for the quantities $\|\rho_n(F;\cdot)\|_{T,q}$ in the case where $F\in L^{\psi}_{\beta,p}(T)$ and $\psi\in\mathfrak{M}'_{\infty}$ established in [3, Chap. II, Sec. 6].

The lower bound in Theorem 2 obviously follows from the established lower bound for the quantity $d_n\big(L^{\psi}_{\beta,p}(\Omega);\tilde{A}_q(\Omega)\big)$ (it can also be obtained directly).

Theorem 2 and, hence, Theorem 1 are proved.

Remark 2. Estimates similar to those presented in Theorem 1 are also valid in the case of approximation of the classes $L^{\psi}_{\beta,p}(\Omega)$ by arbitrary n-dimensional subspaces of the Smirnov space $E_q(\Omega)$ of functions analytic in Ω under the only condition that the closed rectifiable Jordan curve Γ that bounds the domain Ω is regular.

REFERENCES

1. A. I. Stepanets and V. S. Romanyuk, “Classes of $(\psi;\beta)$-differentiable functions of a complex variable and approximation by linear means of their Faber series,” *Ukr. Mat. Zh.,* **44**, No. 11, 1556–1570 (1992).
2. V. S. Romanyuk, “Approximation of classes of analytic functions by algebraic polynomials and Kolmogorov widths,” *Ukr. Mat. Zh.,* **48**, No. 2, 236–250 (1996).
3. A. I. Stepanets, *Classification and Approximation of Periodic Functions* [in Russian], Naukova Dumka, Kiev (1987).
4. B. Muckenhoupt, “Weighted norm inequalities for the Hardy maximal functions,” *Trans. Amer. Math. Soc.,* **165**, No. 1, 207–226 (1972).
5. V. K. Dzyadyk, *Introduction to the Theory of Uniform Polynomial Approximation of Functions* [in Russian], Nauka, Moscow (1977).
6. P. K. Suetin, *Series in Faber Polynomials* [in Russian], Nauka, Moscow (1984).
7. G. M. Goluzin, *Geometric Theory of Functions of Complex Variables* [in Russian], Nauka, Moscow (1966).
8. D. Gaier, *Vorlesungen Über Approximation im Komplexen,* Birkhäuser, Basel (1980).
9. E. M. Dynkin and B. P. Osilenker, “Weight estimates of singular integrals and their applications,” in: *VINITI Series in Mathematical Analysis* [in Russian], Vol. 21 (1983), pp. 42–128.
10. S. M. Nikol’skii, “Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of many variables,” *Tr. Mat. Inst. Akad. Nauk SSSR,* **38**, 244–278 (1951).
11. V. M. Tikhomirov, “Approximation theory,” in: *VINITI Series in Contemporary Problems in Mathematics. Fundamental Trends* [in Russian], Vol. 14 (1987), pp. 105–260.