The Lambert W Function, Laguerre Polynomials, and the Zeros of the QCD Partition Function

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We study solutions of a transcendental equation for the complex chemical potential at which a random-matrix QCD model can undergo a phase transition at zero mass. An explicit solution is obtained in terms of the Lambert W function. We also provide a closed form expression for a QCD random matrix model partition function, as a sum of Laguerre polynomials, for complex chemical potential and non-zero mass.

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I. INTRODUCTION

The study of zeros of the partition function is a classic problem of mathematical physics and an essential tool for the investigation of phase transitions (see [1], section 9.3). Yang and Lee [2, 3] showed that, subject to reasonable assumptions, the thermodynamic limit of the Helmholtz free energy density is an analytic function of the fugacity in any region of the complex fugacity plane (or the complex chemical potential plane) in which the partition function has no zeros.

In lattice quantum chromodynamics (QCD), the connection between phase transitions and the complex chemical potential has received much study; see Aarts [4, 5] for recent status reports. There are difficult issues arising in the simulation of the partition function of lattice QCD for complex chemical potential.

Halasz, Jackson, and Verbaarschot [hereafter HJV] considered [6] a random matrix partition function with the global symmetries of the QCD partition function, and obtained interesting results. They utilized both numerical and analytical methods to determine the Yang-Lee zeros of the partition function $Z_N(m, \mu)$ for the system considered. Here m is the mass parameter, which is allowed to be complex-valued, μ is the complex chemical potential, and N is a "size-of-system" parameter which is analogous to volume. Their study included explorations in each of the complex mass plane and the complex chemical potential plane.

HJV obtained two results which are the focus of our present work. These are found in the original HJV paper [6] as equations (8) and (10), respectively.

Equation (8) of the HJV paper is a transcendental equation, expressing the conditions on μ for a phase transition when the mass m is zero in their model. This equation can be solved explicitly using the Lambert W function. We present the details of that solution in section II. Consideration of the multiple branches of the Lambert W function exhibits some additional solutions to those obtained in the HJV paper.

Equation (10) of the HJV paper is the QCD partition function for their random matrix model. This can be written as a sum of Laguerre polynomials. We present the details in section III.

There are two advantages which may be obtained from the expression of formulas which arise in a model of a particular problem, in terms of well-known functions. Firstly, we can use our knowledge of those standard functions for improved understanding of the behavior of the formulas in the problem at hand, and perhaps gain insight into the underlying physical and analytical relationships. Secondly, in conjunction with highly efficient software for the calculation of standard functions, it may be possible to speed up the computational work for the problem model.

This paper is a particular topic within a more broad-focus study of the concepts of complex chemical potential and complex fugacity. We are interested in the manner in which extending the domain of analysis for chemical potential and fugacity into the complex plane, can provide further insight into physical models. Our inspiration has been the papers of M. Howard Lee [7, 8] regarding chemical potential and the use of polylogarithms in statistical mechanics. Here, we direct our attention to equations (8) and (10) of the HJV paper [6].

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II. LAMBERT W SOLUTION OF HJV EQN (8)

The random matrix model which was explored by HJV involves replacing the matrix elements of the Dirac operator in the continuum QCD partition function, with Gaussian distributed random variables consistent with the global partition function. See Verbaarschot and Wettig [9] for a review of the technique. HJV in [6] consider a single flavor of quarks.

If $m = 0$, then in the thermodynamic limit the points μ at which the phase change occurs are determined by the solutions to the transcendental equation

$$\mathtt{Re}\big[\mu^2 + \mathtt{log}(\mu^2)\big] = -1 \tag{1}$$

which appears as equation (8) in [6]. The solutions of (1) are

$$\mu = W(z)^{1/2} \tag{2}$$

where W is the multi-branch Lambert W function, and z is any point on the circle of radius $1/e$ about the origin.

Before giving the proof, we provide some background. The Lambert W function $w = W(z)$ is the multi-branch analytic function which satisfies

$$we^w = z \tag{3}$$

Here w and z are complex. W_k denotes the k-th branch of W. For details and history of the Lambert W function, see Corless, et al [10]. For an example of its use in quantum statistics, see Valluri, et al [11]. Of possible relevance in QCD contexts, the Lambert W function is also defined for matrix arguments; see Higham [12], page 51.

Now we turn to the proof that equation (2) is the solution of (1). Set $w = \mu^2$. Then

$$e^{\mu^2 + \mathtt{log}(\mu^2)} = e^{w + \mathtt{log}(w)} = we^w \tag{4}$$

Let $z = we^w$ so that $w = W(z)$, where $W()$ denotes the multi-branch Lambert W function, and w is any of the possible function values. Suppose that (1) holds. That is,

$$\mu^2 + \mathtt{log}(\mu^2) = -1 + \mathtt{i}y \tag{5}$$

for some arbitrary real value y. Then

$$e^{\mu^2 + \mathtt{log}(\mu^2)} = we^w = e^{-1 + \mathtt{i}y} = \big(\frac{1}{e}\big)e^{\mathtt{i}y} = \big(\frac{1}{e}\big)v \tag{6}$$

where v is an arbitrary point on the unit circle.

Let C denote the circle of radius $1/e$ about the origin. The solution set of (1) is thus μ^2 in $W(C)$, the image of C under the Lambert W function. Taking the square root (another multivalued function), the solution set of equation (1) is seen to be $\mathtt{sqrt}(W(C))$. That establishes (2).

What does the solution set $S = \mathtt{sqrt}(W(C))$ look like? One has to consider the various branches of the Lambert W function, as well as the two branches of the square root function. We have graphed the solution set with two standard mathematical software packages (Maple and Mathematica) with identical results. In figure 1 we show the solution set points for the principal branch W_0, comprising the two central arcs which meet at $\mu = \mathbf{i}$ and $\mu = -\mathbf{i}$; the solution set arising from the branch W_1, which is the lines going upward-right and downward-left; and the solution set arising from the branch W_{-1}, which is the lines going upward-left and downward-right. The solutions for other branches W_k continue those lines further.

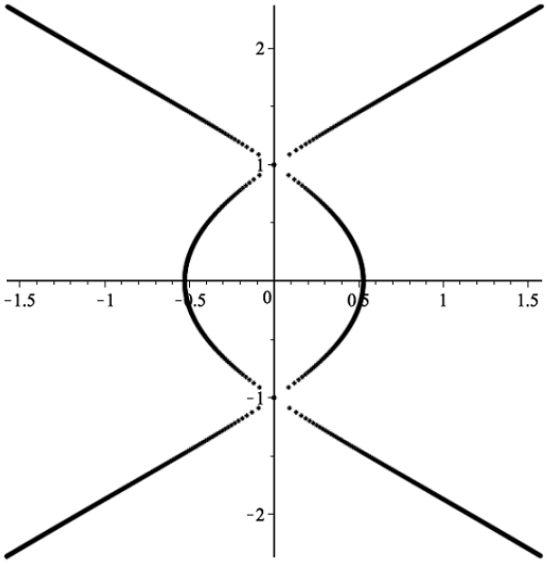

FIG. 1. Solutions of equation (1).

Compare with figure 2 in [6], the top portion for $m = 0$: Only the solutions arising from the principal branch W_0 appear in the figure in [6]. Thus, understanding that equation (1) has a solution in terms of the Lambert W function leads to additional solutions μ which can be considered in the larger model context as the possible locations of phase transitions.

The real valued solutions of (1) are readily calculated with the Lambert W function evaluator provided by standard mathematical software. They are

$$\mu = \mathtt{sqrt}\Big[W_0\Big(\frac{1}{e}\Big)\Big] = \pm\ 0.5276973970 \tag{7}$$

We note that equation (1) can also be viewed in terms of the polylogarithm function $\mathtt{Li}_s$ in this case of order $s = 1$, since $\mathtt{Li}_1(z) = -\mathtt{log}(1 - z)$. There is a potential for generalization to other orders of polylogs. We believe that the inherent structure of the polylogarithm function will be a useful tool for the study of properties of partition functions. See [13] or [14] for background regarding polylogs, and, for instance, [8] for an application.

III. LAGUERRE POLYNOMIAL REWRITE OF HJV EQN (10)

HJV rewrite the partition function explicitly as a polynomial in powers of μ^2 and m^2, to obtain their equation (10), which is

$$Z_N(m,\mu) = \frac{\pi N!}{N^{N+1}} \sum_{k=0}^{N} \sum_{j=0}^{N-k} \frac{(Nm^2)^k}{(k!)^2} \frac{(-N\mu^2)^j}{j!} \frac{(N-j)!}{(N-j-k)!} \tag{8}$$

For simplicity, let $u = Nm^2$, $v = N\mu^2$, let A denote the coefficient $\frac{\pi N!}{N^{N+1}}$, and adopt the convention that j and k are always non-negative. Then (8) becomes

$$Z_N(m,\mu) = A \sum_{k=0}^{N} \sum_{j \le k} \frac{u^k}{k!} \frac{(-v)^j}{j!} \binom{N-j}{N-j-k} \tag{9}$$

Reversing the order of summation and rearranging,

$$Z_N(m,\mu) = A \sum_{j=0}^{N} B_j \tag{10}$$

where

$$B_j = \frac{(-v)^j}{j!} \sum_{k \le j} \frac{u^k}{k!} \binom{N-j}{N-j-k} \tag{11}$$

The Laguerre polynomials $L_r(x)$, for any non-negative integer r, can be defined in a variety of ways. See [13] for background. One expression for the Laguerre polynomials (see, for instance, [15], page 614, formula 21) is

$$L_r(x) = \sum_{s=0}^{r} \binom{r}{s} \frac{(-1)^s}{s!} x^s \tag{12}$$

Hence B_j is seen to be

$$B_j = \frac{(-v)^j}{j!} L_{N-j}(-u) \tag{13}$$

We obtain a representation of the partition function as a sum of Laguerre polynomials,

$$Z_N(m,\mu) = \frac{\pi N!}{N^{N+1}} \sum_{j=0}^{N} \frac{(-N\mu^2)^j}{j!} L_{N-j}(-Nm^2) \tag{14}$$

In particular, for the case of zero chemical potential, since only the $j = 0$ term remains, $Z_N(m,0)$ is explicitly given by a single Laguerre polynomial

$$Z_N(m,0) = \frac{\pi N!}{N^{N+1}} L_N(-Nm^2) \tag{15}$$

These representations offer the possibility of speeding up the computation of the partition function by using standard mathematical software. In this regard, refer to [16] on asymptotic behavior of Laguerre polynomials, and to [17, 18] on the density of zeros of Laguerre polynomials.

The partition function (14) can be expressed in terms of a confluent hypergeometric function or a cylindrical Bessel function. The density of zeros of this partition function can be studied in terms of the zeros of Bessel functions [19].

IV. CONCLUSION

In conclusion, we express the solutions of the transcendental equation (8) of [6] for complex μ in terms of the Lambert W function. We provide an expression of the QCD random matrix partition function, equation (10) of [6], as a Laguerre polynomial in m^2 for zero chemical potential, and as a sum of Laguerre polynomials for non-zero μ. We hope that the analytic results found will be useful in understanding QCD random matrix models, and will result in higher accuracy and a reduction of computer time in further studies.

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