On Beltrami equations and Hölder regularity

Tonia Ricciardi*

Dipartimento di Matematica e Applicazioni "R. Caccioppoli"

Università di Napoli Federico II

Via Cintia, 80126 Napoli, Italy

fax: +39 081 675665

tonia.ricciardi@unina.it

Abstract

We estimate the Hölder exponent α of solutions to the Beltrami equation $\overline{\partial} f = \mu \partial f$, where the Beltrami coefficient satisfies $\|\mu\|_\infty < 1$. Our estimate improves the classical estimate $\alpha \geq \|K_\mu\|^{-1}$, where $K_\mu = (1 + |\mu|)/(1 - |\mu|)$, and it is sharp, in the sense that it is actually attained in a class of mappings which generalize the radial stretchings. Some other properties of such mappings are also provided.

Key Words: linear Beltrami equation, measurable coefficients, Hölder regularity

MSC 2000 Subject Classification: 30C62 (35J25)

1 Introduction and main results

Let Ω be a bounded open subset of $\mathbb{R}^2$ and let $f \in W^{1,2}_{\rm loc}(\Omega, \mathbb{C})$ satisfy the Beltrami equation

$$\overline{\partial} f = \mu\, \partial f \qquad \text{a.e. in } \Omega, \tag{1}$$

where $\mu \in L^\infty(\Omega, \mathbb{C})$ satisfies $\|\mu\|_\infty < 1$, $\overline{\partial} = (\partial_1 + i\partial_2)/2$, $\partial = (\partial_1 - i\partial_2)/2$. By classical results, see, e.g., Iwaniec and Martin [4], there exists $\alpha \in (0, 1)$ such that f is α-Hölder continuous in Ω. Namely, for every compact $T \Subset \Omega$ there exists $C_T > 0$ such that

$$\frac{|f(z) - f(z')|}{|z - z'|^\alpha} \leq C_T \qquad \forall z, z' \in T,\ z \neq z'.$$

Moreover, let

$$K_\mu = \frac{1 + |\mu|}{1 - |\mu|}$$

denote the distortion function. Then, the following estimate holds:

$$\alpha \geq \frac{1}{\|K_\mu\|_\infty}. \tag{2}$$

*Supported in part by the MIUR National Project *Variational Methods and Nonlinear Differential Equations*.

This estimate is sharp, in the sense that it reduces to an equality on the radial stretchings

$$f(z) = |z|^{\alpha-1}z. \tag{3}$$

Indeed, the function f defined above satisfies the Beltrami equation (1) with Beltrami coefficient given by $\mu(z) = -(1-\alpha)/(1+\alpha)z\bar{z}^{-1}$, and a simple computation yields $K_\mu \equiv \alpha^{-1}$. We also note that f defined in (3) satisfies the pointwise equality

$$|Df(z)|^2 = \alpha^{-1}J_f, \tag{4}$$

which is of interest in the context of quasiconformal mappings.

There exists a wide literature concerning the regularity theory for (1), in the degenerate case where $\|\mu\|_\infty = 1$, or equivalently, when the distortion function K_μ is unbounded. See, e.g., [1, 2, 4, 5], and the references therein. Our aim in this note is to extend the classical estimate (2) in another sense, by subtracting a correction term to K_μ.

Theorem 1. *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ satisfy the Beltrami equation (1). Then, f is α-Hölder continuous with*

$$\alpha \geq \left(\operatorname*{ess\,sup}_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \left\{ K_\mu - 2\frac{|\mu| + \mathrm{Re}\,(\mu, n^2)}{1-|\mu|^2} \right\} \right)^{-1}, \tag{5}$$

where n denotes the outer unit normal.

Here, $(\cdot,\cdot)$ denotes the standard inner product in $\mathbb{C}$, and n^2 is understood in the complex multiplication sense. That is, for every $y = x + \rho e^{it} \in S_\rho(x)$, we have $n(y) = e^{it}$, $(\mu, n^2)(y) = \mu(y)\overline{e^{2it}} = \mu(y)e^{-2it}$. In particular, $|\mu| + \mathrm{Re}\,(\mu, n^2) \geq 0$, and therefore

$$\operatorname*{ess\,sup}_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \left\{ K_\mu - 2\frac{|\mu| + \mathrm{Re}\,(\mu, n^2)}{1-|\mu|^2} \right\} \leq \operatorname*{ess\,sup}_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} K_\mu \leq \|K_\mu\|_\infty.$$

It follows that estimate (5) indeed improves (2). Moreover, our estimates are sharp, in the following sense. For every $M > 1$, let

$$c = c(M) = \frac{2}{1+M^{-1}} \tag{6}$$

and let $k_0 : \mathbb{R} \to \mathbb{R}$ be the bounded, 2π-periodic function defined by

$$k_0(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, c\pi/2) \cup [\pi, \pi + c\pi/2) \\ M, & \text{otherwise.} \end{cases} \tag{7}$$

Let $\Theta_1, \Theta_2 : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic Lipschitz functions defined by

$$\Theta_1(\theta) = \begin{cases} \sin(c^{-1}\theta - \pi/4), & \theta \in [0, c\pi/2) \\ \cos(c^{-1}M(\theta - c\pi/2) - \pi/4), & \theta \in [c\pi/2, \pi) \\ -\sin(c^{-1}(\theta - \pi) - \pi/4), & \theta \in [\pi, \pi + c\pi/2) \\ -\cos(c^{-1}M(\theta - \pi - c\pi/2) - \pi/4), & \theta \in [\pi + c\pi/2, 2\pi) \end{cases}$$

and

$$\Theta_2(\theta) = \begin{cases} -\cos(c^{-1}\theta - \pi/4), & \theta \in [0, c\pi/2) \\ \sin(c^{-1}M(\theta - c\pi/2) - \pi/4), & \theta \in [c\pi/2, \pi) \\ \cos(c^{-1}(\theta - \pi) - \pi/4), & \theta \in [\pi, \pi + c\pi/2) \\ -\sin(c^{-1}M(\theta - \pi - c\pi/2) - \pi/4), & \theta \in [\pi + c\pi/2, 2\pi). \end{cases}$$

The sharpness of Theorem 1 is a consequence of the following.

Theorem 2. *Let B the unit disk in $\mathbb{R}^2$ and let $f_0 \in W^{1,2}(B, \mathbb{C})$ be defined in $B \setminus \{0\}$ by*

$$f_0(z) = |z|^{1/c} \left(\Theta_1(\arg z) + i\Theta_2(\arg z)\right).$$

Let

$$\mu_0(z) = \frac{1 - k_0(\arg z)}{1 + k_0(\arg z)} z\bar{z}^{-1}.$$

Then f_0 satisfies (1) *with $\mu = \mu_0$. Furthermore, there exists $\bar{M} > 1$ such that*

$$\operatorname*{ess\,sup}_{S_\rho(x) \subset \Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \left\{ K_{\mu_0} - 2\frac{|\mu_0| + \operatorname{Re}\left(\mu_0, n^2\right)}{1 - |\mu_0|^2} \right\} = c,$$

for every $M \in (1, \bar{M})$.

Our proof of Theorem 1 is based on the equivalence between Beltrami equations and elliptic divergence form equations with unit determinant, to which we can apply some recent results in [6]. On the other hand, in order to prove Theorem 2, we shall use some properties of a class of mappings which generalize the radial stretchings. More precisely, we consider mappings of the form:

$$f(z) = |z|^\alpha (\eta_1(\arg z) + i\eta_2(\arg z)) \tag{8}$$

and coefficients of the form

$$\mu(z) = \frac{1 - k(\arg z)}{1 + k(\arg z)} z\bar{z}^{-1} \tag{9}$$

for some 2π-periodic functions η_1, η_2, k, $k \geq 1$. We prove:

Proposition 1. *Suppose μ is of the form* (9) *and f satisfies* (8). *Then f is a a solution to* (1) *if and only if (η_1, η_2) satisfies the first order system:*

$$\begin{cases} \eta_1' = -\alpha k \eta_2 \\ \eta_2' = \alpha k \eta_1. \end{cases} \tag{10}$$

Furthermore, for every $z \neq 0$ the following equality holds:

$$|Df|^2 = k(\arg z) J_f, \tag{11}$$

where $|Df|$ denotes the operator norm of Df.

We note that mappings of the form (8) generalize the radial stretchings (3). Indeed, when $\eta_1(\theta) = \cos\theta$ and $\eta_2(\theta) = \sin\theta$, (8) reduces to (3). We expect that mappings of the form (8) should be of interest in relation to other results on quasiconformal mapping theory and elliptic equations, as well.

Notation Henceforth, for every measurable function f, $\sup f$ denotes the essential upper bound of f. All integrals are taken with respect to the Lebesgue measure.

2 Proofs

We use some results in [6] for solutions to the elliptic divergence form equation

$$\operatorname{div}(A\nabla\cdot) = 0 \qquad \text{in } \Omega \tag{12}$$

where A is a bounded, symmetric matrix-valued function satisfying

$$\det A = 1 \qquad \text{a.e. in } \Omega. \tag{13}$$

More precisely, let

$$J(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{14}$$

The following results were established in [6].

Theorem 3 ([6]). *The following estimates hold.*

(i) Let $w \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution to (12)*, where A satisfies* (13)*. Then, w is α-Hölder continuous with*

$$\alpha \ge \left(\sup_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \langle n, An\rangle \right)^{-1},$$

where n denotes the outer unit normal.

(ii) Let A_0 be the symmetric matrix-valued function satisfying (13) *defined for every $z \neq 0$ by*

$$A_0(z) = J(\arg z) \begin{pmatrix} k_0(\arg z) & 0 \\ 0 & k_0^{-1}(\arg z) \end{pmatrix} J^*(\arg z),$$

where k_0 is the function defined in (7)*. There exists $\bar{M} > 1$ such that*

$$\sup_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \langle n, A_0 n\rangle = c = \frac{2}{1+M^{-1}}, \tag{15}$$

for every $M \in (1, \bar{M})$. Furthermore, the function $u_0 = |z|^{1/c}\Theta_1(\arg z)$ is a weak solution to (12) *with $A = A_0$.*

The reduction of Beltrami equations to elliptic divergence form equations satisfying (13) is well-known. For the reader's convenience and for uniformity of notation, we sketch it below.

Lemma 1. *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$ satisfy the Beltrami equation* (1)*. Let $A_\mu = (a_{\mu,ij})$ be the bounded, symmetric matrix-valued function satisfying* (13)*, defined in terms of the Beltrami coefficient $\mu = \mu_1 + i\mu_2$, $\mu_1(z), \mu_2(z) \in \mathbb{R}$, by*

$$a_{\mu,11} = \frac{1-2\mu_1+|\mu|^2}{1-|\mu|^2} \qquad a_{\mu,22} = \frac{1+2\mu_1+|\mu|^2}{1-|\mu|^2} \tag{16}$$

$$a_{\mu,12} = a_{\mu,21} = -\frac{2\mu_2}{1-|\mu|^2}.$$

Then $u = \text{Re}f$ *and* $v = \text{Im}f$ *are a weak solutions to the elliptic equation* (12) *with* $A = A_\mu$.

Proof. We have:

$$\overline{\partial} f = \frac{1}{2}\begin{pmatrix} u_x - v_y \\ u_y + v_x \end{pmatrix} \qquad\qquad \partial f = \frac{1}{2}\begin{pmatrix} u_x + v_y \\ -u_y + v_x \end{pmatrix}.$$

Setting

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

for every $z = a + ib = (a,b)^T$ we have

$$Qz = \begin{pmatrix} -b \\ a \end{pmatrix} = iz, \qquad\qquad Rz = \begin{pmatrix} a \\ -b \end{pmatrix} = \overline{z}.$$

Hence, we can write

$$\overline{\partial} f = \frac{1}{2}\left(\nabla u + Q\nabla v\right), \qquad\qquad \partial f = \frac{R}{2}\left(\nabla u - Q\nabla v\right).$$

Setting

$$M = \begin{pmatrix} \mu_1 & -\mu_2 \\ \mu_2 & \mu_1 \end{pmatrix},$$

equation (1) can be written in the form:

$$\nabla u + Q\nabla v = MR\left(\nabla u - Q\nabla v\right).$$

It follows that

$$\left(I - MR\right)\nabla u = -\left(I + MR\right)Q\nabla v$$

and consequently u satisfies

$$\left(I + MR\right)^{-1}\left(I - MR\right)\nabla u = -Q\nabla v$$

and v satisfies

$$-Q\left(I - MR\right)^{-1}\left(I + MR\right)Q\nabla v = Q\nabla u.$$

A straightforward computation yields

$$\left(I + MR\right)^{-1}\left(I - MR\right) = A_\mu = -Q\left(I - MR\right)^{-1}\left(I + MR\right)Q.$$

Now the conclusion follows observing that $\text{div}(Q\nabla\cdot) = 0$. □

Proof of Theorem 1. In view of Lemma 1 and Theorem 3, $\text{Re}f$ and $\text{Im}f$ are α-Hölder continuous with

$$\alpha \geq \left(\sup_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \langle n, A_\mu n\rangle\right)^{-1},$$

where A_μ is defined in (16). Setting $\xi = x + \rho e^{it}$, $t \in \mathbb{R}$ for every $\xi \in S_\rho(x) \subset \Omega$, we have $n(\xi) = e^{it}$. Hence, we compute

$$\begin{aligned}\langle n(\xi), A_\mu(\xi)n(\xi)\rangle = \langle e^{it}, A_\mu(\xi)e^{it}\rangle =& a_{\mu,11}\cos^2 t + 2a_{\mu,12}\sin t\cos t + a_{\mu,22}\sin^2 t \\ =& \frac{1+|\mu|^2}{1-|\mu|^2} - 2\frac{\mu_1\cos 2t + \mu_2 \sin 2t}{1-|\mu|^2}.\end{aligned}$$

Using the identity

$$\frac{1+|\mu|^2}{1-|\mu|^2}+2\frac{|\mu|}{1-|\mu|^2}=\frac{1+|\mu|}{1-|\mu|}=K_\mu,$$

we have on $S_\rho(x)$

$$\langle n, A_\mu n\rangle = K_\mu - 2\frac{|\mu|+\mathrm{Re}\left(\mu, n^2\right)}{1-|\mu|^2} \tag{17}$$

and we conclude the proof. □

Now we prove some properties for the special case where the Beltrami coefficient μ is of the form (9).

Lemma 2. *Suppose μ is of the form* (9)*. Then A_μ as defined in* (16) *is given by*

$$\begin{aligned} A_\mu(z) =& J\begin{pmatrix} k(\arg z) & 0 \\ 0 & k^{-1}(\arg z)\end{pmatrix} J^* \\ =& \begin{pmatrix} k\cos^2\theta + k^{-1}\sin^2\theta & \left(k-k^{-1}\right)\sin\theta\cos\theta \\ \left(k-k^{-1}\right)\sin\theta\cos\theta & k^{-1}\cos^2\theta + k\sin^2\theta \end{pmatrix}. \end{aligned}$$

Proof. Setting $\theta = \arg z$, we have

$$\mu = \frac{1-k}{1+k}e^{2i\theta}$$

and therefore,

$$\mu_1 = \frac{1-k}{1+k}\cos 2\theta \qquad\qquad \mu_2 = \frac{1-k}{1+k}\sin 2\theta.$$

We compute

$$1-|\mu|^2 = \frac{4k}{(1+k)^2} \qquad\qquad 1+|\mu|^2 = 2\frac{1+k^2}{(1+k)^2}.$$

In view of (16), we have

$$\begin{aligned} (1-|\mu|^2)a_{\mu,11} =& 1-2\mu_1+|\mu|^2 = 2\frac{1-\cos 2\theta + k^2(1+\cos 2\theta)}{(1+k)^2} \\ =& 4\frac{\sin^2\theta + k^2\cos^2\theta}{(1+k)^2} \end{aligned}$$

and consequently

$$a_{\mu,11} = k\cos^2\theta + \frac{1}{k}\sin^2\theta.$$

Similarly,

$$a_{\mu,22} = \frac{1}{k}\cos^2\theta + k\sin^2\theta.$$

Finally,

$$\begin{aligned} a_{\mu,12} =& -2\frac{\mu_2}{1-|\mu|^2} \\ =& -2\frac{1-k}{1+k}\sin 2\theta\frac{(1+k)^2}{4} = \left(k-\frac{1}{k}\right)\sin\theta\cos\theta. \end{aligned}$$

□

In what follows it will be convenient to use polar cooordinates $x = \rho\cos\theta$, $y = \rho\sin\theta$. We recall that

$$\begin{aligned}\overline{\partial} =& \frac{1}{2}(\partial_x + i\partial_y) = \frac{1}{2}\left(\frac{z}{|z|}\partial_\rho + i\frac{z}{|z|^2}\partial_\theta\right)\\ \partial =& \frac{1}{2}(\partial_x - i\partial_y) = \frac{1}{2}\left(\frac{\bar z}{|z|}\partial_\rho - i\frac{\bar z}{|z|^2}\partial_\theta\right).\end{aligned}$$

Hence, (1) is equivalent to

$$\frac{z}{|z|}f_\rho + i\frac{z}{|z|^2}f_\theta = \mu\left(\frac{\bar z}{|z|}f_\rho - i\frac{\bar z}{|z|^2}f_\theta\right),$$

from which we derive the polar form of the Beltrami equation (1):

$$(e^{i\theta} - \mu e^{-i\theta})f_\rho = -\frac{i}{\rho}(e^{i\theta} + \mu e^{-i\theta})f_\theta. \tag{18}$$

Finally, we prove Proposition 1.

Proof of Proposition 1. Proof of (10). If μ and f are as given, we have

$$\begin{aligned}e^{i\theta} - \mu e^{-i\theta} =& e^{i\theta}\left(1 - \frac{1-k}{1+k}\right) = \frac{2k}{1+k}e^{i\theta}\\ e^{i\theta} + \mu e^{-i\theta} =& e^{i\theta}\left(1 + \frac{1-k}{1+k}\right) = \frac{2}{1+k}e^{i\theta}.\end{aligned}$$

Since $f_\rho = \alpha\rho^{\alpha-1}(\eta_1 + i\eta_2)$ and $f_\theta = \rho^\alpha(\eta_1' + i\eta_2')$, we derive from (18)

$$\alpha k(\eta_1 + i\eta_2) = -i(\eta_1' + i\eta_2'),$$

which in turn implies (10).

Proof of (11). We regard f as a mapping from $(\Omega, \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2)$ to $(\mathbb{R}^2, \mathrm{d}x^2 + \mathrm{d}y^2)$. Writing $f = \rho^\alpha\eta_1\partial_1 + \rho^\alpha\eta_2\partial_2$, for every $\xi = \xi_\rho\partial_\rho + \xi_\theta\partial_\theta$ we have

$$\mathrm{d}f\xi = \rho^{\alpha-1}\left[(\alpha\eta_1\xi_\rho + \rho\eta_1'\xi_\theta)\partial_1 + (\alpha\eta_2\xi_\rho + \rho\eta_2'\xi_\theta)\partial_2\right].$$

Hence,

$$\rho^{-2(\alpha-1)}|\mathrm{d}f\xi|^2 = \alpha^2(\eta_1^2 + \eta_2^2)\xi_\rho^2 + 2\alpha\rho(\eta_1\eta_1' + \eta_2\eta_2')\xi_\rho\xi_\theta + \rho^2(\eta_1'^2 + \eta_2'^2)\xi_\theta^2.$$

In view of (10), we have $\eta_1\eta_1' + \eta_2\eta_2' = 0$ and $\eta_1'^2 + \eta_2'^2 = \alpha^2k^2(\eta_1^2 + \eta_2^2)$. Therefore,

$$|\mathrm{d}f\xi|^2 = \rho^{2(\alpha-1)}\alpha^2(\eta_1^2 + \eta_2^2)\left(\xi_\rho^2 + \rho^2k^2\xi_\theta^2\right).$$

On the other hand, $|\xi|^2 = \xi_\rho^2 + \rho^2\xi_\theta^2$. It follows that

$$\begin{aligned}|\mathrm{d}f|^2 =& \sup_{\xi\neq 0}\frac{\alpha^2\rho^{2(\alpha-1)}(\eta_1^2 + \eta_2^2)(\xi_\rho^2 + \rho^2k^2\xi_\theta^2)}{\xi_\rho^2 + \rho^2\xi_\theta^2}\\ =& \alpha^2\rho^{2(\alpha-1)}(\eta_1^2 + \eta_2^2)\sup_{t>0}\frac{1 + \rho^2k^2t}{1 + \rho^2t} = \alpha^2k^2\rho^{2(\alpha-1)}(\eta_1^2 + \eta_2^2).\end{aligned}$$

In order to compute J_f, we note that in polar coordinates on $\mathbb{R}^2$ the Hodge star operator satisfies $*1 = \rho \mathrm{d}\rho \wedge \mathrm{d}\theta$ and $** = 1$. Hence,

$$\begin{aligned} J_f = *(\mathrm{dRe}f \wedge \mathrm{dIm}f) =& \alpha\rho^{2\alpha-1}(\eta_1\eta_2' - \eta_1'\eta_2) * (\mathrm{d}\rho \wedge \mathrm{d}\theta) \\ =& \alpha\rho^{2(\alpha-1)}(\eta_1\eta_2' - \eta_1'\eta_2). \end{aligned}$$

In view of (10), we have $\eta_1\eta_2' - \eta_1'\eta_2 = \alpha k(\eta_1^2 + \eta_2^2)$. It follows that

$$J_f = \alpha^2 k \rho^{2(\alpha-1)}(\eta_1^2 + \eta_2^2)$$

and, finally, that

$$|\mathrm{d}f|^2 = kJ_f$$

for every $z \neq 0$. Now (11) is established. □

Proof of Theorem 2. By direct check, (Θ_1, Θ_2) is a solution to system (10) with $k = k_0$ and $\alpha = c^{-1}$, where c is defined in (6). Hence, in view of Proposition 1, the function $f_0 = \rho^{1/c}(\Theta_1 + i\Theta_2)$ is a solution to the Beltrami equation (1) with $\mu = \mu_0$. By Lemma 1, $\mathrm{Re}f_0$ and $\mathrm{Im}f_0$ are solutions to the elliptic equation (12), with $A = A_0$ the matrix-valued funtion defined in Theorem 3–(ii). In view of (17), we have

$$\begin{aligned} c =& \left(\sup_{S_\rho(x) \subset \Omega} \int_{S_\rho(x)} \langle n, A_0 n \rangle \right)^{-1} \\ =& \left(\sup_{S_\rho(x) \subset \Omega} \int_{S_\rho(x)} \left\{ K_{\mu_0} - 2\frac{|\mu_0| + \mathrm{Re}\left(\mu_0, n^2\right)}{1 - |\mu_0|^2} \right\} \right)^{-1} \end{aligned}$$

for every $M \in (1, \bar{M})$. □

Acknowledgments

I thank Professor Kari Astala for interesting and fruitful discussions.

References

[1] K. Astala, T. Iwaniec, P. Koskela and G. Martin, Mappings of *BMO*-bounded distortion, Math. Ann. **317** (2000), 703–726.

[2] V. Gutlyanskiĭ, O. Martio, T. Sugawa and M. Vuorinen, On the degenerate Beltrami equation, Trans. Amer. Math. Soc. **357** No. 3 (2005), 875–900.

[3] T. Iwaniec, P. Koskela, G. Martin and C. Sbordone, Mappings of finite distortion: $L^n \log^\alpha L$-integrability, J. London Math. Soc. **2** No. 67 (2003), 123–136.

[4] T. Iwaniec and G. Martin, *Geometric Function Theory and Non-linear Analysis*, Clarendon Press, Oxford Univeristy Press, New York, 2001.

[5] T. Iwaniec and C. Sbordone, Quasiharmonic fields, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** No. 5 (2001), 519–572.

[6] T. Ricciardi, On the best Hölder exponent for two dimensional elliptic equations in divergence form, preprint 2005; available on arXiv:math.AP/0510606.