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An Averaging Operator on the Dirichlet Space

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Let $\mathscr{D}$ denote the Dirichlet space, the space of all complex-valued functions f such that f is analytic throughout $D \equiv \{z : |z| < 1\}$, $f(0) = 0$, and $|f'|$ is square-integrable with respect to normalized planar Lebesgue measure in D. We formally define the integral operator A on $\mathscr{D}$ by

$$(Af)(z) = \frac{1}{z-1} \int_1^z f(s)\, ds - \int_0^1 f(s)\, ds, \qquad f \in \mathscr{D}, \quad |z| < 1.$$

The integrals both converge since

$$|f(s)| \leqslant \|f\|_{\mathscr{D}} \left[\log \frac{1}{1-|s|^2}\right]^{1/2}.$$

If $e_n(z) = z^n/n^{1/2}$, $n = 1, 2, 3, \ldots$, the set $\{e_n\}_{n=1}^{\infty}$ forms an orthonormal basis for $\mathscr{D}$. We now compute the matrix for A. Since

$$\begin{aligned}(Ae_n)(z) &= \frac{1}{z-1} \int_1^z \frac{s^n}{n^{1/2}}\, ds - \int_0^1 \frac{s^n}{n^{1/2}}\, ds \\ &= \frac{1}{z-1} \frac{1}{n^{1/2}(n+1)} [z^{n+1} - 1] - \frac{1}{n^{1/2}(n+1)} \\ &= \frac{1}{n^{1/2}(n+1)} [n^{1/2} e_n(z) + (n-1)^{1/2} e_{n-1}(z) + \cdots + e_1(z)],\end{aligned}$$

A has matrix entries

$$\alpha_{jk} = \langle Ae_k, e_j \rangle_{\mathscr{D}} = \begin{cases} 0 & \text{if } j > k \geqslant 1 \\ \dfrac{1}{k+1} \left(\dfrac{j}{k}\right)^{1/2} & \text{if } 1 \leqslant j \leqslant k. \end{cases}$$

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We see that

$$A=\begin{bmatrix} \frac{1}{2} & \frac{1}{3}\left(\frac{1}{2}\right)^{1/2} & \frac{1}{4}\left(\frac{1}{3}\right)^{1/2} & \cdots & \frac{1}{(n+1)}\left(\frac{1}{n}\right)^{1/2} & \cdots \\ 0 & \frac{1}{3} & \frac{1}{4}\left(\frac{2}{3}\right)^{1/2} & \cdots & \frac{1}{(n+1)}\left(\frac{2}{n}\right)^{1/2} & \cdots \\ 0 & 0 & \frac{1}{4} & \cdots & \frac{1}{(n+1)}\left(\frac{3}{n}\right)^{1/2} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & \frac{1}{(n+1)} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix}$$

is an upper triangular matrix.

Some information about A can be gathered by modifying the methods of [1]; in fact, it will become evident that, in many ways, A^* behaves like the discrete Cesaro operator.

Schur test. If $\alpha_{jk}\geqslant 0$ $(j, k=1, 2, 3,\ldots)$, if $p_j>0$ $(j=1, 2, 3,\ldots)$, and if β and γ are positive numbers such that

$$\sum_j \alpha_{jk}p_j\leqslant \beta p_k \qquad (k=1, 2, 3,\ldots),$$

$$\sum_k \alpha_{jk}p_k\leqslant \gamma p_j \qquad (j=1, 2, 3,\ldots),$$

then there exists an operator T (on a separable infinite-dimensional Hilbert space) with $\|T\|^2\leqslant\beta\gamma$ and matrix $\langle\alpha_{jk}\rangle$ (with respect to a suitable orthonormal basis).

The proof of the Schur test is found in [2].

THEOREM 1. *A is bounded.*

Proof. We apply the Schur test with $p_j=j^{1/2}/(j+1)$, $j=1, 2, 3,\ldots$.

$$\sum_{j=1}^{\infty}\alpha_{jk}p_j=\sum_{j=1}^{k}\frac{1}{k+1}\left(\frac{j}{k}\right)^{1/2}\frac{j^{1/2}}{j+1}$$

$$=\frac{1}{k^{1/2}(k+1)}\sum_{j=1}^{k}\frac{j}{j+1}\leqslant\frac{k}{k^{1/2}(k+1)}=p_k.$$

$$\sum_{k=1}^{\infty} a_{jk} p_k = \sum_{k=j}^{\infty} \frac{1}{k+1} \left(\frac{j}{k}\right)^{1/2} \frac{k^{1/2}}{k+1} = j^{1/2} \sum_{k=j}^{\infty} \frac{1}{(k+1)^2} \leqslant j^{1/2} \int_j^{\infty} \frac{dx}{x^2}$$
$$= \frac{1}{j^{1/2}} \leqslant 2 \frac{j}{j+1} \frac{1}{j^{1/2}} = 2p_j.$$

It follows from the Schur test that $\|A\|^2 \leqslant 2$.

THEOREM 2. *If* $|\frac{1}{2} - \lambda| < \frac{1}{2}$, *then* λ *is a simple eigenvalue of* A; A *has no other eigenvalues.*

Proof. Observe that

$$(Af)(n) = n^{1/2} \sum_{k=n}^{\infty} \frac{1}{k^{1/2}(k+1)} f(k), \qquad n = 1, 2, 3, \ldots.$$

If $Af = g$, then

$$\frac{1}{n^{1/2}} g(n) - \frac{1}{(n+1)^{1/2}} g(n+1) = \frac{1}{n^{1/2}(n+1)} f(n),$$

so $f(n) = (n+1)\, g(n) - n^{1/2}(n+1)^{1/2} g(n+1)$. If $Af = \lambda f$, then $f(n) = (n+1)\ \lambda f(n) - n^{1/2}(n+1)^{1/2}\ \lambda f(n+1)$ or $\lambda n^{1/2}(n+1)^{1/2} f(n+1) = [(n+1)\lambda - 1] f(n)$. It follows that 0 is not an eigenvalue of A (if $\lambda = 0$, then $f(n) = 0$ for all n). So f must satisfy

$$f(n+1) = \left(\frac{n+1}{n}\right)^{1/2} \left[1 - \frac{1}{\lambda(n+1)}\right] f(n).$$

and hence

$$f(n) = n^{1/2} \left[\prod_{j=2}^{n} \left(1 - \frac{1}{j\lambda}\right)\right] f(1), \qquad n \geqslant 2. \tag{1}$$

From this we conclude that any eigenvalues of A must be simple. We now want to know what nonzero values of λ will result in the convergence of $\sum_{n=1}^{\infty} |f(n)|^2$. Since the ratio test fails, we turn to Raabe's test [3, Theorem II, p. 396]. We find that

$$\frac{|f(n)|^2}{|f(n+1)|^2} - 1 = \frac{-(n+1)|\lambda|^2 + (n+1)(\lambda + \bar{\lambda}) - 1}{(n+1)^2 |\lambda|^2 - (\lambda + \bar{\lambda})(n+1) + 1}$$

and hence

$$\lim_{n \to \infty} n \left[\frac{|f(n)|^2}{|f(n+1)|^2} - 1\right] = \frac{-|\lambda|^2 + (\lambda + \bar{\lambda})}{|\lambda|^2} = \frac{\lambda + \bar{\lambda}}{|\lambda|^2} - 1.$$

By Raabe's test, the series converges for $(\lambda+\bar{\lambda})/|\lambda|^2-1>1$ and diverges for $(\lambda+\bar{\lambda})/|\lambda|^2-1<1$; equivalently, the series converges for $|\lambda-\frac{1}{2}|<\frac{1}{2}$ and diverges for $|\lambda-\frac{1}{2}|>\frac{1}{2}$. Raabe's test fails for $|\lambda-\frac{1}{2}|=\frac{1}{2}$; preparing to use a more refined test for this case, we compute

$$n\left[\frac{|f(n)|^2}{|f(n+1)|^2}-1\right]-1$$
$$=\frac{\left[\begin{array}{c}[-2|\lambda|^2+(\lambda+\bar{\lambda})]\,n^2+[-3|\lambda|^2+2(\lambda+\bar{\lambda})-1]\,n\\ +[-|\lambda|^2+(\lambda+\bar{\lambda})-1]\end{array}\right]}{|\lambda|^2n^2+[2|\lambda|^2-(\lambda+\bar{\lambda})]\,n+|\lambda|^2-(\lambda+\bar{\lambda})+1};$$

since $|\lambda-\frac{1}{2}|=\frac{1}{2}$, we have $\lambda+\bar{\lambda}-2|\lambda|^2=0$ and hence

$$\lim_{n\to\infty}\ \ln n\left\{n\left[\frac{|f(n)|^2}{|f(n+1)|^2}-1\right]-1\right\}$$
$$=\lim_{n\to\infty}\frac{\ln n}{n}\left\{\frac{[-3|\lambda|^2+2(\lambda+\bar{\lambda})-1]\,n+[-|\lambda|^2+(\lambda+\bar{\lambda})-1]}{|\lambda|^2n+[|\lambda|^2-(\lambda+\bar{\lambda})+1](1/n)}\right\}$$
$$=0\cdot\frac{-3|\lambda|^2+2(\lambda+\bar{\lambda})-1}{|\lambda|^2}=0.$$

From [3, Theorem III, p. 396] we conclude that the series $\sum_{n=1}^{\infty}|f(n)|^2$ diverges if $|\lambda-\frac{1}{2}|=\frac{1}{2}$.

We remark that if f is an eigenvector of A with $f(1)=1$, then from (1) we obtain

$$f(z)=z+\sum_{n=2}^{\infty}\left[\prod_{j=2}^{n}\left(1-\frac{1}{j\lambda}\right)\right]z^n=\frac{\lambda}{\lambda-1}\left[(1-z)^{1/\lambda-1}-1\right].$$

Since the eigenvector f corresponding to the eigenvalue $\lambda=1/m$, $m\geqslant 2$, is a polynomial of degree $m-1$, it is easy to see that the eigenvectors of A span $\mathscr{S}$.

THEOREM 3. *The point spectrum of A^* is empty.*

Proof. If $A^*f=g$, then $g(n)=(1/(n+1)\,n^{1/2})\sum_{k=1}^{n}k^{1/2}f(k)$; consequently, $(n+1)\,n^{1/2}\,g(n)-n(n-1)^{1/2}\,g(n-1)=n^{1/2}f(n)$, or $f(n)=(n+1)\,g(n)-n^{1/2}(n-1)^{1/2}\,g(n-1)$ if $n\geqslant 2$. So if $A^*f=\lambda f$ with f nonzero, then $f(n)=\lambda(n+1)f(n)-\lambda n^{1/2}(n-1)^{1/2}f(n-1)$, or

$$[\lambda(n+1)-1]\,f(n)=\lambda n^{1/2}(n-1)^{1/2}f(n-1),\qquad n\geqslant 2. \tag{2}$$

If m is the least positive integer for which $f(m) \neq 0$, then $\lambda = 1/(m+1)$, so $0 < \lambda \leqslant \frac{1}{2}$. It follows from (2) that if $k \geqslant 1$, then

$$\begin{aligned} |f(m+k)| &= (1/k)(m+k)^{1/2}\,(m+k-1)^{1/2}\,|f(m+k-1)| \\ &\geqslant |f(m+k-1)|, \end{aligned}$$

which makes it impossible for $\sum_{n=1}^{\infty} |f(n)|^2$ to converge for a nonzero f.

THEOREM 4. *$\|\frac{1}{2} - A\| = \frac{1}{2}$, $\|A\| = 1$, and the spectrum of A is the closed disk $\{\lambda : |\frac{1}{2} - \lambda| \leqslant \frac{1}{2}\}$.*

Proof. It is routine to compute that

$$A^*A = \begin{bmatrix} \frac{1}{4} & \frac{1}{6\cdot 2^{1/2}} & \frac{1}{8\cdot 3^{1/2}} & \frac{1}{10\cdot 4^{1/2}} & \cdots & \frac{1}{2(n+1)n^{1/2}} & \cdots \\ \frac{1}{6\cdot 2^{1/2}} & \frac{1}{6} & \frac{3}{12\cdot 6^{1/2}} & \frac{3}{15\cdot 8^{1/2}} & \cdots & \frac{3}{3(n+1)(2n)^{1/2}} & \cdots \\ \frac{1}{8\cdot 3^{1/2}} & \frac{3}{12\cdot 6^{1/2}} & \frac{1}{8} & \frac{6}{20\cdot 12^{1/2}} & \cdots & \frac{6}{4(n+1)(3n)^{1/2}} & \cdots \\ \frac{1}{10\cdot 4^{1/2}} & \frac{3}{15\cdot 8^{1/2}} & \frac{6}{20\cdot 12^{1/2}} & \frac{1}{10} & \cdots & \frac{10}{5(n+1)(4n)^{1/2}} & \cdots \\ \vdots & \vdots & \vdots & & & \vdots & \\ \frac{1}{2(n+1)n^{1/2}} & \frac{3}{3(n+1)(2n)^{1/2}} & \frac{6}{4(n+1)(3n)^{1/2}} & \frac{10}{5(n+1)(4n)^{1/2}} & \cdots & \frac{(n/2)(n+1)}{(n+1)^2 n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \ddots \end{bmatrix}.$$

It turns out that $\frac{1}{2}(A^* + A) - A^*A$ is a diagonal matrix with diagonal $\{1/2(n+1)\}_{n=1}^{\infty}$. Therefore $(\frac{1}{2} - A^*)(\frac{1}{2} - A) = \frac{1}{4} - \{\frac{1}{2}(A^* + A) - A^*A\}$ is a diagonal matrix with diagonal $\{(n-1)/4(n+1)\}_{n=1}^{\infty}$. It follows that $\|\frac{1}{2} - A\|^2 = \frac{1}{4}$.

Since $\|\frac{1}{2} - A\| = \frac{1}{2}$, the spectrum of $\frac{1}{2} - A$ is contained in the closed disk$\{\lambda : |\lambda| \leqslant \frac{1}{2}\}$; consequently, the spectrum of A is contained in the closed disk$\{\lambda : |\frac{1}{2} - \lambda| \leqslant \frac{1}{2}\}$. From Theorem 2 we know that the spectrum of A includes the open disk$\{\lambda : |\frac{1}{2} - \lambda| < \frac{1}{2}\}$. Because the spectrum of an operator is a closed set, we are forced to conclude that the spectrum of A is the closed disk$\{\lambda : |\frac{1}{2} - \lambda| \leqslant \frac{1}{2}\}$.

Since the spectral radius of A is 1, we know that $\|A\| \geqslant 1$; from the fact $\|\frac{1}{2} - A\| = \frac{1}{2}$ we conclude that $\|A\| \leqslant 1$. Therefore $\|A\| = 1$.

Let $W(A)$ denote the set of all complex numbers of the form $\langle Af, f\rangle$, where $\|f\| = 1$; $W(A)$ is called the *numerical range* of A.

LEMMA. *If B is an operator and λ is a complex number such that $|\lambda| = \|B\|$ and $\lambda \in W(B)$, then λ is an eigenvalue of B.*

The proof of this lemma appears in [2, p. 319].

THEOREM 5. *The numerical range of A is the open disk $\{\lambda : |\frac{1}{2} - \lambda| < \frac{1}{2}\}$.*

Proof. If λ is an eigenvalue of A, then it is clear that $\lambda \in W(A)$; hence $\{\lambda : |\frac{1}{2} - \lambda| < \frac{1}{2}\} \subseteq W(A)$ by Theorem 2. Assume $|\lambda - \frac{1}{2}| = \frac{1}{2}$; then $|\lambda - \frac{1}{2}| = \|A - \frac{1}{2}\|$ (by Theorem 4) but $\lambda - \frac{1}{2}$ is not an eigenvalue of $A - \frac{1}{2}$ (by Theorem 2). It follows from the lemma that $\lambda - \frac{1}{2} \notin W(A - \frac{1}{2})$ and hence $\lambda \notin W(A)$ when $|\lambda - \frac{1}{2}| = \frac{1}{2}$. If $|\lambda - \frac{1}{2}| > \frac{1}{2}$, then $\lambda \notin W(A)$ since $W(A)$ is a convex set [2, p. 110].

THEOREM 6. *A^* is hyponormal.*

Proof. The matrix AA^* has the form

$$\begin{bmatrix} \alpha_1 & 2^{1/2}\alpha_2 & 3^{1/2}\alpha_3 & \cdots & n^{1/2}\alpha_n & \cdots \\ 2^{1/2}\alpha_2 & 2\alpha_2 & 6^{1/2}\alpha_3 & \cdots & (2n)^{1/2}\alpha_n & \cdots \\ 3^{1/2}\alpha_3 & 6^{1/2}\alpha_3 & 3\alpha_3 & \cdots & (3n)^{1/2}\alpha_n & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ n^{1/2}\alpha_n & (2n)^{1/2}\alpha_n & (3n)^{1/2}\alpha_n & \cdots & n\alpha_n & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{bmatrix}$$

where $\alpha_L = \sum_{k=L}^{\infty} (1/k(k+1)^2)$; the matrix A^*A is displayed in the proof of Theorem 4. The infinite matrix $AA^* - A^*A$ is positive if and only if all of its finite sections S_n have positive determinants; the problem of demonstrating the hyponormality of A^* is therefore reduced to the evaluation of the determinant of S_n, where

$$S_n = \begin{bmatrix} \beta_1 & 2^{1/2}\beta_2 & 3^{1/2}\beta_3 & \cdots & n^{1/2}\beta_n \\ 2^{1/2}\beta_2 & 2\beta_2 & 6^{1/2}\beta_3 & \cdots & (2n)^{1/2}\beta_n \\ 3^{1/2}\beta_3 & 6^{1/2}\beta_3 & 3\beta_3 & \cdots & (3n)^{1/2}\beta_n \\ \vdots & \vdots & \vdots & & \vdots \\ n^{1/2}\beta_n & (2n)^{1/2}\beta_n & (3n)^{1/2}\beta_n & \cdots & n\beta_n \end{bmatrix}$$

with $\beta_L = \alpha_L - 1/2(L+1)L$, $1 \leqslant L \leqslant n$. We multiply the second column by $(\frac{1}{2})^{1/2}$ and subtract from the first, then multiply the third column by $(\frac{2}{3})^{1/2}$ and subtract from the second, and in general multiply column k, $2 \leqslant k \leqslant n$, by $((k-1)/k)^{1/2}$ and subtract from column $k-1$. The resulting matrix has the same determinant as S_n and is upper triangular; its determinant is the product of its diagonal elements $k(\beta_k - \beta_{k+1})$, $1 \leqslant k \leqslant n-1$, and $n\beta_n$. To complete the proof it suffices to show that $\{\beta_k\}_{k=1}^{\infty}$ is positive and decreasing. It is routine to verify that

$$\beta_k - \beta_{k+1} = \frac{1}{k(k+1)^2\,(k+2)}$$

and hence the sequence $\{\beta_k\}_{k=1}^{\infty}$ is (strictly) decreasing. If $k > 1$, then

$$\beta_k = \sum_{j=k}^{\infty} \frac{1}{j(j+1)^2} - \frac{1}{2k(k+1)} \geqslant \int_{k+1}^{\infty} \frac{dx}{x^3} - \frac{1}{2k(k+1)} \geqslant - \frac{1}{2k(k+1)^2}$$

and

$$\beta_k \leqslant \int_{k-1}^{\infty} \frac{dx}{x^3} - \frac{1}{2k(k+1)} \leqslant \frac{1}{2}\left[\frac{3k-1}{(k-1)^2\,k(k+1)}\right],$$

so it is clear that $\{\beta_k\}$ converges to 0. The sequence $\{\beta_k\}$ must be positive since it is a decreasing sequence which converges to 0.

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References

1. A. Brown, P. R. Halmos, and A. L. Shields, Cesaro operators, *Acta. Sci. Math. (Szeged)* **26** (1965), 125–137.
2. P. R. Halmos, "A Hilbert Space Problem Book," Van Nostrand, Princeton, N. J. 1967.
3. J. M. H. Olmsted, "Advanced Calculus," Appleton–Century–Crofts, New York, 1961.