M. Detyniecki, R. R. Yager, and B. Bouchon-Meunier, "Reducing t-norms and augmenting t-conorms," *International Journal of General Systems*, vol. 31 (3), pp. 265-276, 2002.

Reducing t-norms and augmenting t-conorms

Marcin Detyniecki

LIP6 - CNRS -University of Paris VI
4, place Jussieu
75230 Paris Cedex 05, France
Marcin.Detyniecki@lip6.fr

Ronald R. Yager

Machine Intelligence Institute
Iona College
New Rochelle, New York 10801, USA
Yager@panix.com

Bernadette Bouchon-Meunier

LIP6 - CNRS -University of Paris VI
4, place Jussieu
75230 Paris Cedex 05, France
Bernadette.Bouchon-Meunier@lip6.fr

Abstract: We study here how to reduce a t-norm by subtracting a value (computed by a function). We obtain a general form of an operator that we call the reduced t-norm. We study this operator and some interesting generalization properties. We also investigate some interesting particular cases. We complete the study by analyzing the dual operator, which corresponds to augmenting a t-conorms.

1. Introduction

The concept of a triangular norm was introduced by Menger [1] in order to generalize the triangular inequality of a metric. The current notion of a t-norm and its dual operation (t-conorm) is due to Schweizer and Sklar [2,15]. Both of these operations can also be used as a generalization of the Boolean logic connectives to multi-valued logic. The t-norms generalize the conjunctive 'AND' operator and the t-conorms generalize the disjunctive 'OR' operator. This situation allows them to be used to define the intersection and union operation in fuzzy logic. This possibility was first noted by Hohle [3]. Klement [4], Dubois and Prade [5] and Alsina, Trillas, and Valverde [6] very early appreciated the possibilities of this generalization. Bonissone [7] investigated the properties of these operators with the goal of using them in the development of intelligent systems. T-norms and t-conorms have been well-studied and very good overviews and classifications of these operators can be found in the literature, see [8,9,16].

In this paper we start our work by noticing that one well know property of the t-norms is that the minimum operator is the biggest possible t-norm. Taking into account this fact we propose to reduce this operator in order to obtain more drastic t-norms. We study how to do it and we finish by obtaining a general form for what we call the reduced t-norms. Then, we show interesting properties for this kind of operators and we investigate some interesting particular cases, consisting in reducing typical t-norms. After that, we take a look at the dual operator of the reduced t-norm. We obtain an operator, which can be interpreted as an augmented t-conorm. We analyze also this operator.

2. Reduced T-norms and t-conorms

We begin by defining the t-norms and t-conorms:
T-norms: A t-norm is a function T: [0,1]x[0,1] → [0,1], having the following properties

- $T(a,b) = T(b,a)$ (T1) **Commutivity**
- $T(a,b) \le T(c,d)$, if $a \le c$ and $b \le d$ (T2) **Monotonicity (increasing)**
- $T(a,T(b,c)) = T(T(a,b),c)$ (T3) **Associativity**
- $T(a,1) = a$ (T4) **One as identity**

A well know property of t-norms is:

- $T(a,b) \le \text{Min}\ (a,b)$ (1)

It is actually consequence of the axioms (T1, T2, T4).

T-conorm: Formally, a t-conorm is a function S: [0,1]x[0,1] → [0,1], having the following properties:

- $S(a,b) = S(b,a)$ (S1) **Commutivity**
- $S(a,b) \le S(c,d)$, if $a \le c$ and $b \le d$ (S2) **Monotonicity (increasing)**
- $S(a,S(b,c)) = S(S(a,b),c)$ (S3) **Associativity**
- $S(a,0) = a$ (S4) **Zero as identity**

A well known property of t-conorms is:

- $S(a,b) \ge \text{Max}\ (a,b)$ (2)

It is actually consequence of the axioms (S1, S2, S4).

Property: T-norms and t-conorms are related by the following relation:
We say that a t-norm and a t-conorm are associated (or dual) if they satisfy the De Morgan's law.

- $\overline{T(a,b)} = S(\overline{a},\overline{b})$ (De Morgan's law)

where the line over an expression means the negation of the expression. We will use the most typical negation defined by:

- $\overline{a} = 1 - a$ (Negation)

Let us now focus our attention on the t-norms. We indicated that the largest t-norm is the min operator. Let us consider reducing the result obtained from this t-norm operator by subtracting a value from it, inhere the value is computed as $f(a,b)$. This gives us:

$$\min(a,b) - f(a,b) \qquad (3)$$

In order to insure that the resulting operator doesn't give any negative result, we introduce with the help of the max operator an inferior limit:

$$G(a,b) = \max[\min(a,b) - f(a,b),\ 0] \qquad (4)$$

In order to obtain an interesting operator, we consider a function f that has the properties:

- $f(a,b) = f(b,a)$ (F1) **Commutivity**
- $f(a,b) \geq f(c,d)$, if $a \leq c$ and $b \leq d$ (F2) **Monotonicity (decreasing)**
- $f(a,1) = 0$ (F3) **One is a null factor**

We have then that G is commutative, increasingly monotonous way and 1 is a neutral element. We notice that G is almost a t-norm, but this operator is not necessarily associative.

The function f can be expressed in terms of the negation of a t-conorm, a generalization of the classical *'nor'* function (also called the Pierce function):

$$f(a,b) = \beta \cdot \overline{S(a,b)} \tag{5}$$

where $\beta \geq 0$ is a parameter and S is a t-conorm. Using (5), expression (4) becomes:

$$G(a,b) = \max\left[\min(a,b) - \beta \cdot \overline{S(a,b)},\ 0\right] \tag{6}$$

We can generalize this result to other t-norms than the min operator. In that case we will have the general expression:

$$T_{reduced}(a,b) = \max\left[T(a,b) - \beta \cdot \overline{S(a,b)},\ 0\right] \tag{7}$$

We observe that $T_{reduced}$ is almost a t-norm, but this operator is not necessarily associative. However this operator is an interesting aggregation operator. In [14] Mesiar and Komorníková define aggregation operators as a function $Agreg : [0,1]^n \rightarrow [0,1]$ that verifies:

- $Agreg\,(x) = x$ **Identity when unary**
- $Agreg\,(0,\ldots,0) = 0$ and $Agreg\,(1,\ldots,1) = 1$ **Boundary conditions**
- $Agreg\,(x_1,\ldots, x_n) \leq Agreg\,(y_1,\ldots, y_n)$, if $(x_1,\ldots, x_n) \leq (y_1,\ldots, y_n)$ **Non decreasing**

Taking into account this definition the $T_{reduced}$ is an *aggregation operator*.
In fact, we can define $T_{reduced}\,(x) = x$. For the boundary conditions, we have that $T\,(0,0) = 0$ and $S\,(0,0) = 0$ and consequently $T_{reduced}\,(0,0) = 0$. In the same way $T\,(1,1) = 1$ and $S\,(1,1) = 1$ and consequently $T_{reduced}\,(1,1) = 1$. The reduced t-norm is non-decreasing because:

$$T(a,b) - \beta \cdot \overline{S(a,b)} = T(a,b) + \beta \cdot S(a,b) - \beta \tag{8}$$

The t-norm is not decreasing, the t-conorm is not decreasing and the addition or multiplication by a positive constant does not change the monotonicity. Neither does the truncation with the max.

Other interesting basic properties of this aggregation operator is the *commutativity* and having 1 is *a neutral element*.

In fact, the reduced t-norm is commutative, because the t-norm and the t-conorm are commutative. And, 1 is a neutral element, because $T(a,1) = a$. Using the De Morgan's law and using property (2), we have $1 = \max(1,a) \leq S(1,a) \leq 1$ and so $\overline{S(1,a)} = 0$. Consequently :

$$T_{reduced}(1,a) = \max\left[a - \beta \cdot 0,\ 0\right] = a \tag{9}$$

To resume we have that the reduced t-norm is a commutative aggregation operator with neutral element 1. If look more in detail this operator we will discover other interesting properties:

Property 1: The reduced t-norm will be *continuous*, if the underlying t-norm and t-conorm are continuos.
In fact, if the t-norm and the t-conorm are continuous, their linear combination is still continuos. And the truncation by the max gives also a continuous operator.

Property 2: The operator defined in (7) can be written also using only t-norms: Using the De Morgan's laws ($S(a,b)=\overline{T(\bar{a},\bar{b})}$), we can write the expression (7) under the form:

$$T_{reduced}(a,b) = \max\left[T_1(a,b) - \beta \cdot T_2(\bar{a},\bar{b}),\ 0\right] \tag{10}$$

We observe that it is not compulsory to have twice the same t-norm. We can have $T_1 \neq T_2$. This remark applies also to the definition (7) : there are no special constraints on the choice of *S* and *T*.

Property 3: We can consider now that β is a parameter which we vary in order to obtain different aggregation operators. We will have then:

- **For $\beta = 0$**, we obtain for every t-norm T : $\boldsymbol{T_{reduced} = T(a,b)}$.
- **For $\beta \to +\infty$**, we obtain for every t-norm T reduced by t-conorm S without zero divisors (i.e. that if a≠0 and b≠0 then $S(a,b)\neq 1$), that: $\boldsymbol{T_{reduced} \to T_{drastic}}$, the drastic t-norm being defined by:

$$T_{drastic}(a,b) = \begin{cases} a & \text{if } b=1 \\ b & \text{if } a=1 \\ 0 & \text{anywhere else} \end{cases} \tag{11}$$

  In fact, for $a \neq 1$ and $b \neq 1$, we have for a t-norm without zero divisors, that $T(\bar{a},\bar{b}) > 0$. So for β large enough we will have $T(a,b) - \beta \cdot T(\bar{a},\bar{b}) < 0$, and in this way $\max\left[T(a,b) - \beta \cdot T(\bar{a},\bar{b}),\ 0\right] = 0$. For a = 1, we have that $T(\bar{a},\bar{b}) = 0$ and $T(a,b) = b \geq 0$. In this way we obtain $\max\left[T(a,b) - \beta \cdot T(\bar{a},\bar{b}),\ 0\right] = b$. The case b = 1 can be shown with the commutivity property.

- **For $\beta = 1$**, we obtain that for every t-norm and t-conorm pair from the Frank t-norm family [10], $\boldsymbol{T_{reduced} = T_{Lukasiewicz}}$. The Lukasiewicz t-norm is defined by:

$$Luka\,(a,b) = \max(a+b-1,0) \tag{12}$$

  In fact,

$$T(a,b) - T(\bar{a},\bar{b}) \ = T(a,b) - \overline{S(a,b)} \qquad \text{(using De Morgan's law)}$$

$= T(a,b) - (1 - S(a,b))$ (using the definition of negation)

$= T(a,b) + S(a,b) - 1$

The t-norms of the Frank family verify $T(a,b) + S(a,b) = a + b$. So we obtain then that the reduced t-norms $T_{reduced} = \max(a + b - 1, 0)$, which is the Lukasiewicz t-norm.
Note: the Frank family is a very large class of t-norms including the most popular t-norms as the product, the minimum and the Lukasiewicz. The precedent result is also true for ordinal sum of frank t-norms and their dual copulas.

3. Reducing some typical t-norms

Now that we showed some general properties, let us take a look at some particular cases. We will study in this section what happens when we reduce some typical t-norms:

Minimum: As we said, at the beginning of this paper, the biggest t-norm is the minimum. We can reduce this t-norm. We choose here to use the dual couple min, max. Following definition (10), we only need to use the min and we obtain:

$$T_{\min}(a,b) = \max[\min(a,b) - \beta \cdot \min((1-a),(1-b)),\ 0] \qquad (13)$$

Another form for this same operator is (see definition (7)):

$$T_{\min}(a,b) = \max[\min(a,b) - \beta \cdot (1 - \max(a,b)),\ 0] \qquad (14)$$

We observe that we obtain the expected properties:
- for $\beta = 0$, we obtain the minimum.
- for $\beta = 1$, we obtain the Lukasiewicz t-norm.
- for $\beta \to +\infty$, we obtain the drastic t-norm.

An interesting point is that this operator has notable particular cases the same ones as the well-known Yager t-norm [11]. We observe that the computational complexity of the "reduced min" is lower than for the Yager t-norm, but we also remark that the "reduced min" is not associative.

Product: Another very famous t-norm is the product. Reducing the product t-norm by the dual t-conorm and using (10), we obtain the following definition:

$$T_{product}(a,b) = \max[a \cdot b - \beta \cdot ((1-a) \cdot (1-b)),\ 0] \qquad (15)$$

We remark here that the reduced product is associative and in this way it is a t-norm.
We observe that we obtain the expected properties:
- for $\beta = 0$, we obtain the product.
- for $\beta = 1$, we obtain the Lukasiewicz t-norm.
- for $\beta \to +\infty$, we obtain the drastic t-norm.

Here an interesting point is that this operator generalizes the product and the drastic t-norm as the famous Hamacher t-norm [12] does. We also notice that the reduced product generalizes also the Lukasiewicz t-norm, which is not the case for the Hamacher t-norm.

Let us now, compare the reduced product t-norm with the Weber t-norms [13], defined for $\lambda > -1$ by:

$$\max\left(\frac{u+v-1+\lambda\cdot u\cdot v}{1+\lambda},\ 0\right) \tag{16}$$

We observe that both of these operators generalize the same typical operators. And if we take a look closer we will discover that in fact the reduced product is the Weber t-norm for the parameter $\beta = \frac{1}{1+\lambda}$. We notice that in this case the operator is associative and therefore a t-norm.

Lukasiewicz t-norm: The Lukasiewicz t-norm seems to be central because of the theorem affirming that for every t-norm from the Frank family [10], for $\beta = 1$ we obtain the Lukasiewicz t-norm. What happen if we try now to reduce the Lukasiewicz t-norm by the corresponding t-conorm? Using the definition (10), we obtain:

$$T_{Luka} = \max\left[\max(a+b-1,0) - \beta\cdot\max((1-a)+(1-b)-1),\ 0\right] \tag{17}$$

We observe that if $(a + b - 1) \geq 0$, then $((1-a)+(1-b)-1) = (1 - a - b) \leq 0$. So, we can reduce the expression to:

$$T_{Luka} = \max(a+b-1,0) \tag{18}$$

which is exactly the Lukasiewicz t-norm. In other words we cannot reduce the Lukasiewicz t-norm using the associated t-conorm.
Note: We can reduce the Lukasiewicz t-norm by using another t-norm that is not the associated Lukasiewicz t-conorm.

Drastic t-norm: Another interesting case is the drastic t-norm. We know that the drastic t-norm is the smallest t-norm. We have for any t-norm T the following property:

- $T(a,b) \geq T_{drastic}(a,b)$ (19)

So, what happens if we try to reduce the drastic t-norm? Let us try to reduce the drastic t-norm with any t-conorm (not only the associated). Using the definition (7), we obtain:

$$\max\left[T_{drastic}(a,b) - \beta\cdot\overline{S(a,b)},\ 0\right] \tag{20}$$

We observe that for $a \neq 1$ and $b \neq 1$, $T_{drastic}(a,b) = 0$ and so $T_{drastic}(a,b) - \beta\cdot\overline{S(a,b)} \leq 0$, so the reduced drastic t-norm equal 0. For $a = 1$, $T_{drastic}(a,b) = b$ and for any t-conorm we have $\overline{S(a,b)} = 0$. In other words the operator is a drastic t-norm. We can conclude by saying that we cannot reduce the drastic t-norm.

4. The augmented T-conorms

Now that we studied the reduced t-norms, let us take a look at its dual operator. We start with the expression (7) and by using the De Morgan's law, we obtain:

$$F(a,b) = \overline{T_{reduced}\left(\bar{a},\bar{b}\right)}$$

$$= \overline{\max\left[T\left(\bar{a},\bar{b}\right) - \beta \cdot \overline{S\left(\bar{a},\bar{b}\right)},\ 0\right]}$$

$$= \min\left[\overline{T\left(\bar{a},\bar{b}\right) - \beta \cdot T(a,b)}, 1\right]$$

$$= \min\left[1 - T\left(\bar{a},\bar{b}\right) + \beta \cdot T(a,b),\ 1\right]$$

$$= \min\left[S(a,b) + \beta \cdot T(a,b),\ 1\right]$$

The obtained operator can be understood as being a t-conorm to which we add β times a t-norm, the result of this addition being limited to 1. For this reason we will call this operator the augmented t-conorm:

$$S_{augmented}(a,b) = \min\left[S(a,b) + \beta \cdot T(a,b), 1\right] \qquad (21)$$

We remark that this operator is dual of the reduced t-norms by construction. So, it is a commutative aggregation operator and it has zero as neutral element. Once more we do not have necessarily the associativity. If the dual reduced t-norm is associative then the augmented t-conorm is also associative.
Besides these basic properties we have:

Property 1: The augmented t-norm will be *continuous*, if the underlying t-norm and t-conorm are continuos.
In fact, if the t-norm and the t-conorm are continuous, their linear combination is still continuous. And the truncation by the min gives also a continuous operator.

Property 2: The augmented t-conorm can be written using only t-conorms:

$$S_{augmented}(a,b) = \min\left[S(a,b) + \beta \cdot \left(1 - S\left(\bar{a},\bar{b}\right)\right), 1\right] \qquad (22)$$

We used the De Morgan's law to obtain this result.

Property 3: The augmented t-conorm has also the generalization properties:

- **For $\beta = 0$**, we obtain for every t-conorm S and any t-norm T : $\boldsymbol{S_{augmented} = S(a,b)}$.
- **For $\beta \to +\infty$**, we obtain for every t-conorm S augmented by t-norm T without zero divisors (i.e. that if $a \neq 0$ and $b \neq 0$ then $T(a,b) \neq 0$), that: $\boldsymbol{S_{augmented} \to S_{drastic}}$. The drastic t-norm being defined by:

$$S_{drastic}(a,b) = \begin{cases} a \text{ if } b = 0 \\ b \text{ if } a = 0 \\ 1 \text{ anywhere else} \end{cases} \quad (23)$$

- **For $\boldsymbol{\beta}$ = 1**, we obtain for every t-norm and t-conorm pair from the Frank t-norms family [10], we obtain that $S_{augmented} = S_{Lukasiewicz}$. The Lukasiewicz t-conorm is defined by:

$$\min(u+v,1) \quad (24)$$

Property 4: The augmented t-conorm and the reduced t-norms are dual by construction (see the construction of the augmented t-conorm at the beginning of the section) :

$$S_{augmented}(a,b) = 1 - T_{reduced}(\bar{a},\bar{b}) \quad (25)$$

5. Augmenting typical t-conorms

Let us now analyze what happens when we augment typical t-conorms:

Maximum: Using the maximum and augmenting it with the associated t-norm (i.e. minimum), we obtain the following operator (using definition (21)):

$$S_{\max}(a,b) = \min[\max(a,b) + \beta \cdot \min(a,b), 1] \quad (26)$$

We notice that this operator is the dual operator of the reduced minimum. We remark that this operator is not associative, but we observe that it has the same particular cases as the Yager t-conorm [11] and is computational lighter:

- for $\beta = 0$, we obtain the maximum
- for $\beta = 1$, we obtain the Lukasiewicz t-conorm.
- for $\beta \to +\infty$, we obtain the drastic t-conorm.

Product: Using the product t-conorm and augmenting it with the associated t-norm, we obtain the following operator (using definition (21)):

$$\min[a + b + (\beta - 1) \cdot ab, 1] \quad (27)$$

We notice that this operator is the dual operator of the reduced product t-norm. We remark that it is associative and in this way it is a t-conorm. We also observe that this operator generalizes the following t-conorms:

- for $\beta = 0$, we obtain the product t-conorm
- for $\beta = 1$, we obtain the Lukasiewicz t-conorm.
- for $\beta \to +\infty$, we obtain the drastic t-conorm.

We observe that this operator generalizes the product and drastic t-conorm as the Hamacher t-conorm [12] does.

We also notice that the augmented product t-conorm generalizes the same typical t-conorms as the Weber t-conorm [13]. In fact the expression (27) is another form of the Weber t-conorm (where$\beta = \lambda+1$):

$$\min[a+b+\lambda \cdot ab, 1] \qquad (28)$$

These operator is associative and therefore is at-conorm.

Lukasiewicz t-conorm: Like in the dual case, it is impossible to augment the Lukasiewicz t-conorm, using the Lukasiewicz t-norm. This does not mean that it is impossible to augment the Lukasiewicz t-conorm, but we will need to use another t-norm than the Lukasiewicz one (i.e. something like the product or the minimum.

Drastic t-norm: Like in the dual case it is possible to show that the drastic t-conorm can not be augmented. Which is an interesting result since it is the largest t-conorm, but the augmented t-conorm is not necessarily a t-conorm.

6. Mixed reductions and augmentations

Here we are interested in doing reduction and augmentations with pairs of t-norms and t-conorms that are not dual. In other words the reduction of a t-norm will be done using another t-conorm than the dual one. And we will augment the t-conorm with another t-norm than the dual one.
This kind of manipulation is particularly interesting in the case of the Lukasiewicz pair. We saw that it is impossible to reduce the Lukasiewicz t-norm by the Lukasiewicz t-conorm or to augment the Lukasiewicz t-conorm by the Lukasiewicz t-norm. Here we give two examples of reducing the Lukasiewicz t-norm and two of augmenting the Lukasiewicz t-norm.
We can reduce the Lukasiewicz t-norm using the minimum (29) or using the product (30):

$$T_{Luka-\min}(a,b) = \max[\max(a+b-1,0) - \beta \cdot (1-\max(a,b)), 0] \qquad (29)$$

$$T_{Luka-prod}(a,b) = \max[\max(a+b-1,0) - \beta \cdot (1-a) \cdot (1-b), 0] \qquad (30)$$

We observe that both of these operators have the following properties:

- for $\beta = 0$, we obtain the Lukasiewicz t-norm.
- for $\beta \to +\infty$, we obtain the drastic t-norm. (since the max and the product have no zero divisors).

In other words as expected we can reduce the Lukasiewicz t-norm by changing β . And we see that we can go from the Lukasiewicz t-norm until the drastic t-norm.

We have two dual cases that correspond to the augmented Lukasiewicz t-conorm: the first one by the minimum (31) and the second one by the product (32).

$$S_{Luka-\min}(a,b) = \min[\min(a+b,1) + \beta \cdot \min(a,b), 1] \qquad (31)$$

$$S_{Luka-prod}(a,b) = \min[\min(a+b,1) + \beta \cdot a \cdot b,\ 1] \tag{32}$$

We have still for the limit cases :

- for $\beta = 0$, we obtain the Lukasiewicz t-conorm.
- for $\beta \to +\infty$, we obtain the drastic t-conorm (because the min and the product have no zero divisors).

In other words as expected, we can augment the Lukasiewicz t-conorm by augmenting β until we obtain the drastic t-conorm.

In the precedent examples we have two associative operators and two non-associative ones. It seems that knowing in advance which operator is going to be associative or not is not an easy task. To know more about this kind of problem there is an interesting paper of Ling [17].

We have that:

- The Lukasiewicz t-norm reduced by the minimum is *not* associative. Therefore its dual operator the Lukasiewicz t-conorm augmented by the maximum is not associative.

In fact, let us use an counterexample for $\beta = 1$:

$$T_{Luka-\min}(T_{Luka-\min}(0.75,0.8),0.85) = T_{Luka-\min}(0.35,0.85) = 0.05$$

$$and\ \ T_{Luka-\min}(0.75, T_{Luka-\min}(0.8,0.85)) = T_{Luka-\min}(0.75,0.5) = 0$$

- The Lukasiewicz t-conorm augmented by the product is *associative*. Therefore their dual operator the Lukasiewicz t-norm reduced by the product t-conorm is also associative. An immediate result is that these operators are respectively a t-conorm and a t-norm.

Before showing that this operator is associative let us show that it can be simply written by:

$$S_{Luka-prod}(a,b) = \min(a+b+\beta \cdot a \cdot b, 1) \tag{33}$$

In fact, for $a + b > 1$, we have then that $a + b + \beta ab > 1$ since $\beta \geq 0$.

We can show an analogue result for the dual operator:

$$T_{Luka-prod}(a,b) = \max[a+b-1-\beta \cdot (1-a) \cdot (1-b),\ 0] \tag{34}$$

In fact, for $a + b - 1 < 0$, we have then that $a + b - 1 - \beta(1-a)(1-b) < 0$ since $\beta \geq 0$.

Let be $R(a,b) = a + b + \beta \cdot a \cdot b$ the formulation (33) is then:

$$S_{Luka-prod}(a,b) = \min(R(a,b),1) \tag{35}$$

And we have,

$$R(R(a,b),c) \;=\; R(a,b) + c + \beta \cdot R(a,b) \cdot c$$

$$= a + b + c + \beta \cdot (a \cdot b + a \cdot c + b \cdot c) + \beta^2 \cdot a \cdot b \cdot c$$

$$= R(a, R(b, c))$$

In conclusion, augmenting the Lukasiewicz t-conorm by the product gives a parameterized t-conorm. And reducing the Lukasiewicz t-norm by the product t-conorm gives a parameterized t-conorm.

7. Conclusion

In this paper we study how to reduce a t-norm by subtracting to it a value given by a function. Using a commutative, monotone decreasing function that has one as null factor, we obtain an interesting aggregation operator, which is almost a t-norm besides the fact that it is not always associative. This very particular side effect shows the difficulty of reducing t-norms.

If we impose the associativity to the function being subtracted, we discover that it is actually β-times the negation of a t-conorm. Then we study some interesting general properties and after that, we study some particular cases by reducing some typical t-norms. We obtained for each typical t-norm an interesting parameterized family. We compare them to some existing parameterized t-norms.

Then using the De Morgan's law we obtain a dual operator. We find that these kinds of operators are augmented t-conorms. In fact this operators are built by adding to a t-conorm, β times a t-norm. This result is very interesting because it suggests that in order to make more drastic a t-conorm we need to add a t-norm. Here, once again, we do not obtain always a t-conorm, because the resulting operator is not always associative. We also study some interesting particular cases by augmenting typical t-conorms.

Before concluding we take a look at the reduction and augmentation of not dual t-norms and t-conorms. This situation is particularly interesting for the Lukasiewicz case. Since we showed that the Lukasiewicz t-norm (or t-conorm) cannot be reduced (or augmented) by his dual operator. Augmenting and reducing Lukasiewicz by the minimum, we obtain an interesting non-associative aggregation operator. But when augmenting and reducing by the minimum we obtain a parameterized t-conorm and a parameterized t-norm.

Bibliography

[1]. Menger, K., Statistical metrics, *Proc. Nat. Acad. Sci. U.S.A. 8*, 535-537, 1942.

[2]. Schweizer, B. and Sklar, A., Statistical metric spaces, *Pacific J. Math. 10*, 313-334, 1960

[3]. Hohle, U., "Probabilistic uniformization of fuzzy topologies," *Fuzzy Sets and Systems 1*, 1978.

[4]. Klement, E.P., "A theory of fuzzy measures: a survey," in *FuzzyInformation and Decision Processes*, edited by Gupta, M. and Sanchez, E., North Holland: Amsterdam, 59-66, 1982.

[5]. Dubois, D. and Prade H., "Triangular norms for fuzzy sets," in *Proceedings 2 International Symposium of Fuzzy Sets,* Linz, 39 - 68, 1981.

[6]. Alsina, C., Trillas, E. and Valverde, L., "On some logical connectives for fuzzy set theory," *J. Math Anal. & Appl. 93*, 15-26, 1983.

[7]. Bonissone, P.P., "Selecting uncertainty calculi and granularity: An experiment in trading off precision and complexity," in *Proceedings of the first Workshop on Uncertainty in Artificial Intelligence,* Los Angeles, 57 - 66, 1985.

[8]. Klir, G. J. and Folger, T. A., Fuzzy Sets, Uncertainty and Information, *Prentice-Hall: Englewood Cliffs*, N.J., 1988.

[9]. Dubois, D. and Prade, H., "A review of fuzzy sets aggregation connectives," *Information Sciences 36*, 85 - 121, 1985.

[10]. Frank M.J., On the simultaneous associativity of F(x,y) and x+y-F(x,y), *Aequationes Math.,19* (1979), 194--226.

[11]. Yager, R. R., "On a general class of fuzzy connectives," *Fuzzy Sets and Systems 4*, 235-242, 235-242, 1980.

[12]. Hamacher H., "On Logical Connectives of Fuzzy Statements", *Proc. 3rd. European Meeting Cybernetics and Systems*, Vienna, 1976.

[13]. Weber S., "A General Concept of Fuzzy Connectives, Negations and Implications" *Fuzzy Sets and Systems 11*, 115-134, 1983.

[14]. Mesiar R. and Komorníková M., "Aggregation Operators", in Herceg D., Surla K. (eds.), *Proc. XI Conf. On applied Mathematics PRIM' 96,* Institute of Mathematics, Novi Sad, 193-211, 1997.

[15]. Schweizer B. and Sklar A., Probabilistic metric spaces, *North Holland*, New York, 1983.

[16]. Klement E.P., Mesiar R. and Pap, "Triangular norms", Monograph in preparation.

[17]. Ling C., "Representation of associative function", *Publ. Math.,* Debrecen 12, (1965), 189-212.