Reasoning with different levels of uncertainty

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ABSTRACT. *We introduce a family of preferential logics that are useful for handling information with different levels of uncertainty. The corresponding consequence relations are non-monotonic, paraconsistent, adaptive, and rational. It is also shown that the formalisms in this family can be embedded in corresponding four-valued logics with at most three uncertainty levels, and that reasoning with these logics can be simulated by algorithms for processing circumscriptive theories, such as DLS and SCAN.*

KEYWORDS: *reasoning with uncertainty, logical lattices, multiple-valued logics, preferential semantics, paraconsistency, circumscriptive theories.*

1. Motivation

The ability to reason in a 'rational' way with incomplete or inconsistent information is a major challenge, and its significance should be obvious. It is well-known that classical logic is not suitable for this task, thus non-classical formalisms are usually used for handling uncertainty.[1] Such formalisms should be able to distinguish among different types of inconsistent information and partial data with different degrees of uncertainty that may appear in the same theory, since each kind of uncertainty may require a different treatment and may have a different effect on the set of the consequences of the theory. To see this, consider, e.g., the following well-known example:

EXAMPLE 1 (TWEETY DILEMMA). — Given a knowledge-base with the following set of assertions (specified in some appropriate formal language):

1) Flying ability is a default property of birds.
2) A wounded bird might not fly.
3) Animals with wings are birds.
4) Penguins are birds.
5) Penguins cannot fly.

1. See [BAT 00b, BEN 01, CAR 02a, NIE 03] for recent collections of papers on this topic.

Suppose, further, that it is known that Tweety is a penguin, and that Fred is a wounded animal with wings. This knowledge-base contains conflicting evidence regarding the flying abilities of Tweety and Fred: Tweety is a penguin, therefore it is a bird, and since most of the birds can fly, one concludes that Tweety probably flies. On the other hand, since penguins cannot fly, and since Tweety is a penguin, one has also a good reason to believe that Tweety cannot fly. Note that the latter conclusion is stronger than the former, since it is based on a strict rule (*all* the penguins cannot fly) rather than a default assumption. Indeed, most of the formalisms for such theories conclude (or indicate with high certainty) that Tweety can fly, despite the contradictory data.

The case of Fred is more problematic, as the evidence in favor of concluding that Fred flies and the evidence against this conclusion seem to be more 'balanced': normally, one would conclude that Fred can fly (since it has wings and so it is a bird). However, we also know that it is wounded, and so the injury might prevent this ability (as assertion (2) above suggests). It follows, then, that there is a contradictory evidence about whether Fred can fly. This contradiction stems from the partial information about Fred's injury and the consequences of this injury.

As the example above shows, it is often natural and reasonable to attach different levels of uncertainty to different assertions. This kind of information may be used, for instance, by algorithms for consistency restoration, since data with higher degree of inconsistency may be treated (i.e., eliminated) first (see, e.g., [DUB 94, BEN 95]). A proper method of 'grading' uncertain information w.r.t. inconsistent and/or incomplete theories is also vital for many paraconsistent formalisms[2], in which the criteria for drawing conclusions is the robustness of the premises, i.e., the degree of certainty that the premises indeed hold (see, e.g., [ARI 98b, BAT 98, BAT 00a, ARI 02b]).

In this paper we consider a framework that supports these kinds of considerations, and allows to reason with different levels of uncertain information. We show that the logics that are obtained are nonmonotonic, paraconsistent [COS 74], adaptive in the sense of Batens [BAT 98, BAT 00a, BAT 02], and rational in the sense of Lehmann and Magidor [LEH 92]. It is also shown that for each one of these formalisms there is a logically equivalent four-valued logic with at most three different levels of uncertainty. These logics can be simulated by algorithms for processing circumscriptive theories, such as DLS [DOH 97] and SCAN [OHL 96].

The rest of this paper is organized as follows: in the next section we introduce our framework and define the corresponding family of consequence relations for reasoning with graded uncertainty. In Section 3 we give a characterization theorem for this family in terms of four-valued semantics. Then, in Section 4 we show several properties of the underlying formalisms, and in Section 5 we consider some computational aspects of the corresponding reasoning process. In Section 6 we conclude. [3]

2. I.e., reasoning processes that do not become trivial in the presence of inconsistency. See [CAR 02b] for a review of such systems.
3. This paper is a revised and extended version of [ARI 03a].

2. The framework

2.1. *Logical lattices and their consequence relations*

It is well-known that classical logic is not suitable for reasoning with incomplete and inconsistent data. For instance, classical logic is not *paraconsistent* [COS 74], that is: everything classically follows from an inconsistent theory, and therefore it is not possible to draw, in a non-trivial way, plausible conclusions from such theories.

In order to overcome these shortcomings of classical logic and properly handle uncertainty, we turn to multiple-valued logics. This is a common approach that is the basis of many formal systems (see [AVR 02] for a recent survey), including systems that are based on fuzzy logic [HÁJ 98], probabilistic reasoning [PEA 89], possibilistic logics [DUB 94], annotated logics [SUB 90, KIF 92], and fixpoint semantics for extended/disjunctive logic programs (see, e.g., [LUK 01, ARI 02a], and a survey in [DAM 98]). In most of the approaches mentioned above, and here as well, the truth-values are arranged in a lattice structure. In what follows we shall denote by $\mathcal{L}=(L,\leq)$ a bounded lattice that has at least four elements: a $\leq$-maximal element and a $\leq$-minimal element that correspond to the classical values (denoted, respectively, by t and f), and two intermediate elements, denoted by $\top$ and $\bot$, that may intuitively be understood as representing two basic types of uncertainty: inconsistency and incompleteness (respectively). As usual, the meet and the join operations on $\mathcal{L}$ are denoted by $\wedge$ and $\vee$. In addition, we assume that $\mathcal{L}$ has an involution operator $\neg$ (a 'negation') such that $\neg t=f$, $\neg f=t$, $\neg\top=\top$, $\neg\bot=\bot$. We denote by $\mathcal{D}$ the set of the *designated values* of L (i.e., the set of the truth values in L that represent true assertions). We shall assume that $\mathcal{D}$ is a prime filter[4] in $\mathcal{L}$, s.t. $\top\in\mathcal{D}$ and $\bot\notin\mathcal{D}$. A pair $(\mathcal{L},\mathcal{D})$ is called *logical lattice* [ARI 99].

EXAMPLE 2. — The smallest logical lattice, denoted $\mathcal{FOUR}$, is shown in Figure 2 (left). This is the algebraic structure behind Belnap's well-known four-valued logic [BEL 77a, BEL 77b], and it will play an important role here as well (see Section 3). $\mathcal{FOUR}$ consists of the four basic elements of logical lattices, among which two are designated: $\mathcal{D} = \{t,\top\}$. The other structure shown in Figure 2 is $\mathcal{NINE}$; it may be viewed as an extension of $\mathcal{FOUR}$, which is useful, e.g., for default reasoning (and so dt may be attached to formulae that are 'true by default', bt may represent belief that is 'biased' for t, etc.). This lattice depicts three main levels of uncertainty: incomplete data ($\bot$), inconsistent data ($\top$), and a middle level of uncertainty (m). The latter kind of uncertainty may correspond to contradictory default assumptions, so it could be retracted when further information arrives. The decision whether to view m as designated is one of the differences between the two logical lattices that $\mathcal{NINE}$ induces, namely $(\mathcal{NINE},\{t,bt,\top\})$ and $(\mathcal{NINE},\{t,dt,bt,bf,m,\top\})$.

As logical lattices may be infinite, it is possible to consider structures with arbitrarily many different levels of inconsistency. Consider, e.g., the logical lattice $(\mathcal{L},\mathcal{D})$,

4. In particular, $t\in\mathcal{D}$ and $f\notin\mathcal{D}$.

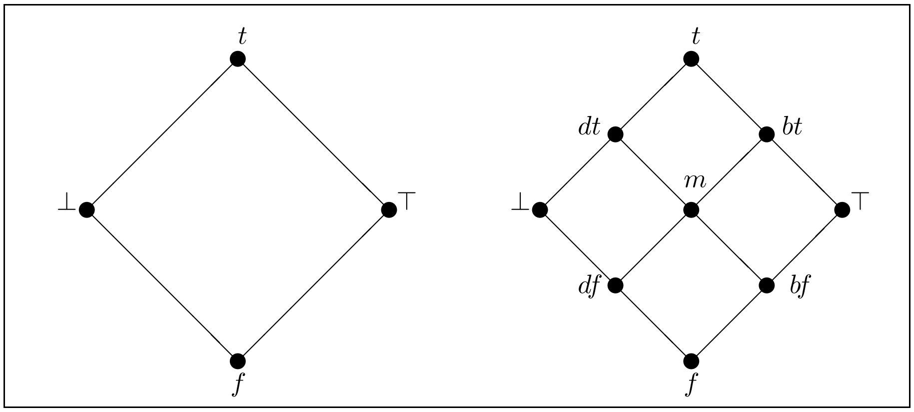

Figure 1. $\mathcal{FOUR}$ *and* $\mathcal{NINE}$.

where $L = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$, $\mathcal{D} = \{(1, y) \mid y \in \mathbb{R},\ 0 \leq y \leq 1\}$, and the lattice operators are defined as follows:

$$(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2)),$$

$$(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2)).$$

In this case, $t = (1, 0)$, $f = (0, 1)$, $\top = (1, 1)$, $\bot = (0, 0)$, and $(x_1, y_1) \leq (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \geq y_2$.[5] One way to intuitively understand the meaning of an element $(x, y) \in L$ is such that x represents the amount of belief for the underlying assertion, and y represents the amount of belief against it. Following this intuition, every element $(x, x) \in L$ may be associated with a different degree of inconsistency.

Given a logical lattice $(\mathcal{L}, \mathcal{D})$, the basic connectives are defined in the standard way: negation corresponds to the lattice involution, conjunction [respectively, disjunction] corresponds to the meet [respectively, join] operator, and the material implication is defined by a combination of negation and disjunction: $p \rightarrow q = \neg p \vee q$. Standard semantic notions are natural generalizations of the classical ones: a (multiple-valued) *valuation* ν is a function that assigns an element of L to each atomic formula. The set of valuations onto L is denoted by $\mathcal{V}^L$. Extension to complex formulae is done in the usual way: $\nu(\neg\psi) = \neg\nu(\psi)$, and $\nu(\psi \circ \phi) = \nu(\psi) \circ \nu(\phi)$ for every $\circ \in \{\vee, \wedge, \rightarrow\}$. A valuation is a *model* of a set of assertions Γ if it assigns a designated value to every formula in Γ. The set of all the models of Γ is denoted by $mod(\Gamma)$.

Note that there are no tautologies in the language of $\{\neg, \vee, \wedge, \rightarrow\}$, since if all the atomic formulae that appear in a formula ψ are assigned $\bot$ by a valuation ν, then $\nu(\psi) = \bot$ as well. It follows that the definition of the material implication $p \rightarrow q$ as

5. See [GIN 88, FIT 90, ARI 00a] for a further discussion on this lattice, in the context of more general structures, called *bilattices*.

$\neg p \vee q$ is not adequate for representing entailments in our semantics. Instead, we use another connective, which does function as an implication in our setting:

DEFINITION 3 ([AVR 91, ARI 96]). — *Let* $(\mathcal{L}, \mathcal{D})$ *be a logical lattice. For every* $x, y \in L$*, define:* $x \supset y = y$ *if* $x \in \mathcal{D}$*, and* $x \supset y = t$ *otherwise.*[6]

The language of $\{\neg, \vee, \wedge, \supset\}$ together with the propositional constants $t, f, \top$ and $\bot$ (which correspond to the four elements that appear in every logical lattice), will be denoted by Σ. Given a set of formulae Γ in Σ, we shall denote by $\mathcal{A}(\Gamma)$ the set of the atomic formulae that appear in the formulae of Γ. Now, a natural definition of a lattice-based consequence relation is the following:

DEFINITION 4. — *Let* $(\mathcal{L}, \mathcal{D})$ *be a logical lattice,* Γ *a set of formulae, and* ψ *a formula. Denote* $\Gamma \models^{\mathcal{L},\mathcal{D}} \psi$ *if every model of* Γ *is a model of* ψ*.*[7]

The relation $\models^{\mathcal{L},\mathcal{D}}$ of Definition 4 is a consequence relation in the standard sense of Tarski [TAR 41]. In [ARI 96] it is shown that this relation is monotonic, compact, paraconsistent [COS 74], and has a corresponding sound and complete cut-free Gentzen-style proof system. The major drawbacks of $\models^{\mathcal{L},\mathcal{D}}$ are that it is strictly weaker than classical logic even for consistent theories (e.g., $\not\models^{\mathcal{L},\mathcal{D}} \neg\psi \vee \psi$), and that it always invalidates some intuitively justified inference rules, such as the Disjunctive Syllogism ($\psi,\ \neg\psi \vee \phi \not\models^{\mathcal{L},\mathcal{D}} \phi$). In the next section we consider a family of logics that preserve the nice properties of $\models^{\mathcal{L},\mathcal{D}}$ and overcome most of its drawbacks.

2.2. *Preferential reasoning and the consequence relation* $\models_c^{\mathcal{L},\mathcal{D}}$

In order to recapture within our framework classical reasoning (where its use is appropriate), as well as standard non-monotonic and paraconsistent methods, we incorporate a concept first introduced by McCarthy [MCC 80] and later considered by Shoham [SHO 88], according to which inferences from a given theory are made with respect to a subset of the models of that theory (and not according to every model of the theory; see also [GAB 85, MAK 89, KRA 90, MAK 94, ARI 99, ARI 00b, SCH 00, LEH 01]). This set of *preferential models* is determined according to some conditions that can be specified by a set of (usually second-order) propositions [ARI 02b], or by some order relation on the models of the theory [PRI 89, PRI 91, ARI 96, ARI 98a, ARI 98b, BES 03]. This relation should reflect some kind of preference criterion on the models of the set of premises. In our case the idea is to give precedence to those valuations that minimize the amount of uncertain information in the premises. The truth values are therefore arranged according to an order relation that reflects differences in the amount of uncertainty that each one of them exhibits. Then we choose those valuations that minimize the amount of uncertainty with respect to this order. The intuition behind this approach is that incomplete or contradictory data correspond

6. Note that on $\{t, f\}$ the material implication ($\rightarrow$) and the new implication ($\supset$) are identical, and both of them are generalizations of the classical implication.
7. When referring to $\mathcal{FOUR}$ we shall abbreviate $\models^{\mathcal{L},\mathcal{D}}$ by $\models^4$.

to inadequate information about the real world, and therefore should be minimized. Next we formalize this idea.

DEFINITION 5. — *A partial order* $<$ *on a set* $\mathcal{S}$ *is called* modular *if* $y < x_2$ *for every* $x_1, x_2, y \in \mathcal{S}$ *s.t.* $x_1 \not< x_2$, $x_2 \not< x_1$, *and* $y < x_1$.

PROPOSITION 6 ([LEH 92]). — *Let* $<$ *be a partial order on* $\mathcal{S}$. *The following conditions are equivalent:*

a) $<$ *is modular.*

b) for every $x_1, x_2, y \in \mathcal{S}$, *if* $x_1 < x_2$ *then either* $y < x_2$ *or* $x_1 < y$.

c) there is a totally ordered set $\mathcal{S}'$ *with a strict order* $\prec$ *and a function* $g : \mathcal{S} \to \mathcal{S}'$ *s.t.* $x_1 < x_2$ *iff* $g(x_1) \prec g(x_2)$.

DEFINITION 7. — *An* inconsistency order $<_c^{\mathcal{L},\mathcal{D}}$ *on a logical lattice* $(\mathcal{L}, \mathcal{D})$ *is a well-founded modular order on* L, *with the following properties:*

a) t *and* f *are minimal and* $\top$ *is maximal w.r.t.* $<_c^{\mathcal{L},\mathcal{D}}$,

b) if $\{x, \neg x\} \subseteq \mathcal{D}$ *while* $\{y, \neg y\} \not\subseteq \mathcal{D}$, *then* $x \not<_c^{\mathcal{L},\mathcal{D}} y$,

c) x *and* $\neg x$ *are either equal or* $<_c^{\mathcal{L},\mathcal{D}}$*-incomparable.*

Inconsistency orders are used here for grading uncertainty in general, and inconsistency in particular. The intuitive meaning of $x <_c^{\mathcal{L},\mathcal{D}} y$ is that formulae that are assigned x are more definite than formulae with a truth value y. Modularity is needed for assuring a proper grading of the truth values.[8] Condition (b) in Definition 7 assures that truth values that intuitively represent inconsistent data will not be considered as more consistent than those ones that correspond to consistent data. The last condition makes sure that any truth value and its negation have the same degree of (in)consistency.

EXAMPLE 8. — $\mathcal{FOUR}$ has four inconsistency orders:

a) The degenerated order, $<_{c_0}^4$, in which $t, f, \bot, \top$ are all incomparable.

b) $<_{c_1}^4$, in which $\bot$ is considered as minimally inconsistent: $\{t, f, \bot\} <_{c_1}^4 \top$.

c) $<_{c_2}^4$, in which $\bot$ is maximally inconsistent: $\{t, f\} <_{c_2}^4 \{\top, \bot\}$.

d) $<_{c_3}^4$, in which $\bot$ is an intermediate level of inconsistency: $\{t, f\} <_{c_3}^4 \bot <_{c_3}^4 \top$.

In the rest of the paper we shall continue to use the notations of Example 8 for denoting the inconsistency orders in $\mathcal{FOUR}$.

Given an inconsistency order $<_c^{\mathcal{L},\mathcal{D}}$ on a logical lattice $(\mathcal{L}, \mathcal{D})$, it induces an equivalence relation on L, in which two elements in L are equivalent iff they are equal or $<_c^{\mathcal{L},\mathcal{D}}$-incomparable. For every $x \in \mathcal{L}$, we denote by $[x]$ the equivalence class of x with respect to this equivalence relation. I.e.,

8. That is, to eliminate orders such as $\{\{t\}, \{f < \bot < \top\}\}$, in which $\top$ and $\bot$ are not comparable with t, while they are comparable with $\neg t$.

$$[x] = \{y \mid y = x, \text{ or } x \text{ and } y \text{ are } <_c^{\mathcal{L},\mathcal{D}}\text{-incomparable}\}.$$

The order relation on these classes is defined as usual by representatives: $[x] \leq_c^{\mathcal{L},\mathcal{D}} [y]$ iff either $x <_c^{\mathcal{L},\mathcal{D}} y$, or x and y are $<_c^{\mathcal{L},\mathcal{D}}$-incomparable.[9] It is easy to verify that this definition is proper, i.e., it does not depend on the choice of the representatives. In what follows we shall write $[x] <_c^{\mathcal{L},\mathcal{D}} [y]$ to denote that $[x] \leq_c^{\mathcal{L},\mathcal{D}} [y]$ and $[x] \neq [y]$.

An inconsistency order on $(\mathcal{L}, \mathcal{D})$ induces the following pre-order on $\mathcal{V}^L$:

DEFINITION 9. — *Let* $<_c^{\mathcal{L},\mathcal{D}}$ *be an inconsistency order on* $(\mathcal{L}, \mathcal{D})$*, and let* $\nu_1, \nu_2 \in \mathcal{V}^L$.

a) $\nu_1 \leq_c^{\mathcal{L},\mathcal{D}} \nu_2$ *iff for every atom p,* $[\nu_1(p)] \leq_c^{\mathcal{L},\mathcal{D}} [\nu_2(p)]$.

b) $\nu_1 <_c^{\mathcal{L},\mathcal{D}} \nu_2$ *if* $\nu_1 \leq_c^{\mathcal{L},\mathcal{D}} \nu_2$ *and there is an atom q s.t.* $[\nu_1(q)] <_c^{\mathcal{L},\mathcal{D}} [\nu_2(q)]$.

DEFINITION 10. — *Let* $<_c^{\mathcal{L},\mathcal{D}}$ *be an inconsistency order on a logical lattice* $(\mathcal{L}, \mathcal{D})$ *and let* Γ *be a set of formulae in* Σ*. The* c-most consistent models *of* Γ *(abbreviation: the* c-mcms *of* Γ*) are the* $\leq_c^{\mathcal{L},\mathcal{D}}$*-minimal models of* Γ*, i.e.,*

$$!(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}}) = \{\nu \in mod(\Gamma) \mid \neg\exists \mu \in mod(\Gamma) \text{ s.t. } \mu <_c^{\mathcal{L},\mathcal{D}} \nu\}.$$

The lattice-based consequence relation $\models^{\mathcal{L},\mathcal{D}}$ (Definition 4) may be refined now such that only the c-most consistent models of the premises are taken into account for drawing conclusions:

DEFINITION 11. — *Let* $<_c^{\mathcal{L},\mathcal{D}}$ *be an inconsistency order on a logical lattice* $(\mathcal{L}, \mathcal{D})$*. Denote:* $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$ *if every c-mcm of* Γ *is a model of* ψ.

2.3. *Examples*

Below are some examples of reasoning with $\models_c^{\mathcal{L},\mathcal{D}}$. In what follows we assume that formulae with free variables are universally quantified. Consequently, a set of assertions Γ, containing a non-grounded formula, ψ, is viewed as representing the corresponding set of ground formulae, formed by substituting for each variable that appears in ψ, every element in the relevant Herbrand universe.

EXAMPLE 12. — Consider one direction of the barber paradox:

$$\Gamma = \{\neg \mathtt{shaves}(\mathtt{x}, \mathtt{x}) \supset \mathtt{shaves}(\mathtt{Barber}, \mathtt{x})\}.$$

Denote by ν_1, ν_2, and ν_3 the valuations that assign t, $\bot$, and $\top$ (respectively) to the assertion $\mathtt{shaves}(\mathtt{Barber}, \mathtt{Barber})$. Using $\mathcal{FOUR}$ as the underlying logical lattice, we have that $!(\Gamma, \leq_{c_2}^4) = !(\Gamma, \leq_{c_3}^4) = \{\nu_1\}$, $!(\Gamma, \leq_{c_1}^4) = \{\nu_1, \nu_2\}$, and $!(\Gamma, \leq_{c_0}^4) = \{\nu_1, \nu_2, \nu_3\}$. Thus, $\Gamma \not\models_{c_i}^4 \mathtt{shaves}(\mathtt{Barber}, \mathtt{Barber})$ when $i = 0, 1$, while $\Gamma \models_{c_i}^4 \mathtt{shaves}(\mathtt{Barber}, \mathtt{Barber})$ when $i = 2, 3$.

9. As usual, we use the same notation to denote the order relation among equivalence classes and the order relation among their elements.

EXAMPLE 13 (TWEETY DILEMMA, REVISITED). — Let's consider the following version of Tweety dilemma, discussed in Example 1.

$$\mathcal{R}ules = \left\{ \begin{array}{l} \texttt{bird(x)} \rightarrow \texttt{fly(x)}, \\ \texttt{wounded(x)} \rightarrow \neg\texttt{fly(x)}, \\ \texttt{has\_wings(x)} \supset \texttt{bird(x)}, \\ \texttt{penguin(x)} \supset \texttt{bird(x)}, \\ \texttt{penguin(x)} \supset \neg\texttt{fly(x)} \end{array} \right\}$$

We are using here different implication connectives according to the strength of each entailment: the first two rules state only default properties of birds and wounded animals. The other three rules, on the other hand, specify characteristic properties of penguins and characterize animals with wings. As there are no exceptions to these rules, they are expressed by a stronger implication connective.

Consider, first, the following set of assertions:

$$\Gamma_1 = \mathcal{R}ules \bigcup \{ \texttt{bird(Tweety)} \}.$$

As shown in Table 1, Γ_1 has 240 four-valued models, among which six are c_1-mcms, and two are both c_2-mcms and c_3-mcms.[10]

Table 1. *The models and the c_i-mcms of* Γ_1

Model No.	bird	fly	penguin	has_wings	wounded
$M_1 - M_{128}$	$\top$	$\top, f$	$\top, t, f, \bot$	$\top, t, f, \bot$	$\top, t, f, \bot$
$M_{129} - M_{160}$	$\top$	$t, \bot$	$f, \bot$	$\top, t, f, \bot$	$\top, f$
$M_{161} - M_{224}$	t	$\top$	$\top, t, f, \bot$	$\top, t, f, \bot$	$\top, t, f, \bot$
$M_{225} - M_{240}$	t	t	$f, \bot$	$\top, t, f, \bot$	$\top, f$

Type	bird	fly	penguin	has_wings	wounded
c_1-mcms	t	t	$f, \bot$	$t, f, \bot$	f
c_2-mcms	t	t	f	t, f	f
c_3-mcms	t	t	f	t, f	f

It follows that with $\models^4_{c_i}$ $(1 \leq i \leq 3)$ one can infer from Γ_1 that `bird(Tweety)`, `fly(Tweety)`, and $\neg$`wounded(Tweety)` (the converse assertions, $\neg$`bird(Tweety)`, $\neg$`fly(Tweety)`, and `wounded(Tweety)`, are, nevertheless, not deducible). This corresponds to the intuitive expectation that, as long as the only data concerning Tweety is that it is a bird, we follow the default assumption that it can fly, and we don't have

10. Recall that we are using here the notations of Example 8 to denote the inconsistency orders in $\mathcal{FOUR}$. In what follows we shall also write $\models^4_{c_i}$ (instead of $\models^{\mathcal{FOUR}}_{c_i}$) to denote the corresponding consequence relations.

any reason to believe that it is wounded. On the other hand, excluding the possibility that Tweety is a penguin seems a more far reaching conclusion than the previous two, and indeed only $\models^4_{c_2}$ and $\models^4_{c_3}$ support this conclusion. Finally, as we do not know anything about animals with wings, except the fact that they are birds, none of $\models^4_{c_i}$, $0 \leq i \leq 3$, allows us to conclude or to rule out the possibility that Tweety has wings.

Suppose now that a new information arrives, and we are informed that Tweety is actually a penguin. Denote the new theory by Γ_2, i.e.,

$$\Gamma_2 = \mathcal{R}ules \bigcup \{ \texttt{bird(Tweety)},\ \texttt{penguin(Tweety)} \}.$$

Clearly, Γ_2 is no longer classically consistent, which implies that everything classically follows from it. On the other hand, as it is shown in Section 4 below, consequence relations of the form $\models^{\mathcal{L},\mathcal{D}}_c$ are paraconsistent, and so they do not have this drawback. Indeed, consider the four-valued models of Γ_2 and its c_i-mcms, shown in Table 2.

Table 2. *The models and the c_i-mcms of* Γ_2

Model No.	bird	fly	penguin	has_wings	wounded
$M_1 - M_{32}$	$\top$	$\top$	$\top, t$	$\top, t, f, \bot$	$\top, t, f, \bot$
$M_{33} - M_{64}$	$\top$	f	$\top, t$	$\top, t, f, \bot$	$\top, t, f, \bot$
$M_{65} - M_{96}$	t	$\top$	$\top, t$	$\top, t, f, \bot$	$\top, t, f, \bot$

Type	bird	fly	penguin	has_wings	wounded
c_1-mcms	$\top$	f	t	$t, f, \bot$	$t, f, \bot$
	t	$\top$	t	$t, f, \bot$	$t, f, \bot$
c_2-mcms	$\top$	f	t	t, f	t, f
	t	$\top$	t	t, f	t, f
c_3-mcms	$\top$	f	t	t, f	t, f
	t	$\top$	t	t, f	t, f

This time, bird(Tweety), penguin(Tweety), and ¬fly(Tweety) are all deducible from Γ_2 with respect to $\models^4_{c_i}$ for $i = 1, 2, 3$, and the complements of these assertions cannot be inferred by any one of these consequence relations, as indeed one expects.

Consider, finally, the following set of assertions:

$$\Gamma_3 = \mathcal{R}ules \bigcup \{ \texttt{has\_wings(Fred)},\ \texttt{wounded(Fred)} \}.$$

Again, Γ_3 is not consistent, and indeed even its c_i-most consistent models $(i=1,2,3)$ assign $\top$ to at least one of its atomic formulae[11] (see Table 3 for the c_i-mcms of Γ_3). However, as already noted in Example 1, the contradiction in this case is more

11. I.e., for every $1 \leq i \leq 3$ and for every $\nu \in !(\Gamma_3, \leq^4_{c_i})$ there is a $p \in \mathcal{A}(\Gamma_3)$ s.t. $\nu(p) = \top$.

fundamental than that of Γ_2, since it is not possible to judge from the information in Γ_3 whether Fred can still fly despite its injury.[12] Indeed, by Table 3 it follows that none of the consequence relations $\models^4_{c_i}$, $i=1,2,3$, allows to conclude that fly(Fred) or that ¬fly(Fred).

Table 3. *The c_i-mcms of* Γ_3

Type	bird	fly	penguin	has_wings	wounded
c_1-mcms	$\top$	f	$t, f, \bot$	t	t
	t	t	$f, \bot$	t	$\top$
	t	$\top$	$t, f, \bot$	t	t
c_2-mcms	$\top$	f	t, f	t	t
	t	t	f	t	$\top$
	t	$\top$	t, f	t	t
c_3-mcms	$\top$	f	t, f	t	t
	t	t	f	t	$\top$
	t	$\top$	t, f	t	t

NOTE 14. — By the last two examples one might get the wrong impression that the set of the c_1-mcms of a given theory always contains the set of the c_2-mcms and the set of the c_3-mcms of the same theory. To see that this is not the case, consider again the last example with two additional (and somewhat more controversial) rules:

$$\mathcal{R}ules' = \mathcal{R}ules \bigcup \left\{ \begin{array}{l} \texttt{has\_wings(x)} \supset \texttt{fly(x)}, \\ \neg\texttt{has\_wings(x)} \supset \neg\texttt{penguin(x)} \end{array} \right\},$$

$$\Gamma_2' = \Gamma_2 \bigcup \mathcal{R}ules'.$$

The (c_i-most consistent) models of Γ_2' are given in Table 4. It follows, for instance, that while with $\models^4_{c_2}$ and $\models^4_{c_3}$ there are indications that Tweety is not a 'typical' penguin (as penguin(Tweety) is assigned $\top$ by some c_2-mcms and c_3-mcms of Γ_2'), the consequence relation $\models^4_{c_1}$ rules out the possibility that Tweety is not a penguin: $\Gamma_2' \models^4_{c_1} \texttt{penguin(Tweety)} \wedge (\neg\texttt{penguin(Tweety)} \supset f)$.

Note also that, unlike the examples above, the set of the c_2-mcms of a theory is in general different than the set of the c_3-mcms of the same theory. This is shown in Proposition 25 below.

3. Embedding in four-valued logics

Four-valued reasoning may be traced back to the 1950's, where is has been investigated by a number of people, including Bialynicki-Birula [BIA 57a], Rasiowa

12. Note, however, the it *is* possible to conclude that Fred is a bird, although this fact is not explicitly mentioned in Γ_3.

Table 4. *The models and the c_i-mcms of* Γ'_2

Model No.	bird	fly	penguin	has_wings	wounded
$M_1 - M_{16}$	$\top$	$\top$	$\top$	$\top, t, f, \bot$	$\top, t, f, \bot$
$M_{17} - M_{24}$	$\top$	f	$\top$	$f, \bot$	t, f
$M_{25} - M_{40}$	t	$\top$	$\top$	$\top, t, f, \bot$	$\top, t, f, \bot$
$M_{41} - M_{48}$	$\top$	$\top$	t	$t, \bot$	$\top, t, f, \bot$
$M_{49} - M_{52}$	$\top$	f	t	$\bot$	$\top, t, f, \bot$
$M_{53} - M_{60}$	t	$\top$	t	$t, \bot$	$\top, t, f, \bot$

Type	bird	fly	penguin	has_wings	wounded
c_1-mcms	$\top$	f	t	$\bot$	$t, f, \bot$
	t	$\top$	t	$t, \bot$	$t, f, \bot$
c_2-mcms	$\top$	f	t	$\bot$	t, f
	$\top$	f	$\top$	f	t, f
	t	$\top$	t	t	t, f
c_3-mcms	$\top$	f	t	$\bot$	t, f
	$\top$	f	$\top$	f	t, f
	t	$\top$	t	t	t, f

[BIA 57b], and Kalman [KAL 58]. Later, Belnap [BEL 77a, BEL 77b] introduced a corresponding four-valued algebraic structure (denoted here by $\mathcal{FOUR}$) for paraconsistent reasoning. As the following theorem shows, this structure is canonical for reasoning with graded uncertainty. Following [ARI 98a, ARI 98b], this is another evidence for the robustness of four-valued logics as representing common-sense reasoning.

THEOREM 15. — *Let $\leq_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order on $(\mathcal{L},\mathcal{D})$. Then there is an inconsistency order $\leq_{c_i}^4$ $(0 \leq i \leq 3)$ on $\mathcal{FOUR}$, such that $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$ iff $\Gamma \models_{c_i}^4 \psi$.*[13]

In the rest of this section we prove Theorem 15. For this, we first need some notations and definitions.

DEFINITION 16. — *$\mathcal{V}^L$ is* stoppered *w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$ if for every Γ and every $\nu \in mod(\Gamma)$, either $\nu \in\, !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$, or there is an $\nu' \in\, !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$ s.t. $\nu' <_c^{\mathcal{L},\mathcal{D}} \nu$.*[14]

Note that if $\mathcal{V}^L$ is well-founded w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$ (i.e., $\mathcal{V}^L$ does not have an infinitely descending chain w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$), then it is in particular stoppered.

PROPOSITION 17. — *Let $\leq_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order on a logical lattice $(\mathcal{L},\mathcal{D})$. Then $\mathcal{V}^L$ is stoppered (w.r.t. the induced order on valuations).*

13. This is a generalization of a similar result that is given in the reduced version of this paper [ARI 03a], in which stopperdness (Definition 16) was assumed.
14. The notion "stopperdness" is due to Makinson [MAK 94]. In [KRA 90] the same property is called *smoothness*.

PROOF. — Let $(\mathcal{L}, \mathcal{D})$ be a logical lattice and $\leq_c^{\mathcal{L},\mathcal{D}}$ an inconsistency order on the elements of L. Consider the corresponding space of valuations $\mathcal{V}^L$, and the order relation on the elements of $\mathcal{V}^L$, denoted also by $\leq_c^{\mathcal{L},\mathcal{D}}$. In [ARI 96, Theorem 3.22] it is shown that the consequence relation $\models^{\mathcal{L},\mathcal{D}}$ that is induced by $(\mathcal{L}, \mathcal{D})$ (Definition 4) is compact, and so in terms of [AVR 01], $\mathcal{V}^L$ is finitary.[15] By Corollary 5.5 of [AVR 01], then, $\mathcal{V}^L$ is stoppered w.r.t. any pointwise order that is defined on its elements.[16] In particular, $\mathcal{V}^L$ is stoppered w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$. ■

DEFINITION 18. — *Given a logical lattice $(\mathcal{L}, \mathcal{D})$, its elements may be divided into the following four sets:*

$$\mathcal{T}_t^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \in \mathcal{D}, \neg x \notin \mathcal{D}\}, \qquad \mathcal{T}_f^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \notin \mathcal{D}, \neg x \in \mathcal{D}\},$$

$$\mathcal{T}_\top^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \in \mathcal{D}, \neg x \in \mathcal{D}\}, \qquad \mathcal{T}_\perp^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \notin \mathcal{D}, \neg x \notin \mathcal{D}\}.$$

Henceforth we shall usually omit the superscripts, and just write $\mathcal{T}_t, \mathcal{T}_f, \mathcal{T}_\top, \mathcal{T}_\perp$.

DEFINITION 19. — *Let $(\mathcal{L}, \mathcal{D})$ be a logical lattice. Denote:*

$$\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x = \{y \in \mathcal{T}_x \mid \neg \exists y' \in \mathcal{T}_x \text{ s.t. } y' <_c^{\mathcal{L},\mathcal{D}} y\} \qquad (\textit{for } x \in \{t, f, \top, \perp\})$$

$$\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}} = \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_t \cup \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_f \cup \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\perp \cup \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top$$

DEFINITION 20. — *Let $(\mathcal{L}_1, \mathcal{D}_1)$ and $(\mathcal{L}_2, \mathcal{D}_2)$ be two logical lattices. Suppose that x_i is some element in L_i and ν_i is a valuation onto L_i $(i = 1, 2)$.*

a) x_1 and x_2 are similar *if $x_1 \in \mathcal{T}_y^{\mathcal{L}_1,\mathcal{D}_1}$ implies that $x_2 \in \mathcal{T}_y^{\mathcal{L}_2,\mathcal{D}_2}$ $(y \in \{t, f, \top, \perp\})$.*

b) ν_1 and ν_2 are similar *if for every atom p, $\nu_1(p)$ and $\nu_2(p)$ are similar.*

PROPOSITION 21. — *Let $(\mathcal{L}_1, \mathcal{D}_1)$ and $(\mathcal{L}_2, \mathcal{D}_2)$ be two logical lattices and suppose that ν_1 and ν_2 are two similar valuations on L_1 and L_2 (respectively). Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are similar.*

PROOF. — By an induction on the structure of ψ.[17] ■

Now we can turn to the proof of Theorem 15.

PROOF (OF THEOREM 15). — In what follows we shall denote by m_x some element in $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x^{\mathcal{L},\mathcal{D}}$ $(x \in \{t, f, \top, \perp\})$, and by $\omega : L \to \{t, f, \top, \perp\}$ the 'categorization' function: $\omega(y) = x$ iff $y \in \mathcal{T}_x$. Also, in the rest of this proof we shall abbreviate $[y] \cap \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ by $[y]$ (so the equivalence classes consist only of elements in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$).

LEMMA 22. — *If $M \in !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$ then for every atom p, $M(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$.*

15. That is, if $\Gamma \models^{\mathcal{L},\mathcal{D}} \psi$ then there is a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models^{\mathcal{L},\mathcal{D}} \psi$.
16. A pre-order $\preceq$ on $\mathcal{V}^L$ is called pointwise if there is a pre-order $\leq$ on L such that for every $\nu_1, \nu_2 \in \mathcal{V}^L$, $\nu_1 \preceq \nu_2$ iff for every atom p, $\nu_1(p) \leq \nu_2(p)$.
17. Note that the fact that $\mathcal{D}$ is a *prime* filter is crucial here.

PROOF. — Suppose that there is some atom p_0 s.t. $M(p_0) \notin \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$. Then, assuming that $M(p_0) \in \mathcal{T}_x$, there is an element $m_x \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ s.t. $m_x <_c^{\mathcal{L},\mathcal{D}} M(p_0)$. Consider the following valuation:

$$N(p) = \begin{cases} m_x & \text{if } p = p_0 \\ M(p) & \text{if } p \neq p_0 \end{cases}$$

N is similar to M, and so, by Proposition 21, N is also a model of Γ. Moreover, $N <_c^{\mathcal{L},\mathcal{D}} M$, thus $M \notin !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$. ■

Now, since $\leq_c^{\mathcal{L},\mathcal{D}}$ is well-founded and since $\mathcal{T}_x$ is nonempty for every $x \in \{t, f, \top, \bot\}$, $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ is nonempty as well, and so there is at least one element of the form m_x for every $x \in \{t, f, \top, \bot\}$. Also, it is clear that for every $m_x, m'_x \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$, $[m_x] = [m'_x]$ (otherwise either $m_x <_c^{\mathcal{L},\mathcal{D}} m'_x$ or $m_x >_c^{\mathcal{L},\mathcal{D}} m'_x$, and so either $m'_x \notin \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ or $m_x \notin \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$). It follows, therefore, that there are no more than the following three equivalence classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$:

1) $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_t \cup \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_f \subseteq [t]$,

2) $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\bot \subseteq [m_\bot]$,

3) $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top \subseteq [m_\top]$,

where $m_\bot$ is some element of $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\bot$, and $m_\top$ is some element of $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top$. By Definition 7, $[t]$ must be a minimal inconsistency class among those in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$, and $[m_\top]$ must be a maximal one. It follows, then, that the inconsistency classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ are arranged in one of the following orders:

0) $[t] = [m_\bot] = [m_\top]$,

1) $[t] = [m_\bot] <_c^{\mathcal{L},\mathcal{D}} [m_\top]$,

2) $[t] <_c^{\mathcal{L},\mathcal{D}} [m_\bot] = [m_\top]$,

3) $[t] <_c^{\mathcal{L},\mathcal{D}} [m_\bot] <_c^{\mathcal{L},\mathcal{D}} [m_\top]$.

If the order relation among the inconsistency classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ corresponds to case i above $(0 \leq i \leq 3)$ we say that the inconsistency order $\leq_c^{\mathcal{L},\mathcal{D}}$ is *of type* i.[18]

LEMMA 23. — *If* $\leq_c^{\mathcal{L},\mathcal{D}}$ *is an inconsistency order of type* i*, then for every* $m, m' \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$, $[m] <_c^{\mathcal{L},\mathcal{D}} [m']$ *iff* $[\omega(m)] <_{c_i}^4 [\omega(m')]$.

PROOF. — Immediate from the definition of inconsistency order of type i, and the definition of $\leq_{c_i}^4$. ■

LEMMA 24. — *If* $\leq_c^{\mathcal{L},\mathcal{D}}$ *is an inconsistency order of type* i *in* $(\mathcal{L}, \mathcal{D})$*, then* $\models_c^{\mathcal{L},\mathcal{D}}$ *is the same as* $\models_{c_i}^4$.

PROOF. — Suppose that $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$ but $\Gamma \not\models_{c_i}^4 \psi$. Then there is a c_i^4-mcm M^4 of Γ s.t. $M^4(\psi) \notin \{t, \top\}$. Now, for every atom p let $M^L(p)$ be some element in

18. In particular, for every $0 \leq i \leq 3$, the inconsistency order $\leq_{c_i}^4$ in $\mathcal{FOUR}$ is of type i.

$\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_{M^4(p)}$. Thus $\omega \circ M^L = M^4$, and M^L is similar to M^4. By Proposition 21, M^L is a model of Γ and it is not a model of ψ. To get a contradiction to $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$, it remains to show, then, that M^L is a c-mcm of Γ in $(\mathcal{L}, \mathcal{D})$. Indeed, otherwise by stopperdness (Proposition 17) there is a c-mcm N^L of Γ s.t. $N^L <_c^{\mathcal{L},\mathcal{D}} M^L$. So for every atom p, $[N^L(p)] \leq_c^{\mathcal{L},\mathcal{D}} [M^L(p)]$, and there is an atom p_0 s.t. $[N^L(p_0)] <_c^{\mathcal{L},\mathcal{D}} [M^L(p_0)]$. Let $N^4 = \omega \circ N^L$. Again, N^4 is similar to N^L, therefore it is a (four-valued) model of Γ. Also, by the definition of M, for every atom p, $M^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ and by Lemma 22, $\forall p\ N^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$. Thus, by Lemma 23,

$$[N^4(p)] = [\omega \circ N^L(p)] \leq_{c_i}^4 [\omega \circ M^L(p)] = [M^4(p)].$$

Also, by the same lemma,

$$[N^4(p_0)] = [\omega \circ N^L(p_0)] <_{c_i}^4 [\omega \circ M^L(p_0)] = [M^4(p_0)].$$

It follows that $N^4 <_{c_i}^4 M^4$, but this contradicts the assumption that M^4 is a c_i^4-mcm of Γ.

For the converse, suppose that $\Gamma \models_{c_i}^4 \psi$, but $\Gamma \not\models_c^{\mathcal{L},\mathcal{D}} \psi$. Then there is a c-mcm M^L of Γ in $(\mathcal{L}, \mathcal{D})$ s.t. $M^L(\psi) \notin \mathcal{D}$. Define, for every atom p, $M^4(p) = \omega \circ M^L(p)$. By the definition of ω, M^4 is similar to M^L and so M^4 is a model of Γ in $\mathcal{FOUR}$, but it is not a model of ψ. It remains to show, then, that M^4 is a c_i^4-mcm of Γ. Indeed, otherwise there is a model N^4 of Γ s.t. $N^4 <_{c_i}^4 M^4$, that is, for every atom p $[N^4(p)] \leq_{c_i}^4 [M^4(p)]$, and there is an atom p_0 for which this inequality is strict: $[N^4(p_0)] <_{c_i}^4 [M^4(p_0)]$. Now, for every atom p, let $N^L(p)$ be some element in $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_{N^4(p)}$. Thus $\omega \circ N^L = N^4$, and N^L is similar to N^4. By Proposition 21, N^L is in particular a model of Γ in $(\mathcal{L}, \mathcal{D})$. Moreover, for every atom p,

$$[\omega \circ N^L(p)] = [N^4(p)] \leq_{c_i}^4 [M^4(p)] = [\omega \circ M^L(p)].$$

Now, by the definition of N^L we have that for every atom p, $N^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$, and by Lemma 22, $M^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ as well. Hence, by Lemma 23, $[N^L(p)] \leq_c^{\mathcal{L},\mathcal{D}} [M^L(p)]$. Similarly,

$$[\omega \circ N^L(p_0)] = [N^4(p_0)] <_{c_i}^4 [M^4(p_0)] = [\omega \circ M^L(p_0)]$$

and again this entails that $[N^L(p_0)] <_c^{\mathcal{L},\mathcal{D}} [M^L(p_0)]$. It follows that $N^L <_c^{\mathcal{L},\mathcal{D}} M^L$, but this contradicts the assumption that M^L is a c-mcm of Γ in $(\mathcal{L}, \mathcal{D})$. ■

Now, by Lemma 24, Theorem 15 is obtained. ■

4. Reasoning with $\models_c^{\mathcal{L},\mathcal{D}}$

In this section we consider some basic properties of $\models_c^{\mathcal{L},\mathcal{D}}$. By Theorem 15, it is sufficient to consider $\mathcal{FOUR}$ and the four corresponding consequence relations $\models_{c_i}^4$ $(i = 0, \ldots, 3)$. Note that the proof of Theorem 15 induces a simple algorithm for determining which one of the basic four-valued consequence relations is the same as a given consequence relation of the form $\models_c^{\mathcal{L},\mathcal{D}}$. See Figure 2 for the details.

```
input:  an inconsistency order ≤_c^{L,D} in (L,D)

choose some m_⊥ ∈ min_{≤_c^{L,D}} T_⊥  and  m_⊤ ∈ min_{≤_c^{L,D}} T_⊤ .

if ([m_⊤] = [t]) then output 0                 /* ⊨_c^{L,D} = ⊨_{c0}^4 */
else if ([m_⊥] = [t]) then output 1            /* ⊨_c^{L,D} = ⊨_{c1}^4 */
else if ([m_⊤] = [m_⊥]) then output 2          /* ⊨_c^{L,D} = ⊨_{c2}^4 */
else output 3                                  /* ⊨_c^{L,D} = ⊨_{c3}^4 */
```

Figure 2. *Finding the equivalent four-valued consequence relation of* $\models_c^{\mathcal{L},\mathcal{D}}$.

4.1. *The relative strength of the basic logics*

PROPOSITION 25. — *Let* Γ *be a set of formulae and* ψ *a formula in* Σ.

a) The consequence relations $\models^4_{c_i}$, $0 \leq i \leq 3$, *are all different.*

b) For every $1 \leq i \leq 3$, *if* $\Gamma \models^4_{c_0} \psi$ *then* $\Gamma \models^4_{c_i} \psi$.

c) No one of $\models^4_{c_1}$, $\models^4_{c_2}$, *and* $\models^4_{c_3}$, *is stronger than the other.*

PROOF. — For the first part, consider the following set:

$$\Gamma = \{\neg q,\ (p \supset q) \vee (\neg q \supset \neg p),\ (\neg p \supset q) \vee (\neg q \supset p)\}.$$

The c^4_i-mcms of Γ are given in Table 5.

Table 5. *The* c^4_i*-mcms of* Γ

	p	q	c^4_0-mcms	c^4_1-mcms	c^4_2-mcms	c^4_3-mcms
M_1	$\bot$	f	+	+	+	+
M_2	$\top$	f	+	–	+	–
M_3	t	$\top$	+	–	+	+
M_4	f	$\top$	+	–	+	+
M_5	$\bot$	$\top$	+	–	–	–
M_6	$\top$	$\top$	+	–	–	–

It is easy to verify that for every $0 \leq i \leq 3$, the consequences of Γ w.r.t. $\models^4_{c_i}$ are different. Let $Th_i(\Gamma) = \{\psi \mid \Gamma \models^4_{c_i} \psi\}$. Then from Table 5 it follows that $Th_0(\Gamma) \subseteq Th_2(\Gamma) \subseteq Th_3(\Gamma) \subseteq Th_1(\Gamma)$. Moreover, $q \supset p \in Th_1(\Gamma) \setminus Th_3(\Gamma)$, $p \supset q \in Th_3(\Gamma) \setminus Th_2(\Gamma)$, and $q \supset (p \vee \neg p) \in Th_2(\Gamma) \setminus Th_0(\Gamma)$, so the inclusions above are proper.

The second part of the claim is obvious. For the last part, note that $p \vee \neg p \in Th_2(\emptyset)$ and $p \vee \neg p \in Th_3(\emptyset)$, while $p \vee \neg p \notin Th_1(\emptyset)$. Thus, by what we have already shown in the first part of this proof, it remains to show that $\models^4_{c_3}$ is not stronger than $\models^4_{c_2}$. For this, consider the following set:

$$\Gamma' = \{p,\ (\neg p \supset q) \supset q,\ q \supset \neg q,\ \neg q \supset q\}.$$

The only c_2^4-mcm of Γ' is $M_1(p) = t$, $M_1(q) = \top$, while the c_3^4-mcms of Γ' are M_1 and $M_2(p) = \top$, $M_2(q) = \bot$. Thus, e.g., $\Gamma' \models^4_{c_2} q$ while $\Gamma' \not\models^4_{c_3} q$. In this case, therefore, $Th_3(\Gamma') \subset Th_2(\Gamma')$. ■

In contrast to the last proposition, in the classical language $\Sigma_{\rm cl}$,[19] the basic consequence relations *are* comparable:

PROPOSITION 26. — *Let Γ be a set of formulae and ψ a formula in $\Sigma_{\rm cl}$. Then:*

a) $\Gamma \models^4_{c_2} \psi$ iff $\Gamma \models^4_{c_3} \psi$.

b) if $\Gamma \models^4_{c_1} \psi$ then $\Gamma \models^4_{c_2} \psi$ and $\Gamma \models^4_{c_3} \psi$.

c) If ψ is a formula in a conjunctive normal form (CNF), none of its conjuncts is a tautology, then $\Gamma \models^4_{c_1} \psi$ iff $\Gamma \models^4_{c_2} \psi$ iff $\Gamma \models^4_{c_3} \psi$.

PROOF. — For item (a), consider the following lemma:

LEMMA 27. — *Let Γ be a set of formulae in $\Sigma_{\rm cl}$, M – a c_2^4-mcm of Γ, and N – a c_3^4-mcm of Γ. Then there is no formula ψ s.t. $M(\psi) = \bot$ or $N(\psi) = \bot$.*

PROOF (OF THE LEMMA). — Since $\{t, f, \top\}$ is closed under $\neg, \vee, \wedge$, it is sufficient to show the lemma only for atomic formulae. We show it for M; the proof for N is similar. Define a transformation $g : \mathcal{FOUR} \to \{t, f, \top\}$ as follows: $g(\bot) = t$, and $g(x) = x$ otherwise. As it is easily verified (by induction on the structure of formulae in $\Sigma_{\rm cl}$), for every formulae ψ in $\Sigma_{\rm cl}$ s.t. $M(\psi)$ is designated, $g \circ M(\psi)$ is designated as well. It follows that $g \circ M$ is also a model of Γ. Since $g \circ M \leq^4_{c_2} M$, necessarily $g \circ M = M$. ■

Now, lemma 27 implies that the set of the c_2^4-mcms of Γ are the same as the set of the c_3^4-mcms of Γ, and so item (a) is obtained.

The first part of item (b) follows from the fact that in $\Sigma_{\rm cl}$, every c_2^4-mcm of Γ is also a c_1^4-mcm of Γ. Indeed, suppose for a contradiction that M is a c_2^4-mcm of Γ and it is not a c_1^4-mcm of Γ, i.e., there is another model N of Γ s.t. $N <^4_{c_1} M$. Define for every atom p a valuation M' as follows: $M'(p) = t$ if $N(p) = \bot$ and $M'(p) = N(p)$ otherwise. Now, it is easy to verify, by induction on the structure of the formulae in $\Sigma_{\rm cl}$, that $M'(\psi) \in \mathcal{D}$ whenever $N(\psi) \in \mathcal{D}$, and so M' is a model of Γ. Also, as for every atom p, the equivalence class w.r.t. $\leq^4_{c_1}$ of $M'(p)$ is the same as that of $N(p)$, and $N <^4_{c_1} M$, it follows that $M' <^4_{c_1} M$. By Lemma 27, then, $M' <^4_{c_2} M$ as well, and this is a contradiction to the assumption that $M \in !(\Gamma, \leq^4_{c_2})$.

The second part of item (b) (the one that is related to $\models^4_{c_3}$) follows from the first part of item (b) together with item (a).

19. I.e., the language of $\{\neg, \wedge, \vee, \to, t, f\}$.

For item (c) it is sufficient to assume that ψ is a disjunction of literals that does not contain an atomic formula and its negation. Assume that $\Gamma \not\models^4_{c_1} \psi$. Then there is a c^4_1-mcm M of Γ s.t. $M(\psi) \notin \{t, \top\}$. Consider the valuation M', defined as follows:

$$M'(p) = \begin{cases} t & \text{if } M(p) = \bot \text{ and } p \text{ does not appear in } \psi \\ f & \text{if } M(p) = \bot \text{ and } p \text{ appears in } \psi \\ M(p) & \text{otherwise} \end{cases}$$

As in part (b), it is easy to verify that M' is a c^4_2-mcm of Γ and $M'(\psi) \notin \{\top, t\}$. Thus, $\Gamma \not\models^4_{c_2} \psi$. Again, the second part of item (c) follows from the first part of this item together with item (a). ■

4.2. *Paraconsistency and relations to classical logic*

In what follows we shall write $\models^2$ for the classical consequence relation, and $\models^4_c$ for any one of $\models^4_{c_i}$, $0 \leq i \leq 3$.

PROPOSITION 28. — $\models^4_c$ *is paraconsistent.*

PROOF. — It is easy to see that reasoning with $\models^4_c$ does not reduce to triviality even when the set of premises is not consistent. For instance, $p, \neg p \not\models^4_c q$. To see that, consider a valuation ν, for which $\nu(p) = \top$ and $\nu(q) = f$. ■

PROPOSITION 29. — *If* $\Gamma \models^4_c \psi$ *then* $\Gamma \models^2 \psi$.

PROOF. — Let M be a classical model of Γ. Since the set $\{t, f\}$ is closed under the operations in Σ, there is no difference between viewing M as a valuation in $\mathcal{FOUR}$ and viewing it as a valuation in $\{t, f\}$. Hence M is also a model of Γ in $\mathcal{FOUR}$. Now, since M assigns only classical truth values to the atomic formulae, M must be a c-mcm of Γ in $\mathcal{FOUR}$. Since $\Gamma \models^4_c \psi$, necessarily $M(\psi) \in \{t, \top\}$. On the other hand, $M(\psi) \in \{t, f\}$, and so $M(\psi) = t$. It follows that M is a classical model of ψ, thus $\Gamma \models^2 \psi$. ■

The converse of Proposition 29 is not true in general. For instance, excluded middle is not valid w.r.t. $\models^4_{c_0}$ and $\models^4_{c_1}$. However, with respect to the other basic four-valued consequence relations, the converse of Proposition 29, applied on classically consistent theories, does hold.

PROPOSITION 30. — *Let* Γ *be a classically consistent theory. Then for every formula* ψ *in* Σ, $\Gamma \models^2 \psi$ *iff* $\Gamma \models^4_{c_2} \psi$ *iff* $\Gamma \models^4_{c_3} \psi$.

PROOF. — Immediately follows from the fact that the set of the c^4_2-mcms and the set of the c^4_3-mcms of a classically consistent theory Γ are the same as the set of the classical models of Γ. ■

By Propositions 28 and 30 it follows that with (any consequence relation of the form $\models^{\mathcal{L},\mathcal{D}}_c$ that is equivalent to) $\models^4_{c_2}$ and $\models^4_{c_3}$ one can draw classical conclusions from (classically) consistent theories, while the set of conclusions is not 'exploded' when the theory becomes inconsistent. Batens [BAT 98, BAT 00a] describes this property as an 'oscillation' between some lower limit (paraconsistent) logic and an upper limit (classical) logic.

4.3. *Monotonicity and transitivity*

PROPOSITION 31. — *$\models^4_{c_0}$ is a monotonic consequence relation, while $\models^4_{c_i}$, $i = 1, 2, 3$, are nonmonotonic relations.*

PROOF. — For the first part, note that $\models^4_{c_0}$ is in fact the same as $\models^4$, which is clearly monotonic. For the other part, consider $\Gamma = \{p, \neg p \vee q\}$. Since $M(p) = t$, $M(q) = t$ is the only c^4_i-mcm of Γ for $i = 1, 2, 3$, it follows that $\Gamma \models^4_{c_i} q$ $(i = 1, 2, 3)$. However, as in the proof of Proposition 28, it is easy to see that $\Gamma, \neg p \not\models^4_{c_i} q$ for $i = 1, 2, 3$.

For another example, consider again Example 13. As we have shown, for every $1 \leq i \leq 3$, $\Gamma_1 \models^4_{c_i}$ $\texttt{fly(Tweety)}$, while $\Gamma_1, \texttt{penguin(Tweety)} \not\models^4_{c_i}$ $\texttt{fly(Tweety)}$. ■

The last proposition implies that unless the inconsistency order under consideration is degenerated, $\models^{\mathcal{L},\mathcal{D}}_c$ is not monotonic, and so it is not a consequence relation in the standard sense of Tarski [TAR 41]. In such cases it is usual to require a weaker condition (see, e.g., [GAB 85, KRA 90]):

PROPOSITION 32. — *$\models^4_c$ satisfies cautious (left) monotonicity: if $\Gamma \models^4_c \psi$ and $\Gamma \models^4_c \phi$, then $\Gamma, \psi \models^4_c \phi$.*

PROOF. — Assume that $\Gamma \models^4_c \psi$, and $\Gamma \models^4_c \phi$. Let M be some c-mcm of $\Gamma \cup \{\psi\}$. In particular, M is a model of Γ. Moreover, it must be a c-mcm of Γ as well, since otherwise there would have been some $N \in mod(\Gamma)$ s.t. $N <^4_c M$. Since $\Gamma \models^4_c \psi$, this N would have been a model of $\Gamma \cup \{\psi\}$ which is strictly $\leq^4_c$-smaller than M. Hence M cannot be a c-mcm of $\Gamma \cup \{\psi\}$, with a contradiction to the choice of M. Therefore, M is a c-mcm of Γ. Now, since $\Gamma \models^4_c \phi$, M is a model of ϕ. Hence $\Gamma, \psi \models^4_c \phi$. ■

A desirable property of non-monotonic consequence relations is the ability to preserve any conclusion when learning about a new fact that has no influence on the set of premises. Consequence relations that satisfy this property are called *rational* [LEH 92]. The next proposition shows that $\models^4_{c_i}$ $(i = 0, \ldots, 3)$ are rational.

PROPOSITION 33. — *If $\Gamma \models^4_c \psi$ and $\mathcal{A}(\Gamma \cup \{\psi\}) \cap \mathcal{A}(\phi) = \emptyset$, then $\Gamma, \phi \models^4_c \psi$.*[20]

Intuitively, the second condition in Proposition 33 guarantees that ϕ is 'irrelevant' for Γ and ψ. The intuitive meaning of Proposition 33 is, therefore, that the reasoner does not have to retract ψ when learning that ϕ holds.

PROOF. — Otherwise, $\Gamma, \phi \not\models^4_c \psi$, so there is an $M \in !(\Gamma \cup \phi, \leq^4_c)$ s.t. $M(\psi) \notin \{t, \top\}$. Let m be some $\leq^4_c$-minimal element. Consider the following valuation:

$$N(p) = \begin{cases} M(p) & \text{if } p \in \mathcal{A}(\Gamma \cup \psi), \\ m & \text{otherwise.} \end{cases}$$

20. Recall that $\mathcal{A}(\Gamma)$ is the set of atomic formulae that appear in some formula of Γ.

Clearly, N is a model of Γ and $N(\psi) \notin \{t, \top\}$. Since $\Gamma \models^4_c \psi$, N cannot be a c-mcm of Γ, and so there is a model N' of Γ s.t. $N' <^4_c N$. By the definition of N, there is some $p_0 \in \mathcal{A}(\Gamma \cup \psi)$ s.t. $N'(p_0) <^4_c N(p_0)$. Now, consider the following valuation:

$$M'(p) = \begin{cases} N'(p) & \text{if } p \in \mathcal{A}(\Gamma \cup \psi), \\ M(p) & \text{otherwise.} \end{cases}$$

Clearly, $M' <^4_c M$, and since M' is the same as N' on $\mathcal{A}(\Gamma)$, M' is also a model of Γ. Moreover, using the facts that $\mathcal{A}(\Gamma \cup \psi) \cap \mathcal{A}(\phi) = \emptyset$ and that M is a model of ϕ, it follows that M' is also a model of ϕ. Hence M' is a model of $\Gamma \cup \{\phi\}$, which is strictly $\leq^4_c$-smaller than M, but this is a contradiction to the choice of M. ■

NOTE 34. — In order to assure rationality, Lehmann and Magidor [LEH 92] introduced the rule of *rational monotonicity:*

$$\text{if } \Gamma \mid\sim \psi \text{ then } \Gamma, \phi \mid\sim \psi, \text{ unless } \Gamma \mid\sim \neg\phi.$$

Rational monotonicity may be considered as too strong for assuring rationality, and there are many general patterns of nonmonotonic reasoning that do not satisfy this rule. For instance, although $\models^4_{c_1}$ is rational (by Proposition 33), it does not satisfy rational monotonicity. To see this consider, e.g., $\Gamma = \{p,\ q \supset \neg p\}$, $\psi = \neg p \supset \neg q$, and $\phi = q$.

PROPOSITION 35. — $\models^4_c$ *is a Tarskian cautious consequence relation in the sense of* [ARI 99, ARI 00b]*, i.e., it is reflexive* ($\Gamma \models^4_c \psi$ *for every* $\psi \in \Gamma$)*, and satisfies cautious monotonicity* (*see Proposition 32*) *and cautious cut* (*if* $\Gamma \models^4_c \psi$ *and* $\Gamma, \psi \models^4_c \phi$*, then* $\Gamma \models^4_c \phi$).

PROOF. — Reflexivity easily follows from the definition of $\models^4_c$, and cautious monotonicity is shown in Proposition 32. It remains to show cautious cut (transitivity): suppose that $\Gamma \models^4_c \psi$ and $\Gamma, \psi \models^4_c \phi$. We shall show that this entails that every $\leq^4_c$-mcm of Γ is also a model of ϕ (and so $\Gamma \models^4_c \phi$). Indeed, let $M \in mod(\Gamma)$. Since $\Gamma \models^4_c \psi$, M is a model ψ, thus $M \in mod(\Gamma \cup \{\psi\})$. Now, M is also a c-mcm of $\Gamma \cup \{\psi\}$, otherwise there would have been some $N \in mod(\Gamma \cup \{\psi\})$ s.t. $N <^4_c M$. In particular, this N would be a model of Γ which is strictly $\leq^4_c$-smaller than M, and this is a contradiction to the choice of M as a c-mcm of Γ. The fact that M is a model of ϕ follows now from the assumption that $\Gamma, \psi \models^4_c \phi$. ■

4.4. *Inconsistency adaptation*

We conclude this section by showing that $\models^4_{c_2}$ and $\models^4_{c_3}$ are, in terms of Batens [BAT 98, BAT 00a, BAT 02], *adaptive*: if it is possible to distinguish between a consistent part and an inconsistent part of a given theory, then every assertion that classically follows from the consistent part, and is not related to the inconsistent part, is also a $\models^4_{c_i}$-consequence $(i = 2, 3)$ of the whole theory. Thus, while $\models^4_{c_2}$ and $\models^4_{c_3}$

handle inconsistent theories in a nontrivial way (Proposition 28), they presuppose a consistency of all the assertions 'unless and until proven otherwise'.

PROPOSITION 36. — *Let* $\Gamma = \Gamma' \cup \Gamma''$ *be a set of formulae in* Σ *s.t.* Γ' *is classically consistent and* $\mathcal{A}(\Gamma') \cap \mathcal{A}(\Gamma'') = \emptyset$*. Then for every* ψ *s.t.* $\mathcal{A}(\psi) \cap \mathcal{A}(\Gamma'') = \emptyset$*, the fact that* $\Gamma' \models^2 \psi$ *entails that* $\Gamma \models^4_{c_2} \psi$ *and* $\Gamma \models^4_{c_3} \psi$*.*

PROOF. — We show here the case of $\models^4_{c_2}$; the proof for $\models^4_{c_3}$ is similar. Suppose that $\Gamma' \models^2 \psi$. By Proposition 30, $\Gamma' \models^4_{c_2} \psi$, and by Proposition 33, since $\mathcal{A}(\Gamma' \cup \psi) \cap \mathcal{A}(\Gamma'') = \emptyset$, we have that $\Gamma \models^4_{c_2} \psi$. ■

EXAMPLE 37. — Consider the set $\Gamma = \{p,\ \neg p,\ q,\ \neg p \vee r,\ \neg q \vee s\}$. A plausible inference system should *not* apply here the Disjunctive Syllogism on $\Gamma'' = \{p, \neg p \vee r\}$ for concluding that r follows from Γ. The reason for this is that $\neg p$ is also true in Γ, and so $\neg p \vee r$ holds even in cases that r is false. On the other hand, applying the Disjunctive Syllogism on the subset $\Gamma' = \{q,\ \neg q \vee s\}$ (for concluding s from Γ) may be justified by the fact that Γ' should not be affected by the inconsistency in Γ, therefore inference rules that are classically valid can be applied on its elements. Now, since Γ can be split-up to two separated subsets, one (Γ') is consistent, and the other (Γ'') is inconsistent, it follows from Proposition 36 that $\Gamma \models^4_{c_2} s$ and $\Gamma \models^4_{c_3} s$. Also, $\Gamma \not\models^4_{c_2} r$ and $\Gamma \not\models^4_{c_3} r$, as indeed intuitively expected.

5. Computability

A general method for reducing questions of consequences in preferential structures to computations of classical entailments is introduced in [ARI 02b, ARI 03b]. This approach is based on a definition of appropriate circumscriptive axioms to capture the notion of minimality and for representing preferential reasoning. In this section we incorporate this method in our framework, and show how reasoning with (graded) uncertainty can be implemented using algorithms for processing circumscriptive theories (such as those of [OHL 96, DOH 97]). As the underlying language of these algorithms is the (propositional or first-order) classical one, our computational method is applied on theories in the classical fragment of Σ, namely: $\Sigma_{\rm cl} = \{\neg, \wedge, \vee, \rightarrow, f, t\}$.

Given a theory Γ, the first step according to the approach of [ARI 02b, ARI 03b], is to apply on it the following transformation, that essentially serves as a separator of negated atoms from affirmed ones:

DEFINITION 38 ([ARI 03B]). — *Let* ψ *be a formula in* $\Sigma_{\rm cl}$*. Denote by* $\overline{\psi}$ *the formula that is obtained from* ψ *by substituting every positive occurrence in* ψ *of an atomic formula* p *by a new symbol* p^+*, and replacing every negative occurrence in* ψ *of an atomic formula* p *by* $\neg p^-$*.*[21] *The language that is obtained from* $\Sigma_{\rm cl}$ *by introducing these new symbols is denoted by* $\Sigma^{\pm}_{\rm cl}$*. Given a set* Γ *of formulae in* $\Sigma_{\rm cl}$*, we denote the set* $\{\overline{\psi} \mid \psi \in \Gamma\}$ *by* $\overline{\Gamma}$*.*

21. An occurrence of p in ψ is called *positive* if it appears in the scope of an even number of negation operators; otherwise, it is a *negative* occurrence.

EXAMPLE 39. — Let $\psi = \neg(p \vee \neg q) \vee \neg q$. The first appearance of q in ψ is positive, and the second appearance of q in ψ as well as the appearance of p in ψ are negative. Thus, $\overline{\psi} = \neg(\neg p^- \vee \neg q^+) \vee \neg\neg q^-$.

Consider now the following transformation between four-valued valuations of formulae in $\Sigma_{\rm cl}$ and two-valued valuations of formulae in $\Sigma^{\pm}_{\rm cl}$:

DEFINITION 40 ([ARI 02B, ARI 03B]). —

$$\nu(p) = t \iff \overline{\nu}(p^+) = 1,\ \overline{\nu}(p^-) = 0, \qquad \nu(p) = f \iff \overline{\nu}(p^+) = 0,\ \overline{\nu}(p^-) = 1,$$
$$\nu(p) = \top \iff \overline{\nu}(p^+) = 1,\ \overline{\nu}(p^-) = 1, \qquad \nu(p) = \bot \iff \overline{\nu}(p^+) = 0,\ \overline{\nu}(p^-) = 0.$$

A key result for the computational method is the following:

LEMMA 41 ([ARI 02B, ARI 03B]). — $\nu \in mod(\psi)$ *iff* $\overline{\nu} \in mod(\overline{\psi})$.

Lemma 41 immediately entails the following result:

PROPOSITION 42 ([ARI 02B, ARI 03B]). — $\Gamma \models^4 \psi$ *iff* $\overline{\Gamma} \models^2 \overline{\psi}$.

Since, by its definition, $\models^4$ is the same as $\models^4_{c_0}$, we actually have a method of computing $\models^4_{c_0}$ by $\models^2$:

COROLLARY 43. — $\Gamma \models^4_{c_0} \psi$ *iff* $\overline{\Gamma} \models^2 \overline{\psi}$.

Corollary 43 implies, in particular, that reasoning with $\models^4_{c_0}$ may be implemented by two-valued theorem provers. Moreover, since $\overline{\Gamma}$ is obtained from Γ in a polynomial time, computing consequences in this case is *polynomially reducible* to computations of classical entailment.

We turn now to the other three basic types of logics of the form $\models^{\mathcal{L},\mathcal{D}}_c$, namely: $\models^4_{c_1}$, $\models^4_{c_2}$, and $\models^4_{c_3}$. Recall that with respect to the language $\Sigma_{\rm cl}$, $\models^4_{c_2}$ and $\models^4_{c_3}$ are actually the same relation (item (a) of Proposition 26). It is sufficient, therefore, to consider only $\models^4_{c_1}$ and $\models^4_{c_2}$.

In order to simulate reasoning with $\models^4_{c_1}$ and $\models^4_{c_2}$ by classical entailment we should first represent the inconsistency orders $\leq^4_{c_i}$ $(i=1,2)$. Following [ARI 02b, ARI 03b], this is accomplished by introducing a circumscription axiom Circ_i that expresses $\leq^4_{c_i}$ objectively, using a formula C_i:

DEFINITION 44. — *For a set* $\vec{p} = \{p_1, p_2, \ldots, p_n\}$ *of atoms in* $\Sigma_{\rm cl}$,[22] *let* $\vec{p}^{\,\pm} = \{p_1^+, p_1^-, p_2^+, p_2^-, \ldots, p_n^+, p_n^-\}$ *be the corresponding set of atoms in* $\Sigma^{\pm}_{\rm cl}$, *and* $\vec{q}^{\,\pm}$ *– a renaming in* $\Sigma^{\pm}_{\rm cl}$ *of* $\vec{p}^{\,\pm}$. *Now,*

$$\mathsf{C}_1(\vec{p}^{\,\pm}, \vec{q}^{\,\pm}) = \bigwedge_{i=1}^{n} \Big((p_i^+ \wedge p_i^-) \rightarrow (q_i^+ \wedge q_i^-) \Big),$$

$$\mathsf{C}_2(\vec{p}^{\,\pm}, \vec{q}^{\,\pm}) = \bigwedge_{i=1}^{n} \Big(\big((p_i^+ \wedge p_i^-) \vee (\neg p_i^+ \wedge \neg p_i^-)\big) \rightarrow \big((q_i^+ \wedge q_i^-) \vee (\neg q_i^+ \wedge \neg q_i^-)\big) \Big).$$

22. In fact, for checking the entailment $\Gamma \models^4_{c_i} \psi$, it is sufficient to take $n = |\mathcal{A}(\Gamma \cup \{\psi\})|$.

For a finite set Γ *of formulae in* $\Sigma_{\rm cl}$ *depending on* $\vec{p}^{\,\pm}$*, and* $i = 1, 2$*, denote:*

$$\mathsf{Circ}_i^{\Gamma}(\vec{p}^{\,\pm}) \;=\; \forall(\vec{q}^{\,\pm}) \Big(\bigwedge_{\psi\,\in\,\Gamma} \overline{\psi}(\vec{q}^{\,\pm}) \;\rightarrow\; \big(\, \mathsf{C}_i(\vec{q}^{\,\pm}, \vec{p}^{\,\pm}) \rightarrow \mathsf{C}_i(\vec{p}^{\,\pm}, \vec{q}^{\,\pm}) \,\big) \Big).$$

Now we are ready to give a general characterization of reasoning with $\models^4_c$ in terms of 'formula circumscription' [MCC 86] and two-valued entailment:

THEOREM 45. — *Let* Γ *be a finite set of formulae and* ψ *a formula in* $\Sigma_{\rm cl}$*. Let* Circ_i^{Γ} $(i = 1, 2)$ *be the formula given in Definition 44. Then* $\Gamma \models^4_{c_i} \psi$ *iff* $\overline{\Gamma}, \mathsf{Circ}_i^{\Gamma} \models^2 \overline{\psi}$*.*

Theorem 45 immediately follows from the following proposition:

PROPOSITION 46. — *Let* Γ *be a finite set of formulae in* $\Sigma_{\rm cl}$*. Then* ν *is a* c^4_i*-mcm of* Γ *(*$i = 1, 2$*) iff* $\overline{\nu}$ *is a model of* $\overline{\Gamma}$ *and* Circ_i^{Γ}*.*

PROOF. — By Lemma 41, ν is a model of Γ iff $\overline{\nu}$ is a model of $\overline{\Gamma}$. It remains to show, then, that the fact that $\overline{\nu}$ satisfies Circ_i^{Γ} is a necessary and sufficient condition for assuring that ν is $\leq^4_{c_i}$-minimal among the models of Γ. Indeed, let $(\vec{p}^{\,\pm} : \overline{\nu},\, \vec{q}^{\,\pm} : \overline{\mu})$ be the two-valued interpretation that interprets the symbols in $\vec{p}^{\,\pm}$ according to $\overline{\nu}$ and the symbols in $\vec{q}^{\,\pm}$ according to $\overline{\mu}$. It is easy to see that $\nu \leq^4_{c_i} \mu$ iff $(\vec{p}^{\,\pm} : \overline{\nu},\, \vec{q}^{\,\pm} : \overline{\mu})$ satisfies $\mathsf{C}_i(\vec{p}^{\,\pm}, \vec{q}^{\,\pm})$. It follows, therefore, that $\overline{\nu}$ satisfies Circ_i^{Γ} iff for every valuation μ that satisfies Γ and for which $\mu \leq^4_{c_i} \nu$, it is also true that $\nu \leq^4_{c_i} \mu$. Thus, $\overline{\nu}$ satisfies Circ_i^{Γ} iff there is no model μ of Γ such that $\mu <^4_{c_i} \nu$, iff $\nu \in\, !(\Gamma, \leq^4_{c_i})$. ■

By Theorem 45 and Proposition 26(a) we also have the following result:

COROLLARY 47. — *Let* Γ *be a finite set of formulae and* ψ *a formula in* $\Sigma_{\rm cl}$*. Then* $\Gamma \models^4_{c_3} \psi$ *iff* $\overline{\Gamma}, \mathsf{Circ}_2^{\Gamma} \models^2 \overline{\psi}$*.*

Now, entailments of the form $\Gamma \models^{\mathcal{L},\mathcal{D}}_c \psi$ (where $\leq^{\mathcal{L},\mathcal{D}}_c$ is an inconsistency order in $(\mathcal{L}, \mathcal{D})$, and $\Gamma \cup \{\psi\}$ is a finite set of formulae in $\Sigma_{\rm cl}$) can be reduced to questions of classical entailments by the algorithm of Figure 3.

compute $\overline{\Gamma}$ and $\overline{\psi}$
execute the algorithm of Figure 2 to determine the type of $\leq^{\mathcal{L},\mathcal{D}}_c$
let i be the type of $\leq^{\mathcal{L},\mathcal{D}}_c$ /* i.e., $\models^{\mathcal{L},\mathcal{D}}_c \;=\; \models^4_{c_i}$ */
- if $i = 0$, check if $\overline{\Gamma} \models^2 \overline{\psi}$
- if $i = 1$, compute Circ_1 and check if $\overline{\Gamma}, \mathsf{Circ}_1^{\Gamma} \models^2 \overline{\psi}$
- otherwise, compute Circ_2 and check if $\overline{\Gamma}, \mathsf{Circ}_2^{\Gamma} \models^2 \overline{\psi}$

Figure 3. *Algorithm for converting* $\models^{\mathcal{L},\mathcal{D}}_c$*-entailments to classical entailments.*

Next we consider the computational complexity of reasoning with $\models^{\mathcal{L},\mathcal{D}}_c$. Again, by Theorem 15 and the algorithm of Figure 2, the problem is polynomially reducible to the four-valued case, so it is sufficient to consider only four-valued logics. As the following proposition shows, checking entailments with respect to such consequence relations is on the second level of the polynomial hierarchy.

PROPOSITION 48. — *Let* Γ, ψ *be in* Σ. *Then*

1) Checking if $\Gamma \models^4_{c_0} \psi$ *is* coNP*-complete; if* Γ *is in CNF and* ψ *is an atom, the same decision problem is in* P.

2) Checking if $\Gamma \models^4_{c_i} \psi$*, where* $1 \leq i \leq 3$*, is* Π^{P}_2*-complete (even if* Γ *is in CNF and* ψ *is an atom).*

NOTE 49. — The results in Proposition 48 remain the same also when ψ and the formulae of Γ are in $\Sigma_{\rm cl}$.

The proof of Proposition 48 and Note 49 is a direct adaptation to our framework of the proof of Proposition 2 in [COS 02]; the interested reader is referred to the appendix of that paper. A computational analysis of the algorithm of Figure 3 provides an alternative proof for Note 49, as it shows that verifying $\models^4_{c_0}$-inferences is equivalent to checking classical entailment, and deciding $\models^4_{c_i}$-consequences of a finite theory in $\Sigma_{\rm cl}$ is equivalent to validity checking of formulae with exactly one quantifier alternation (known to be Π^{P}_2-complete, see [WRA 76]).

6. Summary and concluding remarks

We have introduced a family of preferential logics that are useful for reasoning with different degrees of incomplete and inconsistent information. It is shown that these logics can be characterized in terms of four-valued consequence relations, and that they can be computed by algorithms that process second-order circumscriptive axioms.

As we have shown, the 'basic' non-degenerated consequence relations (i.e., $\models^4_{c_i}$ where $1 \leq i \leq 3$) are all different, and none of them is strictly stronger than the other. The decision which one should be used in practice depends, therefore, on further considerations. For instance, the fact that $\models^4_{c_2}$ and $\models^4_{c_3}$ are the same as $\models^2$ as long as the set of premises is classically consistent (but they are not trivial w.r.t. inconsistent set of premises), may be considered as an advantage for applications that need to draw classical conclusions from (classically) consistent theories. For other languages, the relative strength of the consequence relations may be a possible criterion for choosing the most appropriate formalism (see, e.g., Proposition 26 for a comparison of consequence relations with respect to the classical language).

It is worth noting that the logics that are defined here generalize some other formalisms, considered elsewhere in the literature for similar goals. The next proposition shows some examples of this.

PROPOSITION 50. — [23]

– $\models^4_{c_0}$ *coincides with Belnap's four-valued logic* [BEL 77a, BEL 77b]*, and it can be used for implementing Kleene's* [KLE 50] *and Priest's* [PRI 89, PRI 91] *three-valued logics (denoted, respectively,* $\models^3_{\rm Kl}$ *and* $\models^3_{\rm LP}$*): for* $\mathcal{A}(\Gamma \cup \{\psi\}) = \{p_1, p_2, \ldots\}$*,*

$$\Gamma \models^3_{\rm Kl} \psi \ \textit{ iff } \ \Gamma,\ (p_1 \wedge \neg p_1) \supset f,\ (p_2 \wedge \neg p_2) \supset f,\ \ldots \models^4_{c_0} \psi$$

$$\Gamma \models^3_{\rm LP} \psi \ \textit{ iff } \ \Gamma,\ p_1 \vee \neg p_1,\ p_2 \vee \neg p_2\ \ldots \models^4_{c_0} \psi$$

– $\models^4_{c_1}$ *and* $\models^4_{c_2}$ *are, respectively, the same as the logics* $\models^4_{\mathcal{I}_1}$ *and* $\models^4_{\mathcal{I}_2}$ *of* [ARI 98a, ARI 98b].[24] *It follows, therefore, that these logics capture the notions of inclusion of inconsistency sets and* $\mathcal{I}$*-mcms, introduced in* [ARI 98a, ARI 98b].

– $\models^4_{c_i}$ $(i=1,2,3)$ *can be used for implementing Priest's three-valued preferential logic LPm* $(\models^3_{\rm LPm})$ [PRI 89, PRI 91]*, for reasoning with minimal inconsistency:*[25] *if* $\mathcal{A}(\Gamma \cup \{\psi\}) = \{p_1, p_2, \ldots\}$*, then* $\Gamma \models^3_{\rm LPm} \psi$ *iff* $\Gamma, p_1 \vee \neg p_1, p_2 \vee \neg p_2 \ldots \models^4_{c_i} \psi$*.*

PROOF (OUTLINE). — The consequence relation $\models^4_{c_0}$ and Belnap's four-valued logic are the same simply because in both cases conclusions are based on all the four-valued models of the premises. The relation of $\models^4_{c_0}$ to the three-valued logic $\models^3_{\rm Kl}$ (respectively, $\models^3_{\rm LP}$) follows from the fact that the $\{t, f, \bot\}$-models (respectively, the $\{t, f, \top\}$-models) of Γ are the same as the four-valued models of $\Gamma \cup \{p_1 \wedge \neg p_1 \supset f,\ p_2 \wedge \neg p_2 \supset f, \ldots\}$ (respectively, the four-valued models of $\Gamma \cup \{p_1 \vee \neg p_1,\ p_2 \vee \neg p_2, \ldots\}$).

For the second item, consider an inconsistency set $\mathcal{I}$ (in the sense of [ARI 98a, ARI 98b]) in a logical lattice $(\mathcal{L}, \mathcal{D})$. It induces an inconsistency order $\leq_c$ in $(\mathcal{L}, \mathcal{D})$, where $x \leq_c y$ iff $y \in \mathcal{I}$ and $x \notin \mathcal{I}$. In $\mathcal{FOUR}$, there are two inconsistency sets $\mathcal{I}_1 = \{\top\}$ and $\mathcal{I}_2 = \{\top, \bot\}$, so the corresponding inconsistency orders in this case are $\leq^4_{c_1}$ and $\leq^4_{c_2}$, respectively. Also, it is easy to verify that a four-valued valuation M is an $\mathcal{I}_j$-mcm of a theory Γ iff it is a c^4_j-mcm of Γ $(j=1,2)$, and so for every ψ in Σ and for $j = 1, 2$, we have that $\Gamma \models^4_{\mathcal{I}_j} \psi$ iff $\Gamma \models^4_{c_j} \psi$.

The last item immediately follows from the previous item and Proposition 69 of [ARI 98b], which states a similar property as in our proposition, but for the consequence relations $\models^4_{\mathcal{I}_j}$ $(j = 1, 2)$. ■

A major goal for future work is to carry on the ideas of Section 5, so that *practical* reasoning with $\models^{\mathcal{L},\mathcal{D}}_c$ will become more feasible. As making decisions in the presence of (different types and levels of) uncertainty is a matter of everyday life, this challenge is certainly worthy.

23. In what follows we shall assume that the reader is familiar with the relevant formalisms, and so we shall omit the corresponding definitions. For more details, one may check the references that appear in the proposition.
24. Note, however, that there is no equivalent in [ARI 98a, ARI 98b] to $\models^4_{c_3}$.
25. Also known as the logic $\models_m$ of Besnard et al., see [BES 03].

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