Kansas State University

Department of Mathematics

Real and Complex Analysis Qualifying Exams (New System) Solution Manual

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How to Use this Manual

This solution manual has been constructed so that each chapter represents a specific semester's qualifying exam. Within each chapter, the problems are listed in the order in which they originally appeared, and moreover, they have been transcribed almost verbatim. The primary exception to this is the omission of hints.

Following the statement of each problem, the reader will find a list of "key terms." These are the important definitions and theorems relevant to the given solution. In most cases, these are concepts or results that the test-taker is presumed to be knowledgable of. As such, these concepts have been used freely throughout the solutions.

After the list of key terms, the reader will find a solution to the given problem. These solutions should *not* be preferred over others, although some are canonical.

Notation

$\mathbb{Z} :=$ Integers

$\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$\mathbb{R} :=$ Real numbers

$\mathbb{C} :=$ Complex numbers

$S^+ := S \cap (0, \infty)$ $\quad (S \subseteq \mathbb{C})$

$D(z,r) := \{w \in \mathbb{C} : |z-w| < r\}$ $\quad (z \in \mathbb{C}, r > 0)$

$\bar{D}(z,r) := \{w \in \mathbb{C} : |z-w| \leq r\}$ $\quad (z \in \mathbb{C}, r > 0)$

$D'(z,r) := \{w \in \mathbb{C} : 0 < |z-w| < r\}$ $\quad (z \in \mathbb{C}, r > 0)$

$\mathbb{D} := D(0,1)$

$\bar{\mathbb{D}} := \bar{D}(0,1)$

$\mathbb{D}' := D'(0,1)$

$C(z,r) := \{w \in \mathbb{C} : |z-w| = r\}$ $\quad (z \in \mathbb{C}, r > 0)$

$\mathbb{T} := C(0,1)$

$|\ | :=$ Lebesgue measure on $\mathbb{R}^k$ $\quad (k \in \mathbb{N})$

$E^c := \{x \in X : x \notin E\}$ $\quad (E \subseteq X)$

$$\mathbf{1}_E(x) := \begin{cases} 1 & x \in E \\ 0 & x \in E^c \end{cases} \quad (E \subseteq X)$$

$\coprod :=$ disjoint union

$B(x,r) := \{y \in X : d(x,y) < r\}$ $\quad (d \text{ a metric on } X,\ r > 0)$

Contents

CHAPTER 1

Spring 2004

Problem 1.1. *Let $f : [0,1] \to \mathbb{C}$ be continuous and define $F : \Omega := \mathbb{C}\backslash[0,1] \to \mathbb{C}$ by*

$$F(z) := \int_0^1 \frac{f(t)}{t-z} dt.$$

Prove that F is holomorphic in Ω.

Key terms: Morera's Theorem, Fubini's Theorem, Lebesgue's Dominated Convergence Theorem.

Solution. Let $a_1, a_2, a_3 \in \mathbb{C}$ be such that the closed convex hull Δ of the oriented triangle $T := [a_1, a_2, a_3, a_1]$ is contained in Ω. (Here we are using the notation $[a_1, ..., a_n] := \bigcup_{m=1}^{n-1} [a_m, a_{m+1}]$, where $[a_i, a_j]$ is the oriented complex interval.) Then by Fubini's Theorem, we have

$$\begin{aligned} \int_T F(z)dz &= \int_T \left[\int_0^1 \frac{f(t)}{t-z} dt \right] dz = \int_0^1 \left[\int_T \frac{1}{t-z} dz \right] f(t)dt \\ &= \int_0^1 \left[2\pi i \mathrm{Ind}_T(t) \right] f(t)dt = \int_0^1 0 \cdot f(t)dt = 0. \end{aligned}$$

As this holds for every triangle whose closed convex hull is contained in Ω, Morera's Theorem implies that $F \in H(\Omega)$.

Alternative Solution. This proof does not involve the use of Morera's Theorem. First note that since f is continuous and $[0,1]$ is compact, we may find $M \in \mathbb{N}$ such that $|f(t)| \leq M$, for every $t \in [0,1]$. Now, fix $z_0 \in \Omega$. Since Ω is open, we may find $r > 0$ such that $D := D(z_0, r) \subset \Omega$, and therefore, $D \cap [0,1] = \varnothing$. In consequence, we see that

$$(1.1.1) \qquad |t - z| \geq r \qquad (z \in D, t \in [0,1]).$$

For $z \in D$, define $g_z : [0,1] \to \mathbb{C}$ by $g_z(t) := f(t)/[(t-z)(t-z_0)]$. By (1.1.1), we conclude that the collection each g_z is well-defined, and moreover, $|g_z(t)| \leq M/r^2$, for every $z \in D$ and $t \in [0,1]$. From this, it follows that $g_z \in L^1([0,1])$, for every $z \in D$. Finally, as it is obvious that $\lim_{z \to z_0} g_z(t) = g_{z_0}(t)$, for every $t \in [0,1]$, Lebesgue's Dominated Convergence Theorem ensures that

$$\lim_{z \to z_0} \int_0^1 \frac{f(t)}{(t-z)(t-z_0)} dt = \int_0^1 \frac{f(t)}{(t-z_0)^2} dt. \tag{1.1.2}$$

For each $z \in D'(z_0, r)$, observe that

$$\begin{aligned} \frac{F(z) - F(z_0)}{z - z_0} &= \frac{1}{z - z_0} \left[\int_0^1 \frac{f(t)}{t-z} dt - \int_0^1 \frac{f(t)}{t - z_0} dt \right] \\ &= \frac{1}{z - z_0} \int_0^1 \frac{f(t)(t-z_0) - f(t)(t-z)}{(t-z)(t-z_0)} dt \\ &= \int_0^1 \frac{f(t)}{(t-z)(t-z_0)} dt. \end{aligned} \tag{1.1.3}$$

Letting $z \to z_0$ in (1.1.3) and appealing to (1.1.2) shows that $F'(z_0)$ exists, and in fact,

$$F'(z_0) = \int_0^1 \frac{f(t)}{(t-z_0)^2} dt.$$

As z_0 is an arbitrary point of Ω, we are finished.

□

Problem 1.2. *Let $\Omega := \{z \in \mathbb{C} : 1 < |z| < 2\}$. Show that there does not exist a sequence $\{P_n\}$ of polynomials in z such that $P_n(z) \to 1/z$ uniformly, for every $z \in \Omega$.*

Key terms: uniform convergence, index of closed curve.

Solution. Assume, with a view to reach a contradiction, that $\{P_n\}$ is a sequence of polynomials converging to $1/z$ uniformly. Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be the closed curve given by $\gamma(t) = \frac{3}{2} e^{it}$. Since each polynomial is a continuous derivative in Ω, we see that $\int_\gamma P_n(z) dz = 0$, for every n. On the other hand, $\int_\gamma (1/z) dz = 2\pi i \mathrm{Ind}_\gamma(0) = 2\pi i$. The uniform convergence of the P_n on γ to $1/z$ ensures that $2\pi i = \int_\gamma \frac{dz}{z} = \lim_{n \to \infty} \int_\gamma P_n(z) dz = 0$. This is the desired contradiction.

□

Problem 1.3. *Evaluate* $\displaystyle\int_0^\infty \frac{dx}{1+x^7}$.

Key terms: contour integration, pole, Residue Theorem, residue.

Solution. Let $R > 1$ and C_R be the path given as the sum of the path[1] $t \mapsto Re^{it}$, for $0 \le t \le \frac{2\pi}{7}$ and the oriented intervals $[Re^{\frac{2\pi i}{7}}, 0]$, and $[0, R]$.

Put $f(z) := \frac{1}{1+z^7}$. Note that f has exactly one simple pole lying inside the region bounded by C_R at $z_0 := e^{\frac{\pi i}{7}}$. Since $z_0^7 = -1$, we have

$$\begin{aligned}(1.3.1)\qquad \int_{C_R} f(z)dz &= 2\pi i \mathrm{Res}(f, z_0) = 2\pi i \lim_{z\to z_0}(z - z_0)f(z)\\ &= 2\pi i \lim_{z\to z_0} \frac{z - z_0}{z^7 - z_0^7} = 2\pi i \frac{1}{\frac{d}{dz}z^7\big|_{z=z_0}} = \frac{2\pi i}{7z_0^6}.\end{aligned}$$

Now, since $R > 1$, we have $\left|\frac{R}{1+(Re^{it})^7}\right| \le \frac{R}{R^7-1}$, for all t, whence

$$\left|\int_0^{\frac{2\pi}{7}} \frac{iRe^{it}dt}{1+(Re^{it})^7}\right| \le \int_0^{\frac{2\pi}{7}} \frac{Rdt}{R^7-1} = \frac{2\pi R}{7(R^7-1)}.$$

Thus, $\int_0^{\frac{2\pi}{7}} \frac{iRe^{it}dt}{1+(Re^{it})^7} \to 0$, as $R \to \infty$. Furthermore,

$$\begin{aligned}(1.3.2)\qquad \int_{C_R} f(z)dz &= \int_0^{\frac{2\pi}{7}} \frac{iRe^{it}dt}{1+(Re^{it})^7} - \int_0^R \frac{z_0^2 dr}{1+(z_0^2 r)^7} + \int_0^R \frac{dx}{1+x^7}\\ &= \int_0^{\frac{2\pi}{7}} \frac{iRe^{it}dt}{1+(Re^{it})^7} + \left(1 - z_0^2\right)\int_0^R \frac{dx}{1+x^7}.\end{aligned}$$

Combining (1.3.1) with (1.3.2) and taking the limit as $R \to \infty$ gives

$$\int_0^\infty \frac{dx}{1+x^7} = \frac{2\pi i}{7z_0^6\left(1 - z_0^2\right)} = \frac{\pi}{7\sin\left(\frac{\pi}{7}\right)}.$$

The final equality being obtained from the computation:

$$z_0^6 - z_0^8 = e^{\frac{6\pi i}{7}} - e^{\frac{8\pi i}{7}} = -e^{-\frac{\pi i}{7}} + e^{\frac{\pi i}{7}} = 2i\sin(\pi/7).$$

□

[1] See [**RUD**, 217 - 218] for a brief discussion on what is meant by a sum of paths.

Problem 1.4. *Let $f : [0,\infty) \to \mathbb{C}$ be a Lebesgue measurable function satisfying*

$$(\dagger) \qquad |f(x)| \le ae^{kx},$$

for some $a, k \in (0,\infty)$. Put $\Omega := \{z \in \mathbb{C} : \operatorname{Im} z > k\}$.

(a) *Show that for every $z \in \Omega$, the function $g_z : [0,\infty) \to \mathbb{C}$ defined by $g_z(t) := e^{itz} f(t)$ is in $L^1([0,\infty))$.*

(b) *Prove that the map $F : \Omega \to \mathbb{C}$ defined by $F(z) := \int_0^\infty g_z(t)dt$ is holomorphic in Ω.*

Key terms: Lebesgue measure, Lebesgue's Dominated Convergence Theorem, complex differentiability, (Lebesgue) measurable function.

Solution. (a) Clearly, each g_z is measurable, being the product of measurable functions. Now, fix $z := x + iy \in \Omega$ and observe

$$\begin{aligned}\int_0^\infty |g_z(t)|\, dt &= \int_0^\infty \left| e^{it(x+iy)} f(t)\right| dt \le \int_0^\infty e^{-ty} \cdot ae^{kt} dt \\ &= a\int_0^\infty e^{-t(y-k)} dt = \frac{a}{y-k} < \infty,\end{aligned}$$

since $y = \operatorname{Im} z > k$. This completes the proof of (a).

(b) Fix $z_0 = x_0 + iy_0 \in \Omega$. Choose $r > 0$ such that $D := D(z_0, r) \subset \Omega$. Let $t \in [0,\infty)$ and $z \in \bar{D}(z_0, r/2)$ with $z \ne z_0$. Since $it \int_{[z_0,z]} e^{it\xi} d\xi = e^{itz} - e^{itz_0}$, we have

$$\begin{aligned}\left| \frac{e^{itz} - e^{itz_0}}{z - z_0} - ite^{itz_0} \right| &= \left| \frac{it}{z-z_0} \int_{[z_0,z]} e^{it\xi} d\xi - \frac{it}{z-z_0}\int_{[z_0,z]} e^{itz_0} d\xi \right| \\ &= \left| \frac{it}{z-z_0} \int_{[z_0,z]} \left(e^{it\xi} - e^{itz_0}\right) d\xi \right| \\ &\le \frac{t}{|z-z_0|} \cdot \operatorname{length}[z_0,z] \cdot \sup_{\xi\in[z_0,z]} \left| e^{it\xi} - e^{itz_0}\right| \\ &\le t \sup_{\xi\in[z_0,z]} \left(\left|e^{it\xi}\right| + \left|e^{itz_0}\right| \right) \qquad (1.4.1) \\ &\le 2t \sup_{\xi\in[z_0,z]} \left|e^{it\xi}\right| \\ &= 2t \sup_{\xi\in[z_0,z]} \left|e^{-t\operatorname{Im}\xi}\right| \\ &= 2te^{-tk_0},\end{aligned}$$

for some $k_0 > k$, since $[z_0, z] \subset \bar{D}(z_0, r/2)$ is a compact subset of Ω.[2] By the hypothesis (†), we find that

$$\left|te^{-tk_0}f(t)\right| \le ate^{-(k_0-k)t} \in L^1([0,\infty)). \tag{1.4.2}$$

Combining (1.4.1) and (1.4.2) with Lebesgue's Dominated Convergence Theorem, we find that

$$\begin{aligned}
\lim_{z\to z_0}\left[\frac{F(z)-F(z_0)}{z-z_0} - \int_0^\infty ite^{itz_0}f(t)dt\right] \\
&= \lim_{z\to z_0}\int_0^\infty \left(\frac{e^{itz}-e^{itz_0}}{z-z_0} - ite^{itz_0}\right) f(t)dt \\
&= \int_0^\infty \left(\lim_{z\to z_0}\frac{e^{itz}-e^{itz_0}}{z-z_0} - ite^{itz_0}\right) f(t)dt \\
&= 0.
\end{aligned}$$

Thus, $F'(z_0)$ exists and, in fact, $F'(z_0) = \int_0^\infty ite^{itz_0}f(t)dt$.

□

Problem 1.5. *A Lebesgue measurable subset of* $\mathbb{R}$ *is said to be negligible if it has Lebesgue measure zero. Prove that a subset* A *of* $\mathbb{R}$ *is negligible if and only if there exists a sequence* $\{U_n\}$ *of open subsets of* $\mathbb{R}$ *such that* $\lim_{n\to\infty}|U_n| = 0$ *and* $A \subseteq \bigcap_n U_n$.

Key terms: Lebesgue measure, Lebesgue measurable set, negligible set, complete measure space.

Solution. ($\Rightarrow$) As A has Lebesgue measure zero, it follows from the very definition of Lebesgue (outer) measure that, for every $n \in \mathbb{N}$, we may find a sequence $\{I_{n,m}\}_{m=1}^\infty$ of open intervals such that $A \subset \bigcup_m I_{n,m}$ and $\sum_m |I_{m,n}| < \frac{1}{n}$. Put $U_n := \bigcup_m I_{n,m}$. Then U_n is open, being the union of open sets. Since $A \subseteq U_n$, for each n, it must be that $A \subseteq \bigcap_n U_n$. Now, by countable subadditivity, $|U_n| \le \sum_m |I_{n,m}| < \frac{1}{n}$, for each $n \in \mathbb{N}$, whence $|U_n| \to 0$, as $n \to \infty$.

($\Leftarrow$) Let $\{U_n\}$ be the sequence of open sets given by the hypothesis of this implication. Observe that $0 \le |\bigcap_m U_m| \le |U_n|$, for every n. Since $|U_n| \to 0$, as $n \to \infty$ (by

[2] This is a standard way to circumvent the Mean-Value Theorem, which is unavailable for complex-valued functions.

assumption), it must be that $|\bigcap_m U_m| = 0$. Now, $\bigcap_m U_m$ is Lebesgue measurable, being the countable intersection of open (and therefore Borel) sets. Thus, A is Lebesgue measurable, since it is the subset of a set of measure zero and since Lebesgue measure is complete. Finally, A has measure zero, by monotonicity.

$\square$

Problem 1.6. *Let* $1 \leq q < p \leq \infty$.

(a) *Prove that* $L^p([0,1]) \subset L^q([0,1])$.

(b) *Show (by example) that the inclusion in* (a) *is strict.*

(c) *Give an example of a measure space* $(X, \mathscr{A}, \mu)$ *for which the inclusion* $L^p(X) \supseteq L^q(X)$ *holds.*

Key terms: Lebesgue measure, L^p-space, counting measure.

Solution. (a) See the solution for Problem 5.1(a).

(b) Let us first handle the case when $p = \infty$. Consider the function $f(x) := x^{-\frac{1}{2q}} \in L^q([0,1])$. For $M > 0$, let $b_M := \min\{M^{-2q}, 1\}$. It is easy to check that

$$\{x \in (0,1) : |f(x)| > M\} = (0, b_M),$$

from which it follows that $\operatorname{ess\,sup} f = \infty$, since $|(0, b_M)| = b_M > 0$. Therefore, $f \notin L^\infty([0,1])$. So, the inclusion of (a) is strict in this case.

Now, fix $1 \leq q < p < \infty$ and let $r \in \left(\frac{1}{p}, \frac{1}{q}\right)$. Notice that $1 - qr > 0$, while $1 - pr < 0$. So,

$$\int_0^1 \left|\frac{1}{x^r}\right|^q dx = \lim_{t\to 0^+} \int_t^1 x^{-qr} dx = \lim_{t\to 0^+} \frac{1}{1-qr} x^{1-qr}\Big|_t^1 = \lim_{t\to 0^+} \frac{1}{1-qr}(1-t^{1-qr}) = \frac{1}{1-qr},$$

while

$$\int_0^1 \left|\frac{1}{x^r}\right|^p dx = \lim_{t\to 0^+} \int_t^1 x^{-pr} dx = \lim_{t\to 0^+} \frac{1}{1-pr} x^{1-pr}\Big|_t^1 = \lim_{t\to 0^+} \frac{1}{pr-1}(t^{1-pr} - 1) = \infty.$$

Thus, $1/x^r \in L^q([0,1]) \backslash L^p([0,1])$, showing the inclusion is strict, in this case as well.

(c) One can construct a multitude of trivial examples easily: merely take $X :=$ any set, $\mathscr{A} :=$ any σ-algebra on X (one may pick the power set of X for concreteness), and finally, $\mu :=$ the zero measure on X (i.e., $\mu(A) = 0$, for all $A \in \mathscr{A}$).

Of course, a nontrivial example is to be sought out. Perhaps, the simplest such example is to take $X := \mathbb{N}$, $\mathscr{A} :=$ the power set of $\mathbb{N}$, and $\mu :=$ counting measure. Here if $f \in L^q(X)$, then by definition $\sum_{n=1}^{\infty} |f(n)|^q < \infty$. Consequently, we may find $N \in \mathbb{N}$ such that $|f(n)|^q \leq 1$, for each $n \geq N$. So, for $n \geq N$, we have $|f(n)|^p \leq |f(n)|^q$, whence

$$\sum_{n=1}^{\infty} |f(n)|^p \leq \sum_{n=1}^{N-1} |f(n)|^p + \sum_{n=N}^{\infty} |f(n)|^q < \infty.$$

This shows that $f \in L^p(X)$, and therefore, $L^p(X) \subseteq L^q(X)$.

□

Problem 1.7. *Let* $p \in [1, \infty)$ *and suppose* $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R})$ *is a sequence that converges to* 0 *in the* p *norm. Prove that we may find a subsequence* $\{f_{n_k}\}$ *such that* $f_{n_k} \to 0$ *a.e.*

Key terms: pointwise convergence, p-norm convergence, almost everywhere, complete metric space, Beppo-Levi Theorem.

Solution. Choose $n_1 \in \mathbb{N}$ such that $\|f_n\|_p < \frac{1}{2}$, for all $n \geq n_1$. Having found this n_1, we may find $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $\|f_n\|_p < \frac{1}{2^2}$, for all $n \geq n_2$. Continuing in this manner, we may inductively find $n_k \in \mathbb{N}$ such that $n_k > n_{k-1}$ and $\|f_n\|_p < \frac{1}{2^k}$, whenever $n \geq n_k$. Set $f_{n_0} = 0$. Note that $\|f_{n_k}\|_p < \frac{1}{2^k}$, for all $k \geq 1$.

By the Beppo-Levi Theorem (or Lebesgue's Monotone Convergence Theorem), we have

$$\int_{\mathbb{R}} \left[\sum_{k=1}^{\infty} |f_{n_k}(x)|^p \right] dx = \sum_{k=1}^{\infty} \|f_{n_k}\|_p^p \leq \sum_{k=1}^{\infty} \left(2^{-k}\right)^p = \frac{1}{2^p - 1} < \infty.$$

Consequently, $\sum_{k=1}^{\infty} |f_{n_k}(x)|^p$ is finite for a.e $x \in \mathbb{R}$, and therefore, convergent for a.e. $x \in \mathbb{R}$. For these x, it follows from the convergence of the above series that $|f_{n_k}(x)|^p \to 0$, as $k \to \infty$, or equivalently, that $f_{n_k}(x) \to 0$, as $k \to \infty$.

□

Problem 1.8. *Let* f *be an entire function having the property that*

$$f(z + m + ni) = f(z) \qquad (z \in \mathbb{C}, m, n \in \mathbb{Z}).$$

Prove that f is constant.

Key terms: entire function, Liouville's Theoerem.

Solution. Put $S := [0,1] \times [0,1]$. The periodicity condition on f gives that $f(\mathbb{C}) = f(S)$. Since S is compact and f is continuous (it is holomorphic), it follows that f is bounded on S, and therefore, f is bounded on $\mathbb{C}$. By Liouville's Theorem, we deduce that f is constant.

$\square$

CHAPTER 2

Fall 2004

Problem 2.1. *Compute*

(a) $\int_0^\infty e^{-[x]}dx$, *where* $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$, *the integer part of* x.

(b) $\int_0^{\pi/2} f(x)dx$, *where* $f(x) := \begin{cases} \cos x & x \in \mathbb{R}\backslash\mathbb{Q} \\ \sin x & x \in \mathbb{Q} \end{cases}$.

Key terms: Lebesgue's Monotone Convergence Theorem, Beppo-Levi Theorem.

Solution. (a) Applying the Beppo-Levi Theorem (or Lebesgue's Monotone Convergence Theorem) and the fact that $[x] = n$, on $[n, n+1)$, for $n \in \mathbb{N}_0$, we get

$$\begin{aligned} \int_0^\infty e^{-[x]}dx &= \int_0^\infty e^{-[x]}\left[\sum_{n=0}^\infty \mathbf{1}_{[n,n+1)}(x)\right]dx = \sum_{n=0}^\infty \int_0^\infty e^{-[x]}\mathbf{1}_{[n,n+1)}(x)dx \\ &= \sum_{n=0}^\infty \int_0^\infty e^{-n}\mathbf{1}_{[n,n+1)}(x)dx = \sum_{n=0}^\infty e^{-n} = e/(e-1). \end{aligned}$$

(b) Put $I := [0, \pi/2]$. Since integrals over sets of measure zero are 0, we find that

$$\begin{aligned} \int_0^{\pi/2} f(x)dx &= \int_{I\backslash\mathbb{Q}} + \int_{I\cap\mathbb{Q}} f(x)dx = \int_{I\backslash\mathbb{Q}} \cos x dx + \int_{I\cap\mathbb{Q}} \sin x dx \\ &= \int_{I\backslash\mathbb{Q}} \cos x dx = \int_{I\backslash\mathbb{Q}} \cos x dx + \int_{I\cap\mathbb{Q}} \cos x dx \\ &= \int_0^{\pi/2} \cos x dx = 1. \end{aligned}$$

Note that the above functions are measurable, since in a complete measure space we may arbitrarily redefine functions over sets of measure zero and retain measurability.

□

Problem 2.2. (a) *Does the function*

$$f(x) := \begin{cases} x \sin \frac{1}{x} & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

have bounded variation?

(b) *Compute* $\mathrm{Var}_{[0,50]}(e^x)$*, the total variation of* e^x *on the interval* $[0, 50]$.

Key terms: bounded variation, total variation.

Solution. (a) No. For each integer $N \geq 3$, let P_N denote the partition $0 := x_0 < x_1 < ... < x_N := 1$, where $x_n := \frac{1}{\frac{\pi}{2}+\pi(N-n)}$, for $n = 1, ..., N-1$. Since $\sin \frac{1}{x_n} = (-1)^{N-n}$, for each $n = 1, ..., N-1$, the total (or absolute) variation T_N corresponding to this partition satisfies

$$\begin{aligned} T_N &\geq \sum_{n=2}^{N-1} \left| \tfrac{1}{\frac{\pi}{2}+\pi(N-n)}(-1)^{N-n} - \tfrac{1}{\frac{\pi}{2}+\pi[N-(n-1)]}(-1)^{N-n+1} \right| \\ &= \sum_{n=2}^{N-1} \tfrac{1}{\frac{\pi}{2}+\pi(N-n)} + \tfrac{1}{\frac{\pi}{2}+\pi[N-(n-1)]} \geq \sum_{n=2}^{N-1} \tfrac{1}{\pi}\left(\tfrac{1}{N-n+1} + \tfrac{1}{N-n+2}\right) \\ &\geq \sum_{n=2}^{N-1} \tfrac{2}{\pi(N-n+2)} = \sum_{n=3}^{N} \tfrac{2}{\pi n}. \end{aligned} \tag{2.2.1}$$

Now, the total variation T satisfies $T \geq T_N$, for all $N \in \mathbb{N}$. Hence, letting $N \to \infty$ in (2.2.1) verifies that f is not of bounded variation.

(b) Let $P_N := 0 = x_0 < x_1 < ... < x_N = 50$ be a partition of $[0, 50]$. By the Mean-Value Theorem, we may find $x_n^* \in [x_{n-1}, x_n]$ such that $e^{x_n^*} = \frac{e^{x_n} - e^{x_{n-1}}}{x_n - x_{n-1}}$. Hence,

$$\sum_{n=1}^{N} |e^{x_n} - e^{x_{n-1}}| = \sum_{n=1}^{N} e^{x_n^*}(x_n - x_{n-1}).$$

Letting our partitions become finer and finer, one version of the Riemannian theory of integration ensures the sum on the right converges to $\int_0^{50} e^x dx = e^{50} - 1$. On the other hand, the left sum converges to the total variation.

More generally, the above argument shows that one has $\mathrm{Var}_{[a,b]} f = \int_a^b |f'(x)|\, dx$, for every continuously differentiable function f on a compact interval $[a, b]$.

□

Problem 2.3. *Suppose $f \in H(\Omega \setminus \{0\})$ and that 0 is either a pole of order m for f or a removable singularity whose removal results in 0 being a zero of order m for f. Show that 0 is a first order pole of f'/f having residue either $-m$ or m.*

Key terms: pole, removable singularity, residue

Solution. In either case, we may write $f(z) = z^n g(z)$, where $n := \pm m$, $g \in H(\Omega)$ and $g(0) \neq 0$. Since $g(0) \neq 0$ and g is continuous there (it is differentiable), we may find $r > 0$ such that g is zero-free on $D := D(0, r)$. Thus,

$$\frac{f'(z)}{f(z)} = \frac{z^n g'(z) + n z^{n-1} g(z)}{z^n g(z)} = \frac{g'(z)}{g(z)} + n z^{-1} \qquad (z \in D \setminus \{0\}). \tag{2.3.1}$$

As $g \in H(D)$ and g is zero-free on D, it follows that $g'/g \in H(D)$, whence (2.3.1) ensures that f'/f has a simple pole at 0 with residue $n = \pm m$.

□

Problem 2.4. *Show that a nonconstant entire function maps the plane onto a dense subset of the plane.*

Key terms: Casorati-Weierstrass Theorem, entire function

Solution. The proof to follow mimics the canonical proof of the Casorati-Weierstrass Theorem. Let f be a nonconstant entire function. And suppose, in order to reach a contradiction, that $f(\mathbb{C})$ is not dense in $\mathbb{C}$. Then we may find $z_0 \in \mathbb{C}$ and $r > 0$ such that $f(\mathbb{C}) \cap D(z_0, r) = \varnothing$. Consequently,

$$|f(z) - z_0| \geq r, \tag{2.4.1}$$

for every $z \in \mathbb{C}$. This implies that the function $g : \mathbb{C} \to \mathbb{C}$ defined by $g := \frac{1}{f - z_0}$ is entire. Now, (2.4.1) also implies that $|g| \leq \frac{1}{r} < \infty$, and so, Liouville's Theorem ensures that g is constant. This, in turn, forces f to be constant. (Specifically, $f = \frac{1}{c} + z_0$, where $g = c$.) This contradiction implies $f(\mathbb{C})$ is dense in the plane.

□

Problem 2.5. *Let Ω be a bounded region. Suppose that $f \in C(\bar{\Omega}) \cap H(\Omega)$ is zero-free and $|f| = C$ on $\partial\Omega$.*

(a) *Prove that f must be constant.*

(b) *Is the boundedness condition imposed on Ω essential?*

Key terms: Maximum Modulus Principle.

Solution. (a) First note that since f is zero-free, it must be that $C > 0$. By the Maximum Modulus Principle,

$$|f(z)| \leq \|f\|_{\partial\Omega} := \max\{|f(z)| : z \in \partial\Omega\} = C \qquad (z \in \Omega). \tag{2.5.1}$$

Now, since f is zero-free, we evidently have $1/f \in C(\bar{\Omega}) \cap H(\Omega)$. Consequently, the Maximum Modulus Principle applies to $1/f$, and therefore,

$$|1/f(z)| \leq \|1/f\|_{\partial\Omega} = 1/C \qquad (z \in \Omega). \tag{2.5.2}$$

Inequalities (2.5.1) and (2.5.2) combine to show that $|f(z)| = C$ on Ω. Thus, $|f|$ attains its maximum at *every* point of Ω, and so, the Maximum Modulus Principle forces f to be constant on Ω.

(b) Yes. Consider the region $\Omega := \{z \in \mathbb{C} : 0 < \operatorname{Re} z\}$. The exponential map e^z is zero-free and holomorphic on Ω, as well as, continuous and identically equal to 1 in modulus on the boundary of Ω. Yet, this map fails to be constant on Ω. In fact, the continuity of f shows that this constant must be C.

□

Problem 2.6. *Does there exists a Lebesgue measurable subset E of $\mathbb{R}$ such that $\frac{|E\cap I|}{|I|} = \frac{1}{2}$, for every interval $I \subset \mathbb{R}$.*

Key terms: Lebesgue measure, Lebesgue point, almost everywhere

Solution. No. To verify this, we will need the following general results applied to the one-dimensional scenario:

LEMMA 2.6.1. *Let E be Lebesgue measurable subset of $\mathbb{R}^k$. Then for a.e. $x \in \mathbb{R}^k$, the limit*

$$\lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|}$$

exists and is equal to 1 for a.e. point of E and 0 for a.e. point of E^c.

Proof. Assume for a moment that the lemma holds for measurable sets having finite measure. For each $n \in \mathbb{N}$, put $E_n := E \cap B(0,n)$. Each E_n has finite measure, being a subset of a set of finite measure, namely $B(0,n)$. Thus, the lemma applies to each E_n. For each n, let A_n be the set of points $x \in E_n$ such that $\lim_{r\to 0} \dfrac{|E_n \cap B(x,r)|}{|B(x,r)|}$ exists and is equal to 1. Likewise, let B_n be the set of points $x \in \mathbb{R}^k \backslash E_n$ for which the limit exists equal to 0. Since the lemma is assumed to hold for finite measured sets, we have that $\left|\mathbb{R}^k \backslash A_n\right| = 0$ and $\left|\mathbb{R}^k \backslash B_n\right| = 0$, for each n. Put $A := \bigcap_{n=1}^\infty A_n$ and $B := \bigcap_{n=1}^\infty B_n$ and notice that $\left|\mathbb{R}^k \backslash A\right| = \left|\mathbb{R}^k \backslash B\right| = 0$, being the countable union of null sets.

Fix $x \in A$. Since $E_n \cap B(x,r) \subseteq E_{n+1} \cap B(x,r)$, for all n, and since $\bigcup_{n=1}^\infty E_n \cap B(x,r) = E \cap B(x,r)$, we conclude that

$$\frac{|E \cap B(x,r)|}{|B(x,r)|} = \lim_{n\to\infty} \frac{|E_n \cap B(x,r)|}{|B(x,r)|}. \tag{2.6.1}$$

Let $\epsilon > 0$ be given. By (2.6.1), we may find $N \geq 1$ so large so that

$$\left|\frac{|E \cap B(x,r)|}{|B(x,r)|} - \frac{|E_N \cap B(x,r)|}{|B(x,r)|}\right| < \epsilon/2. \tag{2.6.2}$$

Notice that the quantity on the left is well-defined, since $0 < |B(x,r)| < \infty$. As $x \in A \subseteq A_N$, we may find $r_0 > 0$ such that $0 < r < r_0$ implies $\left|\frac{|E_N \cap B(x,r)|}{|B(x,r)|} - 1\right| < \epsilon/2$. This fact along with (2.6.2) and the triangle inequality shows that $\left|\frac{|E \cap B(x,r)|}{|B(x,r)|} - 1\right| < \epsilon$, whenever $0 < r < r_0$. Hence, $\lim_{r\to 0} \dfrac{|E \cap B(x,r)|}{|B(x,r)|}$ exists and is equal to 1 for every $x \in A$. In a similar (in fact, easier) fashion, we see that $\lim_{r\to 0} \dfrac{|E \cap B(x,r)|}{|B(x,r)|}$ exists and is equal to 0 for every $x \in B$. Let F and G be for E and $\mathbb{R}^k \backslash E$ as A_n and B_n are for E_n and $\mathbb{R}^k \backslash E_n$, respectively. Then we have just proved that $A \subseteq F$ and $B \subseteq G$ so that $\mathbb{R}^k \backslash F \subseteq A^c$ and $\mathbb{R}^k \backslash G \subseteq B^c$, and therefore, $\left|\mathbb{R}^k \backslash F\right| = \left|\mathbb{R}^k \backslash G\right| = 0$.

The above argument guarantees that it suffices to prove the lemma with the extra condition that $|E| < \infty$. In this case, put

$$v_r(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} |\mathbf{1}_E(y) - \mathbf{1}_E(x)|\, dy$$

for $0 < r$ and $x \in \mathbb{R}^k$. Since E has finite measure, $\mathbf{1}_E \in L^1(\mathbb{R}^k)$, whence almost every point of $\mathbb{R}^k$ is a Lebesgue point of $\mathbf{1}_E$; i.e., $\lim_{r\to 0} v_r(x) = 0$, for a.e. $x \in \mathbb{R}^k$.

Case 1: $x \in E$. Then $|\mathbf{1}_E(y) - \mathbf{1}_E(x)| = \mathbf{1}_{E^c}(y)$, and therefore,

$$v_r(x) = \frac{|E^c \cap B(x,r)|}{|B(x,r)|} = \frac{|B(x,r)| - |E \cap B(x,r)|}{|B(x,r)|} = 1 - \frac{|E \cap B(x,r)|}{|B(x,r)|} \qquad (\forall r > 0).$$

Case 2: $x \in E^c$. Then we directly have

$$v_r(x) = \frac{|E \cap B(x,r)|}{|B(x,r)|} \qquad (\forall r > 0).$$

Since $\lim_{r\to 0} v_r(x)$ exits and is equal 0 for a.e. $x \in \mathbb{R}^k$, the proof is complete.

□

Proposition 2.6.2. *Let $E \subset \mathbb{R}^k$ be Lebesgue measurable and let $\varepsilon > 0$. Then there exists $x \in \mathbb{R}^k$ and $r > 0$ such that either $\varepsilon > \frac{|E\cap B(x,r)|}{|B(x,r)|}$ or $1 - \varepsilon < \frac{|E\cap B(x,r)|}{|B(x,r)|}$.*

Proof. Since $\mathbb{R}^k$ is the disjoint union of E and E^c, either $|E| > 0$ or $|E^c| > 0$. In the former case, the preceding lemma implies the existence of an $x \in E$ such that $\lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 1$. Thus, we may find some $r > 0$ such that $1 - \varepsilon < \frac{|E\cap B(x,r)|}{|B(x,r)|}$. So, we are finished with this situation.

If, however, $|E^c| > 0$, then applying the preceding lemma again shows that there exists an $x \in E^c$ such that $\lim_{r\to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} = 0$. So, we may find some $r > 0$ such that $\varepsilon > \frac{|E\cap B(x,r)|}{|B(x,r)|}$. This finishes the proof.

□

Returning to Problem 2.6, the answer is no. To see this, we simply take $k = 1$ and $\varepsilon = \frac{1}{2}$ in the preceding proposition.

□

Problem 2.7. *Let* $f(x,y) := \begin{cases} \dfrac{x^2 - y^2}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$

(a) *Show that*

$$\int_0^1 dx \int_0^1 f(x,y)dy \neq \int_0^1 dy \int_0^1 f(x,y)dx.$$

(b) *To save Fubini's Theorem, use polar coordinates to verify that*

$$\int_0^1 \int_0^1 |f(x,y)|\, dxdy = \infty.$$

Key terms: polar coordinates, Fubini's Theorem

Solution. (a) First, some clarification is due regarding the meaning of the iterated integrals. The quantity

$$\int_A dx \int_B f(x,y)dy,$$

is the Lebesgue integral of the function given by $F(x) := \int_B f(x,y)dy$, provided this function is well-defined and Lebesgue integrable. Similarly, for the other iterated integrals in (a). In our present situation,

$$F(x) := \int_0^1 \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) dy = \frac{y}{x^2+y^2}\Big|_{y=0}^{y=1} = \frac{1}{x^2+1},$$

so that the measurability of F is evident. (F is, in fact, continuous on $(0,1]$.) Thus,

$$\int_0^1 F(x)dx = \arctan x\Big|_0^1 = \pi/4.$$

On the other hand, since $f(x,y) = -f(y,x)$, we see that the remaining iterated integral exists and evaluates to $-\pi/4$.

(b) Note that the circular sector $S = \left\{re^{i\theta} : 0 \leq r \leq 1, 0 \leq \theta \leq \pi/2\right\}$ is contained in the rectangle $R = \{(x,y) : 0 \leq x, y \leq 1\}$. Thus,

$$\begin{aligned}
\int_0^1 \int_0^1 |f(x,y)|\, dxdy &= \int\int_R |f(x,y)|\, dxdy \\
&\geq \int\int_S |f(x,y)|\, dxdy \\
&= \int_0^1 \left[\int_0^{\pi/2} |f(r\cos\theta, r\sin\theta)|\, d\theta\right] rdr \\
&= \int_0^1 \left(\int_0^{\pi/2} \left|\frac{\cos 2\theta}{r^2}\right| d\theta\right) rdr \\
&= \int_0^1 \left(\int_0^{\pi/4} - \int_{\pi/4}^{\pi/2} \cos 2\theta d\theta\right) \frac{1}{r} dr \\
&= \int_0^1 \frac{1}{r} dr \\
&= \infty.
\end{aligned}$$

Problem 2.8. *Prove that $f \equiv 0$, whenever $f \in H(\mathbb{C}) \cap L^1(\mathbb{R}^2)$.*

Key terms: Cauchy's Formula, polar coordinates, L^p-space.

Solution. For $r > 0$ and $z \in \mathbb{C}$, let $\gamma_{r,z}$ be the parameterization of the boundary of the circle centered at z with radius r given by $\gamma_{r,z}(t) := z + re^{it}$, $0 \leq t \leq 2\pi$. Since f is entire and $\mathbb{C}$ is convex, we may apply Cauchy's formula to find that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\gamma_{r,z}} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z + re^{it})}{(z + re^{it}) - z} (rie^{it} dt) \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) dt,
\end{aligned}$$

and so,

$$|f(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left|f(z + re^{it})\right| dt. \tag{2.8.1}$$

Integrating both ends of (2.8.1) with respect to rdr over $[0, R]$ and appealing to Fubini's Theorem gives

$$
\begin{aligned}
\frac{R^2}{2}\,|f(z)| &\leq \frac{1}{2\pi}\int_0^R \left[\int_0^{2\pi} \left|f(z+re^{it})\right| dt\right] rdr \\
&= \frac{1}{2\pi}\int_E |f(z+w)|\, d\lambda_2(w) \qquad (2.8.2)\\
&\leq \frac{1}{2\pi}\int_{\mathbb{R}^2} |f| = \frac{1}{2\pi}\,\|f\|_1\,,
\end{aligned}
$$

where $E := \{(x, y) \in D(0, r) : 0 < x, y\}$ and λ_2 is Lebesgue measure in $\mathbb{R}^2$. Since (2.8.2) is clearly equivalent to $|f(z)| \leq \frac{\|f\|_1}{\pi R^2}$, which holds for all $z \in \mathbb{C}$ and $R > 0$, we may deduce that $|f(z)| = 0$, for all z. Equivalently, $f \equiv 0$.

□

CHAPTER 3

Spring 2005

Problem 3.1. *Compute*

(a) $\displaystyle\int_0^{2\pi} \frac{dt}{\cos t - 2}$.

(b) $\operatorname{Res}(z^n e^{10/z}, \infty)$, $n \in \mathbb{N}$.

(c) $\operatorname{Res}\left(\exp(z^2)/z^{2n+1}, 0\right)$, $n \in \mathbb{N}$.

Key terms: residue, contour integration, Residue Theorem, pole.

Solution. (a) Let γ be the curve defined by $t \overset{\gamma}{\mapsto} e^{it}$, for $0 \leq t \leq 2\pi$. If we write $z = \gamma(t)$, then $\cos t = (z + z^{-1})/2$, $dt = -idz/z$, and so,

$$\int_0^{2\pi} \frac{dt}{\cos t - 2} = \int_\gamma \frac{(-idz/z)}{(z + z^{-1})/2 - 2} = -2i \int_\gamma \frac{dz}{z^2 - 4z + 1}. \tag{3.1.1}$$

Let $f(z)$ be the function in the integrand of (3.1.1). The quadratic formula shows that f has simple poles at $z_0 = 2 - 2\sqrt{3}$ and $z_1 = 2 + 2\sqrt{3}$. Among these, z_0 lies in the region bounded by the image of γ, while z_1 lies outside this region. Thus, the integral on the far right of (3.1.1) is equal to

$$2\pi i \operatorname{Res}(f, z_0) = 2\pi i \lim_{z \to z_0} (z - z_0) f(z) = 2\pi i \lim_{z \to z_0} \frac{1}{z - z_1} = \frac{2\pi i}{z_0 - z_1} = -\frac{\sqrt{3}\pi i}{6}.$$

Substitution of this into (3.1.1) yields

$$\int_0^{2\pi} \frac{dt}{\cos t - 2} = -\frac{\sqrt{3}\pi}{3}.$$

(b) Recall that $\operatorname{Res}(f, \infty) := \operatorname{Res}(f(1/z), 0)$. Now,

$$(1/z)^n e^{10/(1/z)} = e^{10z}/z^n = (1/z^n) \sum_{k=0}^{\infty} (10z)^k/k! = \sum_{k=0}^{\infty} (10^k/k!) z^{k-n}.$$

Hence, the coefficient of the z^{-1} term is $\operatorname{Res}(z^n e^{10/z}, \infty) = 10^{n-1}/(n-1)!$.

(c) Observe that

$$\exp(z^2)/z^{2n+1} = (1/z^{2n+1})\sum_{k=0}^{\infty}(z^2)^k/k! = \sum_{k=0}^{\infty} z^{2k-2n-1}/k!.$$

So, the coefficient of the z^{-1} term is $\mathrm{Res}(\exp(z^2)/z^{2n+1}, 0) = 1/n!$.

□

Problem 3.2. *Let $\{E_j\}_{j=1}^m$ be a collection of measurable subsets of $[0,1]$. Assume that this collection has the property that every $x \in [0,1]$ belongs to at least n of the E_j, where $n \le m$. Prove that $|E_j| \ge \frac{n}{m}$, for some j.*

Key terms: Lebesgue measure.

Solution. Since every $x \in [0,1]$ belongs to at least n of the E_j, it must be that $\sum_{j=1}^m \mathbf{1}_{E_j}(x) \ge n$, for every $x \in [0,1]$. Now, suppose, in order to reach a contradiction, that $|E_j| < \frac{n}{m}$, for every j. (As all of our integrals will be over $[0,1]$ we will suppress the integrating set.)

$$n = n \cdot |[0,1]| = \int n \le \int \sum_{j=1}^m \mathbf{1}_{E_j} = \sum_{j=1}^m \int \mathbf{1}_{E_j} = \sum_{j=1}^m |E_j| < \sum_{j=1}^m \frac{n}{m} = n.$$

The appearance of the strict inequality is the sought contradiction, and therefore, the desired conclusion follows.

□

Problem 3.3. *Given $1 \le p < q < r < \infty$, prove that*

$$L^p(X,\mu) \cap L^r(X,\mu) \subseteq L^q(X).$$

Key terms: L^p-space.

Solution. Let $f \in L^p(X) \cap L^r(X)$. Set $E := \{x : |f(x)| > 1\}$ and notice that $|f|^p < |f|^q < |f|^r$ on E, while $|f|^r \le |f|^q \le |f|^p$ on E^c. Therefore,

$$\int_X |f|^q = \int_E |f|^q + \int_{E^c} |f|^q \le \int_E |f|^r + \int_{E^c} |f|^p \le \int_X |f|^r + \int_X |f|^p < \infty.$$

□

Problem 3.4. *Let $f \in L^1(X, \mu)$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\left| \int_A f d\mu \right| < \epsilon,$$

whenever A is a measurable subset of X with $\mu(A) < \delta$.

Key terms: L^p-space, Lebesgue's Monotone Convergence Theorem.

Solution. Let $\epsilon > 0$ be given. We will actually prove the existence of $\delta > 0$ such that

$$\int_A |f| \, d\mu < \epsilon,$$

for every measurable $A \subset X$ with $\mu(A) < \delta$. Once this is shown, we will be finished because the inequality $\left|\int_A f d\mu\right| \leq \int_A |f| \, d\mu$ holds. Since we are only concerned with $|f|$, we may assume that $f = |f|$; that is, we assume f is nonnegative.

For $n \in \mathbb{N}$, define $f_n := \min\{f, n\}$. Notice that $0 \leq f_n(x) \leq f_{n+1}(x) \leq f(x)$ and $f_n(x) \uparrow f(x)$ holds for every $x \in X$ and $n \in \mathbb{N}$. So, by Lebesgue's Monotone Convergence Theorem, we have $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$, or equivalently, $\int_X (f - f_n) d\mu \to 0$, as $n \to \infty$. Hence, we may choose N so large so that $\int_X (f - f_N) d\mu < \frac{\epsilon}{2}$. Having chosen this N, we now choose $\delta > 0$ such that $N\delta \leq \frac{\epsilon}{2}$. So, for measurable $A \subset X$ with $\mu(A) < \delta$, we get

$$\begin{aligned} \int_A f d\mu &= \int_A (f - f_N) d\mu + \int_A f_N d\mu \leq \int_X (f - f_N) d\mu + \int_A N d\mu \\ &< \frac{\epsilon}{2} + N\mu(A) < \frac{\epsilon}{2} + N\delta < \epsilon. \end{aligned}$$

□

Problem 3.5. *Is there an $f \in H(\mathbb{D})$ such that $m_n(f) \to \infty$, as $n \to \infty$, where $m_n(f) = \inf\left\{|f(z)| : 1 - \frac{1}{n} < z < 1\right\}$?*

Key terms: zero-set, Bolzano-Weierstrass, Maximum Modulus Principle.

Solution. No. To see this, assume the contrary. That is, suppose there exists $f \in H(\mathbb{D})$ such that $m_n(f) \to \infty$, as $n \to \infty$. Choose n so large so that $m_n(f) > 0$. We conclude from the definition of $m_n(f)$ that f is zero-free on the annulus $A :=$

$\left\{z : 1 - \frac{1}{n} < z < 1\right\}$. Thus, all the zeros of the nonconstant function f must lie in the compact set $\bar{D}\left(0, 1 - \frac{1}{n}\right)$. Since the zero-set of f contains no limit point of $\mathbb{D}$, we conclude (by Bolzano-Weierstrass) that f has only finitely many zeros, say $\{z_1, ..., z_k\}$.

By iterated use of Theorem 10.18 in [**RUD**], we may write

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_k)^{m_k} g(z),$$

where $g \in H(\mathbb{D})$ and g has no zero in $\mathbb{D}$. Observe that

$$\begin{aligned} |f(z)| &= |(z - z_1)^{m_1} \cdots (z - z_k)^{m_k} g(z)| = |z - z_1|^{m_1} \cdots |z - z_k|^{m_k} |g(z)| \\ &\leq 2^{m_1 + \cdots + m_k} |g(z)|, \end{aligned}$$

for every $z \in \mathbb{D}$. From this it follows that $m_n(g) \to \infty$, as $n \to \infty$. Since

$$\left\{\left|g\left[\left(1 - \frac{1}{2n}\right) e^{i\theta}\right]\right| : 0 \leq \theta \leq 2\pi\right\} \subset \left\{|g(z)| : 1 - \frac{1}{n} < z < 1\right\},$$

for each n, it must be that

$$\min_{0 \leq \theta \leq 2\pi} \left|g\left[\left(1 - \frac{1}{2n}\right) e^{i\theta}\right]\right| \geq m_n(g),$$

for each n. By the corollary to Theorem 10.24 in [**RUD**] (i.e. the *Minimum* Modulus Principle, which is just the Maximum Modulus Principle applied to $1/g$, g being zero-free), we may deduce that $|g(0)| \geq m_n(g)$, for each n, which is impossible if $m_n(g) \to \infty$. This is the desired contradiction. □

Problem 3.6. *Let $f : \Omega \to \mathbb{C}$, where Ω is open.*

(a) *True or false:* (i) *If f is holomorphic and e^f is constant, then f is constant.*

(ii) *Does the answer change if f is only assumed continuous?*

(b) *Is a bounded function that is harmonic on all of $\mathbb{C}$ constant?*

Key terms: harmonic function, entire function, Liouville's Theorem

Solution. Recall that we are assuming Ω to be a region, and so, it is connected.

(a) It suffices to show the following: If f is continuous and e^f is constant, then f is constant, since holomorphic functions are continuous. The non-zero constant function e^f is e^w, for some $w \in \mathbb{C}$. Hence, $e^{f(z)-w} = 1$, or equivalently, $f(z) - w \in 2\pi i\mathbb{Z}$, for all $z \in \Omega$. As $f(z) - w$ is a continuous mapping of the connected region Ω into the discrete space $2\pi i\mathbb{Z}$, it must be that $f - w = 2\pi i n$, for some fixed integer n, whence $f(z) = w + 2\pi n$. That is, f is constant.

(b) As the real and imaginary parts of f satisfy the same conditions as f does, we may without loss of generality assume that f is real-valued. As such, on each closed disk $\bar{D}(0, N)$, $N \in \mathbb{N}$, f is the real part of a holomorphic function that is defined uniquely up to a pure imaginary additive constant (see [**RUD**, 235 - 236]). Adjusting these constants at each radius, we may conclude that f is the real part of an entire function, say $F = f + iv$. Now, since the exponential function is nonnegative and strictly increasing on the real axis, we find that

$$\left|e^F\right| = \left|e^{f+iv}\right| = \left|e^f\right|\left|e^{iv}\right| = \left|e^f\right| = e^f \leq e^{|f|} \leq e^M,$$

where M is a (finite) bound for f. Consequently, we may apply Liouville's Theorem to the entire function e^F to conclude that e^F is constant. From part (a), F must be constant, and therefore, f is constant.

□

Problem 3.7. *Let I be a nonempty (compact) interval in $\mathbb{R}$ and let $f : I \to \mathbb{C}$ be real-analytic, by which it is meant that in a neighborhood of every point of I, f is represented by a convergent power series. Show that we may extend f to a holomorphic function on some open subset of $\mathbb{C}$ that contains I. That is, prove that there exists open $\Omega \subset \mathbb{C}$ and $F \in H(\Omega)$ such that $I \subset \Omega$ and $F|_I \equiv f$.*

Key terms: power series, Uniqueness Theorem for Holomorphic Functions, uniform convergence.

Solution. It turns out that the assumption that I be compact is not essential. So, we will dispense with it. Now, for each $p \in I$, let $\sum_{n=0}^{\infty} c_n(p)(x-p)^n$ be the convergent power series representation of f about p, given by the hypothesis, and let R_p denote the radius of convergence of this power series. Define $f_p(z) := \sum_{n=0}^{\infty} c_n(p)(z-p)^n$, for $z \in D(p, R_p)$. By assumption, $R_p > 0$. Choose r_p such that $0 < r_p < R_p$. By elementary power series theory, we have that the series for f_p converges uniformly and absolutely on $D_p := D(p, r_p)$. Note that $f_p(x) = f(x)$, for every $x \in I \cap [p - r_p, p + r_p]$.

Put $\Omega := \bigcup_{p \in I} D_p$ and define $F : \Omega \to \mathbb{C}$ by $F(z) := f_p(z)$, whenever $p \in I$ is such that $z \in D_p$. Provided that F is well-defined, it is obvious that $F \in H(\Omega)$, since F is representable by a power series about every point. Furthermore, that $F|_I \equiv f$ is also self-evident, by construction. Hence, we seek to show that F is well-defined.

Suppose $p, q \in I$ are such that $z \in D_p \cap D_q$. Without loss of generality, we may assume that $p < q$. Note that the triangle inequality ensures that $q - p < r_q + r_p$. So, if there were an x with $p < x < q$ and $x \notin D_p \cup D_q$, then we would have

$$q - p = (q - x) + (x - p) \geq r_q + r_p > q - p.$$

This contradiction forces $(p, q) \subset D_p \cup D_q$. Hence, if $(p, q) \cap D_p \cap D_q = \varnothing$, then $\{D_p, D_q\}$ would constitute a separation of the connected interval (p, q), see [**RUD**, 196]. From this impossibility, we may conclude that there exists $x \in (p, q) \cap D_p \cap D_q$. It follows that we may find $\epsilon > 0$ so small so that $(x - \epsilon, x + \epsilon) \subset D_p \cap D_q$. Since the interval $(x - \epsilon, x + \epsilon)$ lies inside I, we see that f_p and f_q agree on $(x - \epsilon, x + \epsilon)$. As this interval obviously has a limit point in the region $D_p \cap D_q$, the Uniqueness Theorem for Holomorphic functions guarantees that f_p and f_q agree on all of $D_p \cap D_q$, whence F is well-defined.

□

Problem 3.8. *Let $\{r_k\}_{k=1}^{\infty}$ be one-to-one enumeration of the rational numbers in $(0, 1]$. For each $k \in \mathbb{N}$, let $p_k, q_k \in \mathbb{N}$ be relatively prime positive integers such that $r_k = p_k/q_k$. Define $f_k : [0, 1] \to \mathbb{R}$ by $f_k(x) := \exp[-(p_k - xq_k)^2]$. Prove that $f_k \to 0$ in measure, as $k \to \infty$, yet, for each $x \in [0, 1]$, the pointwise limit of the the sequence $\{f_k(x)\}_{k=1}^{\infty}$ does not exist.*

Key terms: convergence in measure, pointwise convergence

Solution. Let us prove some helpful lemmas:

LEMMA 3.8.1. *Suppose $a, b, c, d \in \mathbb{N}$ are such that $\gcd(a, b) = \gcd(c, d) = 1$ and $a/b = c/d$. Then $a = c$ and $b = d$.*

PROOF. Since $\gcd(a,b)=1$, we may find $m,n\in\mathbb{Z}$ such that $am+bn=1$. Now, combined with the (assumed) fact that $a/b=c/d$, we get

$$c=c(am+bn)=acm+bcn=acm+adn=a(cm+dn).$$

Thus, a divides c. Symmetrically, we find that c divides a, whence $a=c$. Analogously, we get that $b=d$, completing the proof of the lemma.

□

COROLLARY 3.8.2. *Define $\rho:\mathbb{Q}^+\to\mathbb{N}\times\mathbb{N}$ by the condition that $\rho(r):=(\rho_1(r),\rho_2(r))$ if and only if $r=\rho_1(r)/\rho_2(r)$ and $\gcd(\rho_1(r),\rho_2(r))=1$. Then ρ is a well-defined injective function.*

PROOF. For $r,s\in\mathbb{Q}^+$, the following are equivalent:

(1) $r=s$.
(2) $\rho_1(r)/\rho_2(r)=\rho_1(s)/\rho_2(s)$.
(3) $\rho_1(r)=\rho_1(s)$ and $\rho_2(r)=\rho_2(s)$.
(4) $(\rho_1(r),\rho_2(r))=(\rho_1(r),\rho_2(r))$.
(5) $\rho(r)=\rho(s)$.

Hence, ρ is well-defined and injective.

□

LEMMA 3.8.3. *For each $N\in\mathbb{N}$, put $S_N:=\{r\in\mathbb{Q}\cap(0,1]:\rho_2(r)=N\}$. Then $\coprod_N S_N=\mathbb{Q}\cap(0,1]$ and $|S_N|\le N$, for each N.*

PROOF. That $\coprod_N S_N$ is all of $\mathbb{Q}\cap(0,1]$ is obvious, since every positive rational r has a natural number as its denominator $\rho_2(r)$. That the S_N are disjoint follows just as easily, since ρ_2 is a function and so must assign only one value for any given input. To see that the cardinality of each S_N is at most N, suppose $r\in S_N$. In order that $r\le 1$, we must have $\rho_1(r)\le\rho_2(r)=N$. There are precisely N natural numbers having this property, so that there are at most N possible numerators for r. Hence, there are at most N possible values for r. That is, there at most N elements in S_N.

□

COROLLARY 3.8.4. $\lim\limits_{k\to\infty} q_k=\infty$.

Proof. It is a matter of definition (and the uniqueness element of Lemma 3.8.1) that $q_k = \rho_2(r_k)$. For $M \in \mathbb{N}$, put $T_M := \{r \in \mathbb{Q} \cap (0,1] : \rho_2(r) \leq M\}$. Then $T_M = \coprod_{N=1}^{M} S_N$, where the S_N are as above. Since $|S_N| \leq N$ and T_M is the disjoint union, we get $|T_M| \leq \sum_{N=1}^{M} N = M(M+1)/2$. In particular, T_M has finite cardinality for each M. So, for each $M \in \mathbb{N}$, the fact that $k \mapsto r_k$ is injective guarantees that we may set $K := \max\{k \in \mathbb{N} : r_k \in T_M\}$. Thus, for each $k > K$, $r_k \notin T_M$, and therefore, $q_k > M$. This proves the corollary.

□

Lemma 3.8.5. *Given $x \in (0,1]$ and prime n, there exists l_n such that $|x - r_{l_n}| < 1/q_{l_n}$.*

Proof. Put $I_j := \left(\frac{j-1}{n}, \frac{j}{n}\right]$, for $j = 1, ..., n$. Then $(0,1] = \coprod_{j=1}^{n} I_j$. So, $x \in I_j$, for some j, and therefore $|x - j/n| < 1/n$. By hypothesis, the rational number j/n is r_{l_n} for a unique $l_n \in \mathbb{N}$. Since n is prime $\gcd(j,n) = 1$, and so, if $r_{l_n} = j/n$, then $p_{l_n} = j$ and $q_{l_n} = n$. Hence, $|x - r_{l_n}| = |x - j/n| < 1/n = 1/q_{l_n}$.

□

We are finally prepared to tackle the proof of Problem 3.8. Notice that we must formally verify that each f_k is well-defined, but this is handled by Corollary 3.8.2.

Now, fix $x \in (0,1]$. Choose ϵ so that $0 < \epsilon < x$. Since there are infinitely many rational numbers in $(0, \epsilon)$, we may find a subsequence $\{r_{k_m}\} \subset (0, \epsilon)$. Notice that $0 < |x - \epsilon| < |x - r_{k_m}|$, for each m. Multiplying through by q_{k_m}, squaring, and then taking exponentials gives $f_{k_m}(x) < \exp[-q_{k_m}(x-\epsilon)^2]$. Letting $m \to \infty$ and appealing to Corollary 3.8.4 shows that $f_{k_m}(x) \to 0$.

On the other hand, Lemma 3.8.5 yields a (different) subsequence $\{r_{l_n}\}$ such that $|x - r_{l_n}| < 1/q_{l_n}$, for each n. In consequence, $-1 < -(p_{l_n} - xq_{l_n})^2$ so that $f_{l_n}(x) > 1/e$. It follows that the pointwise limit $f_k(x)$ cannot exist. And this is true of every $x \in (0,1]$.

It remains to be seen that the f_k converge in measure to the zero function. To this end, let $0 < \epsilon < 1$ be given. Observe that if $x \in (0,1]$ is such that $\exp[-(p_k - xq_k)^2] = |f_k(x)| > \epsilon$, then $|x - r_k| < (1/q_k)\sqrt{\ln(1/\epsilon)}$. Thus, $\{x : f_k(x) > \epsilon\} \subseteq (r_k - \delta_k, r_k + \delta_k)$, where $\delta_k := (1/q_k)\sqrt{\ln(1/\epsilon)}$. Hence, $|\{x : |f_k(x)| > \epsilon\}| \le |(r_k - \delta_k, r_k + \delta_k)| = 2\delta_k$. By Corollary 3.8.4, $\delta_k \to 0$ as $k \to \infty$, whence we may choose K so large so that $2\delta_k < \epsilon$, for every $k \ge K$. And so, $f_k \to 0$ in measure.

$\square$

Problem 3.9. *Let $(X, \mathscr{A}, \mu)$ be a finite measure space and f a nonnegative measurable function on X. Then $f \in L^1(X)$ if and only if $\sum_{n=0}^{\infty} 2^n \mu(S_n) < \infty$, where $S_n := \{x \in X : f(x) \ge 2^n\}$.*

Key terms: finite measure space, L^p-space.

Solution. ($\Leftarrow$) For $n \in \mathbb{N}_0$, let $T_n := \{x \in X : 2^n \le f(x) < 2^{n+1}\}$. Also, put $A := \{x : 0 \le f(x) < 1\}$. Note that $X = A \amalg (\coprod_n T_n)$ so that $\mathbf{1}_X = \mathbf{1}_A + \sum_{n=0}^{\infty} \mathbf{1}_{T_n}$. Also, notice that $\mathbf{1}_{T_n} \le \mathbf{1}_{S_n}$, since $T_n \subseteq S_n$. Thus,

$$\begin{aligned} f &= f \cdot \left(\mathbf{1}_A + \sum_{n=0}^{\infty} \mathbf{1}_{T_n} \right) = f \cdot \mathbf{1}_A + \sum_{n=0}^{\infty} f \cdot \mathbf{1}_{T_n} \le \mathbf{1}_A + \sum_{n=0}^{\infty} 2^{n+1} \mathbf{1}_{T_n} \\ &\le \mathbf{1}_X + 2 \sum_{n=0}^{\infty} 2^n \mathbf{1}_{S_n} \in L^1(X), \end{aligned}$$

since $\mu(X)$ and $\sum_{n=0}^{\infty} 2^n \mu(S_n)$ are both assumed finite.

($\Rightarrow$) Let T_m be as above and notice that $S_n = \coprod_{m=n}^{\infty} T_m$, for each $n \in \mathbb{N}_0$. Define $a : \mathbb{N}_0 \times \mathbb{N}_0 \to \{0, 1\}$ by $a_{n,m} := \begin{cases} 0 & m < n \\ 1 & m \ge n \end{cases}$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} 2^n \mathbf{1}_{S_n} &= \sum_{n=0}^{\infty} 2^n \left[\sum_{m=n}^{\infty} \mathbf{1}_{T_m} \right] = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} 2^n \mathbf{1}_{T_m} a_{n,m} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} 2^n \mathbf{1}_{T_m} a_{n,m} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m-1} 2^n \right) \mathbf{1}_{T_m} = \sum_{m=0}^{\infty} (2^m - 1) \mathbf{1}_{T_m} \le \sum_{m=0}^{\infty} 2^m \mathbf{1}_{T_m} \le \sum_{m=0}^{\infty} f \mathbf{1}_{T_m} \\ &= f \cdot \left(\sum_{m=0}^{\infty} \mathbf{1}_{T_m} \right) \le f \cdot \mathbf{1}_X = f \in L^1(X). \end{aligned}$$

$\square$

CHAPTER 4

Fall 2005

Problem 4.1. *Complete the following:*

(a) *State and prove the Schwarz lemma for holomorphic self-maps of* $\mathbb{D}$.

(b) *From* (a)*, conclude that any conformal automorphism of* $\mathbb{D}$ *that fixes zero must be a rotation.*

Key terms: Schwarz Lemma, conformal mapping, rotation, removable singularity, Maximum Modulus Principle, Chain Rule.

Solution. (a) Here essentially is the version found in [**RUD**, 254] and its proof:

THEOREM 4.1.1 (Schwarz Lemma). *Suppose* $f \in H(\mathbb{D})$ *is such that* $\|f\|_\infty \leq 1$ *and zero is fixed by* f. *Then*

$$|f(z)| \leq |z| \qquad (z \in \mathbb{D}) \tag{4.1.1}$$

and

$$|f'(0)| \leq 1. \tag{4.1.2}$$

If equality holds in (4.1.1) *for some* $z \in \mathbb{D}'$ *or if equality holds in* (4.1.2)*, then there exists* $u \in \mathbb{T}$ *such that* $f(z) = uz$*, for every* $z \in \mathbb{D}$.

PROOF. Since f has a zero at 0, it follows that the map $f(z)/z$ has a removable singularity there. (Write f as power series about the origin and factor.) Thus, there exists $g \in H(\mathbb{D})$ such that $f(z) = zg(z)$. Now, fix $z \in \mathbb{D}\setminus\{0\}$. Then for any r such that $|z| < r < 1$ (such an r exists, as $\mathbb{D}$ is open), we have, by the Maximum Modulus Principle, that

$$\frac{|f(z)|}{|z|} = |g(z)| \leq \max_\theta \left|g(re^{i\theta})\right| = \max_\theta \frac{\left|f(re^{i\theta})\right|}{r} \leq \frac{1}{r},$$

since $|f(z)| \leq 1$. Letting $r \to 1$ and multiplying through by $|z|$, gives $|f(z)| \leq |z|$, provided $z \in \mathbb{D}\setminus\{0\}$. Since this inequality is evidently also true at $z = 0$, we get (4.1.1). Now, (4.1.2) follows from the fact that $f'(0) = g(0)$, by the product rule. Finally, if equality holds in (4.1.1) for some $z \in \mathbb{D}'$ or if equality holds in (4.1.2), then application of Maximum Modulus Principle to g again shows that $f(z)/z$ is a constant of unit modulus.

(b) Since f is conformal automorphism of $\mathbb{D}$, f^{-1} is as well (see Theorem 10.33 in [**RUD**]). Thus, f and f^{-1} both satisfy the hypothesis of the Schwarz Lemma, and therefore,

$$|f'(0)|, \left|(f^{-1})'(0)\right| \leq 1. \tag{4.1.3}$$

Since $z = (f \circ f^{-1})(z)$, for all $z \in \mathbb{D}$, we get

$$1 = f'[f^{-1}(0)](f^{-1})'(0) = f'(0)(f^{-1})'(0), \tag{4.1.4}$$

by the Chain Rule. Combining (4.1.3) and (4.1.4), we must have $|f'(0)| = 1$, allowing us to deduce from the second part of the Schwarz Lemma that f is a rotation.

□

Problem 4.2. *Let* $a \in \mathbb{D}$ *and define* $f_a(z) := \dfrac{a - z}{1 - \bar{a}z}$, *for* $z \in \bar{\mathbb{D}}$.

(a) *Show that* f_a *is an automorphism of* $\mathbb{D}$ *and is its own inverse.*
(b) *Show that for every conformal automorphism* f *of* $\mathbb{D}$ *there exists* $u \in \mathbb{T}$ *and* $a \in \mathbb{D}$ *such that* $f = u \cdot f_a$.
(c) *Verify that* (b) *implies that every conformal automorphism of* $\mathbb{D}$ *extends to a homeomorphism of* $\bar{\mathbb{D}}$.
(d) *Show that the values* a *and* u *are uniquely determined by* f.

Key terms: conformal mapping, Maximum Modulus Principle, Chain Rule, homeomorphism.

Solution. (a) If $a = 0$, then $f_a(z) = -z$, and so, (a) is immediate, in this case. So, we may assume that $a \in \mathbb{D}\setminus\{0\}$. Note that this forces $\bar{a}^{-1} \in \mathbb{C}\setminus\bar{\mathbb{D}}$. The computation

$$\begin{aligned}(f_a \circ f_a)(z) &= \frac{a - \frac{a-z}{1-\bar{a}z}}{1 - \bar{a} \cdot \frac{a-z}{1-\bar{a}z}} = \frac{\frac{a(1-\bar{a}z)-(a-z)}{1-\bar{a}z}}{\frac{1-\bar{a}z-\bar{a}(a-z)}{1-\bar{a}z}} = \frac{a - |a|^2 z - a + z}{1 - \bar{a}z - |a|^2 + \bar{a}z} \\ &= \frac{z - |a|^2 z}{1 - |a|^2} = z\end{aligned}$$

shows simultaneously that, on $\mathbb{C}\setminus\{a^{-1}\}$, f_a acts as both left and right inverse to itself. Thus, f_a is a bijection on $\mathbb{C}\setminus\{\bar{a}^{-1}\}$. It remains to show that f_a maps $\bar{\mathbb{D}}$ onto $\bar{\mathbb{D}}$. We will, in fact, prove a stronger result: f_a maps $\mathbb{T}$ onto $\mathbb{T}$ and $\mathbb{D}$ onto $\mathbb{D}$. To see this, let $u \in \mathbb{T}$. Then

$$|f_a(u)| = \left|\frac{a-u}{1-\bar{a}u}\right| = \frac{|a-u|}{|-u(\bar{a}-u^{-1})|} = \frac{|a-u|}{|-u| \cdot |\overline{a-u}|} = 1.$$

It follows that $f_a(\mathbb{T}) \subset \mathbb{T}$, and so, $\mathbb{T} = (f_a \circ f_a)(\mathbb{T}) \subset f_a(\mathbb{T})$. These inclusions combine to give $f_a(\mathbb{T}) = \mathbb{T}$. Since is f_a is nonconstant, we may deduce from the Maximum Modulus Principle that $|f_a(z)| < \sup\{|f_a(w)| : w \in \partial\mathbb{D} = \mathbb{T}\} = 1$, for every $z \in \mathbb{D}$. Thus, $f_a(\mathbb{D}) \subset \mathbb{D}$. Applying f_a to both sides reverses the inclusion, so that $f_a(\mathbb{D}) = \mathbb{D}$.

(b) Put $a := f^{-1}(0) \in \mathbb{D}$. Since f and f_a are both conformal automorphisms of $\mathbb{D}$, it follows from the Chain Rule that $f \circ f_a$ is also a conformal automorphism of $\mathbb{D}$. Moreover, $(f \circ f_a)(0) = f[f_a(0)] = f(a) = f[f^{-1}(0)] = 0$; i.e., f fixes 0. Part (b) of Problem 4.1 above yields $u \in \mathbb{T}$ such that $(f \circ f_a)(z) = uz$, for every $z \in \mathbb{D}$. In consequence, for every $z \in \mathbb{D}$, we have

$$f(z) = f[(f_a \circ f_a)(z)] = (f \circ f_a)[f_a(z)] = uf_a(z).$$

(c) f_a and multiplication by u are both homemorphisms of $\bar{\mathbb{D}}$, and therefore, their composition is a homeomorphism of $\bar{\mathbb{D}}$.

(d) This amounts to proving that if $u, v \in \mathbb{T}$ and $a, b \in \mathbb{D}$ and

$$uf_a(z) = vf_b(z) \qquad (\forall z \in \mathbb{D}), \tag{4.2.1}$$

then $u = v$ and $a = b$. To verify that this is true, first observe that since (4.2.1) holds for $z = b$, $u\frac{a-b}{1-\bar{a}b} = 0$. Since $u \neq 0$, we conclude that $a - b = 0$, and therefore, $a = b$.

Thus, for every $z \in \mathbb{D}$, we have $uf_a(z) = vf_a(z)$, or equivalently, $(u - v)f_a(z) = 0$, for every $z \in \mathbb{D}$. Since f_a is not the zero function, it must be that $u - v = 0$, and so, $u = v$.

□

Problem 4.3. *Suppose the series* $\sum_{n=0}^{\infty} c_n z^n$ *converges in* $\mathbb{D}$ *and the function* $f(z)$ *that it defines vanishes at* $\frac{1}{k}$*, for each* $k \in \mathbb{N}$*. Prove that* $f \equiv 0$*.*

Key terms: power series, Well-Ordering Principle in $\mathbb{N}_0$.

Solution. Assume, with a view to reach a contradiction, that there exists $n \in \mathbb{N}_0$ such that $c_n \neq 0$. By the Well-Ordering Principle, the set $\{n \in \mathbb{N}_0 : c_n \neq 0\}$ has a least element, say N. Put $g(z) := \sum_{n=0}^{\infty} c_{N+n} z^n$. By the very definition of N, it must be that $c_n = 0$, for $n = 0, 1, ..., N - 1$. In consequence,

$$f(z) = \sum_{n=N}^{\infty} c_n z^n = z^N g(z).$$

From this and the fact that $f\left(\frac{1}{n}\right) = 0$, for every $n \in \mathbb{N}$, we conclude that $g\left(\frac{1}{n}\right) = 0$, for every $n \in \mathbb{N}$. As g is represented by a convergent power series in $\mathbb{D}$, it must be that $g \in H(\mathbb{D})$, whence g is continuous. Thus,

$$c_N = g(0) = g\left(\lim_{n\to\infty} \frac{1}{n}\right) = \lim_{n\to\infty} g\left(\frac{1}{n}\right) = \lim_{n\to\infty} 0 = 0,$$

contradicting our choice of N. So, our initial assumption is false, from which we infer that $c_n = 0$, for every $n \in \mathbb{N}_0$, and so, $f \equiv 0$.

□

Problem 4.4. *Complete the following:*

(a) *State the Open-Mapping Theorem for holomorphic functions.*

(b) *State the Maximum Modulus Principle for holomorphic functions.*

(c) *Provide a short deduction of* (b) *from* (a)*.*

(d) *Show that the image of any closed subset (of the plane) under a nonconstant polynomial is closed.*

(e) *From* (a) *and* (d) *deduce the Fundamental Theorem of Algebra.*

Key terms: Open-Mapping Theorem, Maximum Modulus Principle, Fundamental Theorem of Algebra, Bolzano-Weierstrass Theorem.

Solution. (a) Here is the terse version found in [**RUD**, 214]:

THEOREM 4.4.1 (The Open-Mapping Theorem). *If Ω is a region and $f \in H(\Omega)$, then $f(\Omega)$ is either a region or a point.*

(b) Again, we provide Rudin's version found on [**RUD**, 253]:

THEOREM 4.4.2 (The Maximum Modulus Principle). *If $f \in H(\Omega) \cap C(\bar{\Omega})$, where Ω is a bounded region, then the inequality*

$$|f(z)| \leq \|f\|_{\partial\Omega} := \sup_{w \in \partial\Omega} |f(w)|$$

holds for every $z \in \Omega$. Moreover, equality occurs at some point in Ω if and only if f is constant on $\bar{\Omega}$.

(c) We may deduce (b) from (a) since no nonempty open subset of $\mathbb{C}$ contains an element of maximum modulus. To see this, suppose Ω is an open subset of $\mathbb{C}$ and $z_0 \in \Omega$. Since Ω is open, we may find $r > 0$ such that $D(z_0, r) \subset \Omega$. Put

$$z_1 := \begin{cases} z_0 + \frac{z_0 r}{2|z_0|} & z_0 \neq 0 \\ \frac{r}{2} & z_0 = 0 \end{cases}.$$

Direct computation shows that $|z_1 - z_0| = \frac{r}{2}$, and therefore, $z_1 \in D(z_0, r) \subset \Omega$. Moreover, we also have that $|z_1| > |z_0|$. In summary, for any point $z_0 \in \Omega$, we may find another point z_1 in Ω that has larger modulus than z_0; i.e., Ω does not contain a point of maximum modulus.

(d) Let $p(z) := \sum_{k=0}^{m} a_k z^k$ be a nonconstant polynomial, with $a_m \neq 0$ $(m > 0)$, and let K be a closed subset of the plane. Let $w \in \overline{p(K)}$. Choose a sequence $\{w_n\} \subset p(K)$ such that $w_n \to w$. As, each $w_n \in p(K)$, we may find $z_n \in K$ such that $p(z_n) = w_n$. Now, the triangle inequality gives

$$|a_m| |z|^m = \left| p(z) - \sum_{k=0}^{m-1} a_k z^k \right| \leq |p(z)| + \sum_{k=0}^{m-1} |a_k| |z|^k,$$

and therefore,

$$|z|^m \left(|a_m| - \sum_{k=0}^{m-1} \frac{|a_k|}{|z|^{m-k}} \right) \leq |p(z)| \qquad (z \neq 0). \tag{4.4.1}$$

It follows easily from (4.4.1) that $|p(z)| \to \infty$, as $z \to \infty$, since $a_m \neq 0$. In consequence, the sequence $\{z_n\}$, must be bounded, for otherwise there would be infinitely many $n \in \mathbb{N}$ such that $|w_n| = |p(z_n)| > |w| + 1$, contradicting that $|w_n| \to |w|$. Thus, by the Bolzano-Weierstrass Theorem, we may find a subsequence $\{z_{n_l}\}$ that converges to some element, say z, of K, since K is closed. Finally, from the continuity of p, we get

$$w = \lim_{l\to\infty} w_{n_l} = \lim_{l\to\infty} p(z_{n_l}) = p\left(\lim_{l\to\infty} z_{n_l}\right) = p(z),$$

and so, $w \in p(K)$, proving $p(K)$ is closed.

(e) Here is the form of the Fundamental Theorem of Algebra that we shall prove:

THEOREM 4.4.3 (The Fundamental Theorem of Algebra). *If $p(z)$ is a polynomial of positive degree m over $\mathbb{C}$, then there are precisely m zeros of p in $\mathbb{C}$, provided one accounts for multiplicity.*

PROOF. Write $p(z) := \sum_{k=0}^{m} a_k z^k$. Choose $r > 0$ such that $|a_m|\, r^m - \sum_{k=0}^{m-1} |a_k|\, r^k > |a_0| = |p(0)|$. Then, as in the proof of (d), the triangle inequality ensures that $\left|p(re^{i\theta})\right| > |p(0)|$, for all θ. If p does not have any zeros, then the function $f(z) := 1/p(z)$ would be entire and satisfy $|f(0)| > \left|f(re^{i\theta})\right|$, for all θ. This contradicts the Maximum Modulus Principle. Thus p has at least one zero, say z_0, and so, we may write $p(z) = (z - z_0)q(z)$, where q is some polynomial of degree $m-1$. By induction, we may deduce that p has exactly m zeros, counting multiplicities.

□

Problem 4.5. *Let X be a Lebesgue measurable subset of $\mathbb{R}$ with $|X| = \infty$. Construct a function f such that $f \in L^p(X)$, for every $p \geq 1$, but $f \notin L^\infty(X)$.*

Key terms: Lebesgue measure, L^p-space, uniform continuity, Lebesgue's Monotone Convergence Theorem, Beppo-Levi Theorem, ratio test, essential supremum

Solution. We will use the following two general results:

THEOREM 4.5.1. *Let E be a Lebesgue measurable subset of $\mathbb{R}$. Then the function defined by $f_E(x) := |E \cap (-x, x)|$, for $x \in [0, \infty)$, is a uniformly continuous map taking $[0, \infty)$ onto $[0, |E|)$. Furthermore, $\lim_{x\to\infty} f_E(x) = |E|$.*

PROOF. Notice that we may write $f_E(x) = \int_{\mathbb{R}} \mathbf{1}_E \cdot \mathbf{1}_{(-x,x)}$. Let $x, y \in [0, \infty)$. Without loss of generality, assume $x \leq y$. Observe

$$\begin{aligned} |f_E(x) - f_E(y)| &= \left| \int_{\mathbb{R}} \mathbf{1}_E \cdot \mathbf{1}_{(-x,x)} - \int_{\mathbb{R}} \mathbf{1}_E \cdot \mathbf{1}_{(-y,y)} \right| \leq \int_{\mathbb{R}} \mathbf{1}_E \left| \mathbf{1}_{(-x,x)} - \mathbf{1}_{(-y,y)} \right| \\ &= \int_{\mathbb{R}} \mathbf{1}_E \left[\mathbf{1}_{(-y,-x)} + \mathbf{1}_{(x,y)} \right] = |E \cap (-y, -x)| + |E \cap (x, y)| \\ &\leq 2(y - x) = 2\,|x - y|\,. \end{aligned}$$

The uniform continuity of f is now immediate.

Applying Lebesgue's Monotone Convergence Theorem to the sequence $\left\{\mathbf{1}_E \mathbf{1}_{(-n,n)}\right\}$ shows that $\lim_{n\to\infty} f_E(n) = \int_{\mathbb{R}} \mathbf{1}_E = |E|$. As $f_E(0) = 0$, we may deduce that f_E maps $[0, \infty)$ onto $[0, |E|)$ from the Intermediate-Value Theorem.

□

PROPOSITION 4.5.2. *Let $\{x_n\} \subset [0, \infty)$ and X a Lebesgue measurable set of infinite measure. Then there exists a sequence $\{X_n\}$ of pairwise disjoint Lebesgue measurable subsets of X such that $|X_n| = x_n$, for each $n \in \mathbb{N}$.*

PROOF. Let f_X be as in Theorem 4.5.1. Since f_X maps $[0, \infty)$ onto $[0, |X|) = [0, \infty)$, we may find $x \in [0, \infty)$ such that $X_1 = X \cap (-x, x) \subset X$ has measure x_1. Assume inductively that we have found pairwise disjoint measurable subsets $X_1, ..., X_n$ of X such that $|X_k| = x_k$, for $1 \leq k \leq n$. By additivity, we have that $|X| = |X \cap (\bigcap_{k=1}^n X_k^c)| + \sum_{k=1}^n |X_k|$. Since $|X| = \infty$ and $\sum_{k=1}^n |X_k| = \sum_{k=1}^n x_k < \infty$, it must be that $|X \cap (\bigcap_{k=1}^n X_k^c)| = \infty$. Mimicking the construction of X_1, we may find $X_{n+1} \subset X \cap (\bigcap_{k=1}^n X_k^c) \subseteq X$ such that $|X_{n+1}| = x_{n+1}$. That X_{n+1} is disjoint from each X_k, $1 \leq k \leq n$ is obvious. This completes the induction and yields the desired sequence of sets.

□

Returning to Problem 4.5, by the results above, we may find a sequence $\{X_n\}$ of pairwise disjoint measurable subsets of X such that $|X_n| = 2^{-n}$, for every $n \in \mathbb{N}$. Put $f := \sum_{n=1}^{\infty} n\mathbf{1}_{X_n}$. Since f is nonnegative and since the X_n are pairwise disjoint, it follows that $|f|^p = f^p = \sum_{n=1}^{\infty} n^p \mathbf{1}_{X_n}$. Thus, the Beppo-Levi Theorem and the ratio test for series give

$$\int_X |f|^p = \int_X \sum_{n=1}^{\infty} n^p \mathbf{1}_{X_n} = \sum_{n=1}^{\infty} \int_X n^p \mathbf{1}_{X_n} = \sum_{n=1}^{\infty} n^p |X_n| = \sum_{n=1}^{\infty} n^p 2^{-n} < \infty,$$

whence $f \in L^p(X)$.

To verify that $f \notin L^\infty(X)$, set $S_M := \{x \in X : |f(x)| > M\}$, for $M \geq 0$. Since we may always find $n \in \mathbb{N}$ so large so that $M < n$ and since $|f| = f$, we have that

$$S_M \supseteq \{x \in X : f(x) = n\} = X_n.$$

By monotonicity, $|S_M| \geq |X_n| = 2^{-n} > 0$, and consequently, the set $\{M : |S_M| = 0\}$ is empty. Hence, the essential supremum $\|f\|_\infty = \infty$, and therefore, $f \notin L^\infty(X)$.

□

Problem 4.6. *Let $(X, \mathscr{A}, \mu)$ be a finite positive measure space. Suppose $f \in L^\infty(X)$ is such that $\|f\|_\infty > 0$. For $n \in \mathbb{N}$, put $\alpha_n := \int_X |f|^n \, d\mu = \|f\|_n^n$. Show that*

(a) *$\|f\|_n \to \|f\|_\infty$, as $n \to \infty$ and*

(b) *$\alpha_{n+1}/\alpha_n \to \|f\|_\infty$, as $n \to \infty$.*

Key terms: essential supremum, finite measure space, positive measure.

Solution. (a) $f \in L^\infty(X)$ means that $\|f\|_\infty < \infty$, where

$$\|f\|_\infty := \inf \{M : \mu(\{x : |f(x)| > M\}) = 0\}.$$

So, for every $\epsilon > 0$, the set $S_\epsilon := \{x \in X : |f(x)| > \|f\|_\infty - \epsilon\}$ is nonempty, and moreover, $\mu(S_\epsilon) > 0$. Since $|f| > \|f\|_\infty - \epsilon$ on S_ϵ, we have the following

$$\begin{aligned} 0 &< (\|f\|_\infty - \epsilon)^n \mu(S_\epsilon) = \int_{S_\epsilon} (\|f\|_\infty - \epsilon)^n \, d\mu \leq \int_{S_\epsilon} |f|^n \, d\mu \\ &\leq \int_X |f|^n \, d\mu = \|f\|_n^n, \end{aligned}$$

for every $n \in \mathbb{N}$, provided $\epsilon < \|f\|_\infty$. Taking the n^{th} root gives

$$[\mu(S_\epsilon)]^{\frac{1}{n}} \left(\|f\|_\infty - \epsilon\right) \leq \|f\|_n \,.$$

Since $0 < \mu(S_\epsilon) \leq \mu(X) < \infty$, we have that $\lim_{n\to\infty} [\mu(S_\epsilon)]^{\frac{1}{n}} = 1$, whence

$$\|f\|_\infty - \epsilon \leq \liminf_{n\to\infty} \|f\|_n \,.$$

As this is true for arbitrary $\epsilon > 0$, we get

$$\|f\|_\infty \leq \liminf_{n\to\infty} \|f\|_n \,. \tag{4.6.1}$$

For the reverse inequality, we recall that $|f| \leq \|f\|_\infty$ a.e. Hence, $|f|^n \leq \|f\|_\infty^n$ a.e., for each $n \in \mathbb{N}$. So, we obtain

$$\|f\|_n^n = \int_X |f|^n \, d\mu \leq \int_X \|f\|_\infty^n \, d\mu = \|f\|_\infty \mu(X).$$

Taking the n^{th} root, gives

$$\|f\|_n \leq \|f\|_\infty [\mu(X)]^{\frac{1}{n}} \,. \tag{4.6.2}$$

Now, since $0 < \|f\|_\infty$, we must necessarily have $0 < \mu(X)$, and so, $\lim_{n\to\infty} [\mu(X)]^{\frac{1}{n}} = 1$. By taking the limit superior in (4.6.2), we get $\limsup_{n\to\infty} \|f\|_n \leq \|f\|_\infty$. This with (4.6.1) proves that $\lim_{n\to\infty} \|f\|_n$ exists and is equal to $\|f\|_\infty$.

(b) Combining $\int_X |f|^n \, d\mu \leq \|f\|_\infty^n \mu(X) < \infty$ with (4.6.1), we see that $\alpha_n \in (0, \infty)$, for every $n \in \mathbb{N}$. As

$$\int_X |f|^{n+1} \, d\mu = \int_X |f| \cdot |f|^n \, d\mu \leq \|f\|_\infty \int_X |f|^n \, d\mu,$$

we obtain $\alpha_{n+1} \leq \|f\|_\infty \alpha_n$, for every n, so that

$$\limsup_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty \,. \tag{4.6.3}$$

Now, Hölder's Inequality yields

$$\begin{aligned} \alpha_n &= \int_X |f|^n \, d\mu \leq \left[\int_X \left(|f|^n\right)^{\frac{n+1}{n}} \, d\mu\right]^{\frac{n}{n+1}} \left[\int_X \mathbf{1}_X^{n+1} d\mu\right]^{\frac{1}{n+1}} = \left(\alpha_{n+1}\right)^{\frac{n}{n+1}} [\mu(X)]^{\frac{1}{n+1}} \\ &= \frac{\alpha_{n+1} [\mu(X)]^{\frac{1}{n+1}}}{\|f\|_{n+1}}, \end{aligned}$$

or equivalently,

$$\frac{\|f\|_{n+1}}{[\mu(X)]^{\frac{1}{n+1}}} \leq \frac{\alpha_{n+1}}{\alpha_n} \qquad (n \in \mathbb{N}). \tag{4.6.4}$$

Part (a) along with the fact that $\lim_{n\to\infty} [\mu(X)]^{\frac{1}{n+1}} = 1$ (as $\mu(X) \in (0, \infty)$) yields

$$\|f\|_\infty = \lim_{n\to\infty} \frac{\|f\|_{n+1}}{[\mu(X)]^{\frac{1}{n+1}}} \leq \liminf_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n}, \tag{4.6.5}$$

by (4.6.4). Consideration of (4.6.5) along side (4.6.2) shows that $\lim_{n\to\infty} \alpha_{n+1}/\alpha_n$ exists, and moreover, $\|f\|_\infty = \lim_{n\to\infty} \alpha_{n+1}/\alpha_n$.

Problem 4.7. *For $p \in \mathbb{R}$, define $h_p := \sum_{n=1}^\infty n^p \mathbf{1}_{I_n}$, where $I_n := \left(\frac{1}{n+1}, \frac{1}{n}\right]$. Prove*

(a) *$h_p \in L^1(\mathbb{R})$, provided $p < 1$.*

(b) *$h_1 \in L^1_{\text{weak}}(\mathbb{R}) \backslash L^1(\mathbb{R})$.*

(c) *$h_p \notin L^1_{\text{weak}}(\mathbb{R})$, for $p > 1$.*

Key terms: L^p-space, Weak L^p, Beppo-Levi Theorem, p-test.

Solution. (a) Observe that

$$\begin{aligned} \int h_p &= \int \sum n^p \mathbf{1}_{I_n} = \sum n^p \int \mathbf{1}_{I_n} = \sum n^p \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \sum \frac{n^p}{n(n+1)} = \sum \frac{n^{p-1}}{n+1} \leq \sum \frac{1}{n^{2-p}}, \end{aligned} \tag{4.7.1}$$

where the change of integration and summation is justified by the Beppo-Levi Theorem. The series on the right of (4.7.1) converges by the p-test for series, since $p < 1$.

(b) For $\lambda \in \mathbb{R}$, put $S_\lambda := \{x \in \mathbb{R} : |h_1(x)| > \lambda\}$. Fix $\lambda \geq 0$. Define $N := \max\{n \in \mathbb{N} : n \leq \lambda\}$. Note that $N \leq \lambda$, while $N + 1 > \lambda$.

It follows that $S_N \supseteq S_\lambda$ so that $|S_\lambda| \leq |S_N|$. Now, from the definition of h_1 we readily see that $h_1(x) = |h_1(x)| > N$ if and only if $x \in \coprod_{n=N+1}^\infty I_n$. Thus, $S_N = \coprod_{n=N+1}^\infty I_n$, and therefore,

$$|S_N| = \sum_{n=N+1}^\infty \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{N+1} - \lim_{n\to\infty} \frac{1}{n+1} = \frac{1}{N+1} < \frac{1}{\lambda}.$$

In summary,

$$\lambda\,|S_\lambda| \leq \lambda\,|S_N| < \lambda(1/\lambda) = 1.$$

As $\lambda \geq 0$ is arbitrary, we may take the supremum over all such λ to conclude that $h_1 \in L^1_{\text{weak}}(\mathbb{R})$.

To see that $h_1 \notin L^1(\mathbb{R})$, we note that the computation involved in (4.7.1) is valid for h_1. That is, $\int h_1 = \sum \frac{1}{n+1}$. Since the series is not finite, neither is the integral.

(c) First, let us set some notation. For $x \in \mathbb{R}$, we define $[x] := \max\{n \in \mathbb{Z} : n \leq x\}$. Now, analogous to the above, let $S_\lambda := \{x \in \mathbb{R} : |h_p(x)| > \lambda\}$. Fix p and λ and set $N := \min\{n \in \mathbb{N} : n^p \geq \lambda\}$. By definition, $N^p \geq \lambda$, while $(N-1)^p < \lambda$. Now, $x \in S_N$ if and only if $h_p(x) = |h_p(x)| > N$ if and only if $x \in \coprod_{n=[N^{1/n}]+1}^{\infty} I_n$ so that $S_N = \coprod_{n=[N^{1/n}]+1}^{\infty} I_n$, and therefore, $|S_N| = \frac{1}{[N^{1/p}]+1}$. Since $N \geq \lambda$, $S_N \subseteq S_\lambda$, and so,

$$\lambda\,|S_\lambda| \geq \lambda\,|S_N| > \frac{(N-1)^p}{[N^{\frac{1}{p}}]+1} \geq \frac{(N-1)^p}{[N^{\frac{1}{p}}]+1} \geq \frac{(N/2)^p}{2N^{\frac{1}{p}}} = \frac{N^{p-\frac{1}{p}}}{2^{p+1}}.$$

Now, since $p > 1$, $N \to \infty$, as $\lambda \to \infty$. Hence, we conclude that $h_p \notin L^1_{\text{weak}}(\mathbb{R})$.

□

Problem 4.8. *Choose intervals* $J_n \subset (0,1)$ *in such a way that* $U = \bigcup_n J_n$ *is dense in* $(0,1)$ *and yet the set* $K := (0,1)\backslash U$ *has positive measure.*

Key terms: Lebesgue measure

Solution. Let $0 < \epsilon < 1$ and put $Q := \mathbb{Q} \cap (0,1)$. Note that Q is dense in $(0,1)$, yet has measure zero, since it is countable. By the definition of Lebesgue (outer) measure, we may find open intervals I_n such that $\cup_n I_n$ covers Q and $\sum_n |I_n| < \epsilon$. Put $J_n := I_n \cap (0,1)$. Then each J_n is an interval (or empty),$J_n \subset (0,1)$, for each n, and moreover, $U = \bigcup_n J_n \subset \bigcup_n I_n$. Hence, $|U| =\leq |\bigcup_n I_n| \leq \sum_n |I_n| < \epsilon$. Since $(0,1)$ has finite measure, we get

$$|K| = |(0,1)\backslash U| = |(0,1)| - |U| = 1 - |U| > 1 - \epsilon > 0.$$

□

CHAPTER 5

Spring 2006

Problem 5.1. *Let* $(X, \mathscr{A}, \mu)$ *be a finite measure space. Prove:*

(a) *For* $1 \leq r \leq s \leq \infty$*, we have* $L^s(X) \subseteq L^r(X)$*.*

(b) *The result of* (a) *can fail if* $\mu(X) = \infty$*.*

Key terms: finite measure space, L^p-space, Hölder's Inequality, Lebesgue Measure, conjugate exponent.

Solution. (a) Clearly, we may assume that $r < s$, so that $1 < \frac{s}{r}$. Put $p := \frac{s}{r}$ and let q be its conjugate exponent.

For $f \in L^s(X)$, Hölder's inequality gives

$$\begin{aligned}\int_X |f|^r \, d\mu &= \int_X |f|^r \cdot \mathbf{1}_X d\mu \leq \left[\int_X (|f|^r)^p\right]^{1/p} \left[\int_X \mathbf{1}_X^q d\mu\right]^{1/q} \\ &= \left[\int_X |f|^{rp}\right]^{1/p} \left[\int_X \mathbf{1}_X d\mu\right]^{1/q} = \|f\|_s^r \cdot [\mu(X)]^{1/q} < \infty.\end{aligned}$$

Consequently, $f \in L^r(X)$. As this is true for each $f \in L^s(X)$, the conclusion follows.

(b) Let $X := [1, \infty)$, $\mathscr{A} :=$ Lebesgue measurable subsets of $[1, \infty)$, and $\mu :=$ Lebesgue measure. Then for $f(x) := \frac{1}{x}$, we have

$$\int_X |f|^2 \, d\mu = \int_1^\infty \frac{1}{x^2} dx = 1,$$

while

$$\int_X |f| \, d\mu = \int_1^\infty \frac{1}{x} dx = \infty.$$

Hence, in this situation, we see that $L^2(X) \not\subseteq L^1(X)$, even though $1 \leq 2$.

□

Problem 5.2. *Let* $A, B \subseteq \mathbb{R}$ *be Lebesgue measurable and define* $h(x) := |(A - x) \cap B|$*. Show that* (a) h *is a Lebesgue measurable function; and,* (b) $\int_{\mathbb{R}} h(x)dx = |A|\,|B|$*.*

Key terms: measurable function, measurable set, convolution, Lebesgue's Monotone Convergence Theorem, Fubini's Theorem, translation invariance.

Solution. For $n \in \mathbb{N}$, define $A_n := A \cap (-n, n)$ and $B_n := B \cap (-n, n)$. Notice that $|A_n|, |B_n| \leq 2n < \infty$ so that $\mathbf{1}_{A_n}, \mathbf{1}_{B_n} \in L^1(\mathbb{R})$. Consequently, by Theorem 8.14 in [**RUD**] (the Convolution Theorem), the function given by

$$h_n(x) := \int_{\mathbb{R}} \mathbf{1}_{A_n}(x+y)\mathbf{1}_{B_n}(y)dy$$

is Lebesgue integrable. In particular, each h_n is Lebesgue measurable.

Fix $x \in \mathbb{R}$ and observe that $\left\{\mathbf{1}_{(A_n - x)\cap B_n}\right\}_{n=1}^{\infty}$ is a monotone increasing sequence of measurable functions that converge everywhere to $\mathbf{1}_{(A-x)\cap B}$, whence Lebesgue's Monotone Convergence Theorem yields

$$\begin{aligned}\int_{\mathbb{R}} \mathbf{1}_{(A-x)\cap B}(y)dy &= \lim_{n\to\infty} \int_{\mathbb{R}} \mathbf{1}_{(A_n-x)\cap B_n}(y)dy = \lim_{n\to\infty} \int_{\mathbb{R}} \mathbf{1}_{A_n}(x+y)\mathbf{1}_{B_n}(y)dy \\ &= \lim_{n\to\infty} h_n(x).\end{aligned}$$

On the other hand,

$$h(x) = |(A-x)\cap B| = \int_{\mathbb{R}} \mathbf{1}_{A-x}(y)\mathbf{1}_B(y)dy = \int_{\mathbb{R}} \mathbf{1}_{(A-x)\cap B}(y)dy,$$

so that $h(x) = \lim_{n\to\infty} h_n(x)$. Since x is arbitrary, we conclude that h is the pointwise limit of a sequence of measurable functions, and so, h is measurable.

To show (b), we first notice that

$$\mathbf{1}_{A-x}(y) = \mathbf{1}_A(x+y) = \mathbf{1}_{A-y}(x)$$

holds for each $x, y \in \mathbb{R}$. Combining this with Fubini's Theorem and the translation-invariance of Lebesgue measure gives

$$\begin{aligned}\int_{\mathbb{R}} h(x)dx &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \mathbf{1}_{A-x}(y)\mathbf{1}_B(y)dy\right]dx = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \mathbf{1}_{A-x}(y)\mathbf{1}_B(y)dx\right]dy \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \mathbf{1}_{A-y}(x)dx\right]\mathbf{1}_B(y)dy = \int_{\mathbb{R}} |A-y|\,\mathbf{1}_B(y)dy \\ &= \int_{\mathbb{R}} |A|\,\mathbf{1}_B(y)dy = |A|\int_{\mathbb{R}} \mathbf{1}_B(y)dy = |A|\,|B|.\end{aligned}$$

□

Problem 5.3. *Let $\mathscr{A}$ be a σ-algebra on a set X and assume $\mu : \mathscr{A} \to [0, \infty]$ has the following properties:*

(i) *If $A_1, A_2 \in \mathscr{F}$ with $A_1 \cap A_2 = \varnothing$, then $\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$.*

(ii) *If $\{A_n\}_{n=1}^{\infty}$ is a sequence in $\mathscr{A}$ such that $A_{n+1} \subset A_n$, for all $n \in \mathbb{N}$, and $\bigcap\limits_{n=1}^{\infty} A_n = \varnothing$, then $\lim\limits_{n\to\infty} \mu(A_n) = 0$.*

Prove that μ is a positive measure on X.

Key terms: σ-algebra, positive measure, finitely additive, countably additive.

Solution. Since $\mathscr{A}$ is a σ-algebra, we have $\varnothing \in \mathscr{A}$. For $n \in \mathbb{N}$, let $A_n = \varnothing$. The sequence $\{A_n\}$ satisfies the hypothesis of (ii), whence

$$\mu(\varnothing) = \lim_{n\to\infty} \mu(\varnothing) = \lim_{n\to\infty} \mu(A_n) = 0.$$

So, μ is not identically infinite.

It remains to be shown that μ is countably additive. Now, condition (i) combined with induction readily shows that μ is finitely additive; i.e.,

$$\mu\left(\bigcup_{n=1}^{k} A_n\right) = \sum_{n=1}^{k} \mu(A_n),$$

whenever $A_1, ..., A_k \in \mathscr{A}$ are pairwise disjoint.

To see that μ is, in fact, countably additive, let $\{B_m\}_{m=1}^{\infty}$ be a sequence of measurable pairwise disjoint sets. For $n \in \mathbb{N}$, put $A_n := \bigcup\limits_{m=n}^{\infty} B_m$. Clearly, $A_{n+1} \subset A_n$, for each n. Now, suppose, in order to reach a contradiction, that there exists $x \in \bigcap\limits_{n=1}^{\infty} A_n$. Then $x \in A_1 = \bigcup\limits_{m=1}^{\infty} B_m$. Since the sets B_m are pairwise disjoint, there is a *unique* $n(x) \in \mathbb{N}$ such that $x \in B_{n(x)}$. Hence, $x \notin \bigcup\limits_{m=n(x)+1}^{\infty} B_m = A_{n(x)+1}$. This contradicts that $x \in \bigcap\limits_{n=1}^{\infty} A_n$. Thus, $\bigcap\limits_{n=1}^{\infty} A_n = \varnothing$, and therefore, $\{A_n\}$ has the requisite properties to invoke (ii). In consequence, $\lim\limits_{n\to\infty} \mu(A_n) = 0$. Now, notice that as the B_m are pairwise disjoint, the sets $\bigcup_{m=1}^{n-1} B_m$ and A_n are disjoint, provided $n \geq 2$. So, for each $n \geq 2$, we have

$$\mu\left(\bigcup_{m=1}^{\infty} B_m\right) = \mu\left[\left(\bigcup_{m=1}^{n-1} B_m\right) \cup A_n\right] = \mu\left(\bigcup_{m=1}^{n-1} B_m\right) + \mu(A_n) = \sum_{m=1}^{n-1} \mu(B_m) + \mu(A_n).$$

Taking the limit as $n \to \infty$ completes the proof.

□

Problem 5.4. *Let h be a bounded Lebesgue measurable function on $\mathbb{R}$ having the property that $\lim_{n\to\infty} \int_E h(nx)dx = 0$, for every Lebesgue measurable subset E of finite measure. Show that for every $f \in L^1(\mathbb{R})$, we have $\lim_{n\to\infty} \int_{\mathbb{R}} f(x)h(nx)dx = 0$.*

Key terms: Lebesgue measure, L^p-space, abstract integration, simple function.

Solution. Let s be a complex, measurable, simple function on $\mathbb{R}$ that vanishes outside a set of finite measure. Write $s = \sum_{k=1}^m \alpha_k \mathbf{1}_{A_k}$, where $\alpha_k \in \mathbb{C}\setminus\{0\}$ and A_k are Lebesgue measurable and pairwise disjoint. Note that since s vanishes outside a set of finite measure, each A_k has finite measure. Thus, for every such simple function, we have

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}} s(x)h(nx)dx &= \lim_{n\to\infty} \int_{\mathbb{R}} \left[\sum_{k=1}^m \alpha_k \mathbf{1}_{A_k}(x)\right] h(nx)dx \\ &= \sum_{k=1}^m \alpha_k \left[\lim_{n\to\infty} \int_{\mathbb{R}} \mathbf{1}_{A_k}(x)h(nx)dx\right] \\ &= 0. \end{aligned} \tag{5.4.1}$$

Now, let $f \in L^1(\mathbb{R})$ and let $M \geq 0$ be a finite bound for h. Fix $\epsilon > 0$. By Theorem 3.13 in [**RUD**], we may find a complex, measurable, simple function s that vanishes outside a set of finite measure such that $\|f - s\|_1 < \epsilon/M$. Now, for each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\left|\int_{\mathbb{R}} f(x)h(nx)dx\right| &= \left|\int_{\mathbb{R}}[f(x)-s(x)]h(nx)dx+\int_{\mathbb{R}} s(x)h(nx)dx\right| \\
\text{(5.4.2)} \qquad &\leq \int_{\mathbb{R}} |f(x)-s(x)|\,|h(nx)|\,dx+\left|\int_{\mathbb{R}} s(x)h(nx)dx\right| \\
&\leq M\,\|f-s\|_1+\left|\int_{\mathbb{R}} s(x)h(nx)dx\right| \\
&< \epsilon+\left|\int_{\mathbb{R}} s(x)h(nx)dx\right|.
\end{aligned}
$$

Letting $n \to \infty$ in (5.4.2) and appealing to (5.4.1) gives

$$\left|\lim_{n\to\infty}\int_{\mathbb{R}} f(x)h(nx)dx\right| < \epsilon,$$

by the continuity of the absolute value function. The proof is completed by letting $\epsilon \to 0$.

□

Problem 5.5. *Let $L : \Omega \to \mathbb{C}$ be a continuous function having the property that $e^{L(z)} = z$, for all $z \in \Omega$. Prove that*

(a) $L \in H(\Omega)$.

(b) *There is no continuous map on $\mathbb{D}'$ that acts as a right inverse for the exponential map on $\mathbb{D}'$.*

Key terms: exponential map, complex differentiability, index of a closed curve (with respect to a point), Fundamental Theorem of Calculus, complex logarithm.

Solution. (a) Note that since e^z is never zero, we may deduce that $0 \notin \Omega$. Now, fix $z \in \Omega$ and $h \in \mathbb{C}\setminus\{0\}$. Notice that L must be injective, for if $L(a) = L(b)$, then $a = e^{L(a)} = e^{L(b)} = b$. Thus, $\frac{L(z+h)-L(z)}{h} \neq 0$, and therefore,

$$
\begin{aligned}
\frac{L(z+h)-L(z)}{h} &= \left[\frac{(z+h)-z}{L(z+h)-L(z)}\right]^{-1} = \left[\frac{e^{L(z+h)}-e^{L(z)}}{L(z+h)-L(z)}\right]^{-1} \\
&= \left[\frac{e^{L(z)+H(h)}-e^{L(z)}}{H(h)}\right]^{-1},
\end{aligned}
$$

where $H(h) := L(z+h) - L(z)$. Since L is assumed continuous, we have $H(h) \to 0$, as $h \to 0$. Combining this with the continuity of the map $w \mapsto w^{-1}$ on $\mathbb{C} \setminus \{0\}$ and the fact that e^z is its own derivative yields

$$\begin{aligned} \lim_{h\to 0} \frac{L(z+h)-L(z)}{h} &= \lim_{h\to 0} \left[\frac{e^{L(z)+H(h)} - e^{L(z)}}{H(h)} \right]^{-1} = \left[\lim_{h\to 0} \frac{e^{L(z)+H(h)} - e^{L(z)}}{H(h)} \right]^{-1} \\ &= \left[e^{L(z)} \right]^{-1} = z^{-1}. \end{aligned}$$

This shows that L is differentiable on Ω, and moreover, $L'(z) = z^{-1}$.

(b) We will actually show that under the assumptions on L, $D'(0,r) \not\subseteq \Omega$, for all $r > 0$. To verify this, suppose contrarily that, for some $0 < r < \infty$, L is a continuous right inverse for the exponential map in $D(0,r)$, that is, $e^{L(z)} = z$, for all $0 < |z| < r$. Put $\rho := \frac{r}{2}$. Let $\gamma : [0,1] \to \mathbb{C}$ be the closed curve given by $\gamma(t) := \rho e^{2\pi i t}$. By part (a) L is differentiable and $L'(z) = 1/z$. Therefore,

$$\int_\gamma L'(z)dz = \int_\gamma z^{-1}dz = \mathrm{Ind}_\gamma(0) = 2\pi i.$$

On the other hand, by the Fundamental Theorem of Calculus, $\int_\gamma L'(z)dz = 0$, since γ is closed. This contradiction completes the proof.

□

Problem 5.6. *Let f and g be entire functions such that $|f| \le |g|$. Prove*

(a) *If z_0 is a zero of g having multiplicity m, then z_0 is a zero of f having multiplicity at least m.*

(b) *f is a constant multiple of g.*

Key terms: entire function, Liouville's Theorem, zero-set, removable singularity.

Solution. It is enough to prove (b). To this end, note that it is clear that f vanishes whenever g does. Thus, we may assume that g is not identically equal to zero. As such, the set of zeros of g, $Z(g)$, contains no limit point in $\mathbb{C}$. In consequence, for each (fixed) zero $z \in Z(g)$, we may find $r_z > 0$ such that $D'(z, r_z) \cap Z(g) = \varnothing$. Hence, the function $\frac{f}{g}$ is holomorphic and bounded (by 1) on the punctured disk $D'(z, r_z)$, and therefore, has a removable singularity at z. (See Theorem 10.20 in [**RUD**].) Let

h_z be an extension of $\frac{f}{g}$ to a holomorphic function on $D(z,r_z)$. Note that $|h_z| \leq 1$, since this is true for $\frac{f}{g}$ on $D'(z,r_z)$ and since h_z is continuous (it is holomorphic) on $D(z,r_z)$.

Now, define $h : \mathbb{C} \to \mathbb{C}$ by $h(z) = \begin{cases} \frac{f(z)}{g(z)} & z \notin Z(g) \\ h_z(z) & z \in Z(g) \end{cases}$. Clearly, h is differentiable at every z in the *open* set $\mathbb{C}\backslash Z(g)$. Furthermore, for $z \in Z(g)$, h and h_z agree in the neighborhood $D(z,r_z)$ of z, where h_z is holomorphic, and so, h is also differentiable at z. We conclude that h is an entire function. Moreover, the conditions on f and g and the fact that $|h_z| \leq 1$, for all $z \in Z(g)$, ensure that $|h| \leq 1$. By Liouville's Theorem, we deduce that h is some constant c. The very definition of h thus gives that $f(z) = cg(z)$, for all $z \notin Z(g)$. Yet $|f| \leq |g|$ implies that $f(z) = 0 = c \cdot 0 = cg(z)$, for all $z \in Z(g)$. So, we actually have $f(z) = cg(z)$, for all $z \in \mathbb{C}$.

□

Problem 5.7. *Verify that*

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\sqrt{2}}{2}\pi.$

(b) $\displaystyle\int_0^{2\pi} e^{-i\theta} \exp(e^{i\theta}) d\theta = 2\pi.$

Key terms: contour integration, Residue Theorem, pole, Laurent series.

Solution. (a) Put $f(z) := \frac{1}{z^4+1}$ and note that f has simple poles at $z_k = \exp\left[i\frac{(2k+1)\pi}{4}\right]$, $k = 0, ..., 3$. For $R > 1$, let $\gamma_R : [0,\pi] \to \mathbb{C}$ be the path $t \mapsto Re^{it}$. Among the four simple poles, only z_0 and z_1 lie inside the region bounded by γ_R and the x-axis. Hence,

$$\int_{-R}^{R} f(x)dx + \int_{\gamma_R} f(z)dz = 2\pi i \sum_{k=0}^{3} \text{Res}(f, z_k),$$

or equivalently,

$$\int_{-R}^{R} \frac{1}{x^4+1} dx = -\int_{\gamma_R} f(z)dz + 2\pi i \left[\text{Res}(f, z_0) + \text{Res}(f, z_1)\right]. \tag{5.7.1}$$

Now, observe that

$$\left|\int_{\gamma_R} f(z)dz\right| = \left|\int_0^{\pi} \frac{1}{(Re^{it})^4+1} Rie^{it} dt\right| \leq \int_0^{\pi} \frac{R}{R^4-1} dt = \frac{\pi R}{R^4-1}.$$

Clearly, the above inequality implies that $\left|\int_{\gamma_R} f(z)dz\right| \to 0$, as $R \to \infty$. Applying this fact to (5.7.1) we may conclude that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx = 2\pi i\left[\mathrm{Res}(f, z_0) + \mathrm{Res}(f, z_1)\right]. \tag{5.7.2}$$

At this stage, it is appropriate to calculate $\mathrm{Res}(f, z_0)$ and $\mathrm{Res}(f, z_1)$. The procedure is the same for both:

$$\mathrm{Res}(f, z_k) = \lim_{z\to z_k} \frac{z-z_k}{z^4+1} = \frac{1}{\lim\limits_{z\to z_k} \dfrac{z^4-z_k^4}{z-z_k}} = \frac{1}{\frac{d}{dz}z^4\big|_{z=z_k}} = \frac{1}{4z_k^3}.$$

So,

$$\mathrm{Res}(f, z_0) + \mathrm{Res}(f, z_1) = \frac{\exp\left(-i\frac{3\pi}{4}\right) + \exp\left(-i\frac{9\pi}{4}\right)}{4} = \frac{-2i\sin(\pi/4)}{4} = -\frac{i}{2\sqrt{2}}.$$

Substitution of this into (5.7.2) gives the desired result.

(b) Let γ be the closed curve given by $\theta \mapsto e^{i\theta}$, for $\theta \in [0, 2\pi]$. Then

$$\int_0^{2\pi} e^{-i\theta}\exp(e^{i\theta})d\theta = -i\int_\gamma \frac{e^z}{z^2}dz. \tag{5.7.3}$$

By the Residue Theorem, one has

$$\int_\gamma \frac{e^z}{z^2}dz = 2\pi i \mathrm{Res}\left(\frac{e^z}{z^2}, 0\right) = 2\pi i,$$

since $\frac{e^z}{z^2}$ has Laurent expansion $\frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{3!} + \ldots$ about the point $z = 0$. Substituting this into equation (5.7.3) completes the solution.

□

Problem 5.8. *Let $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}\, z < 2\}$ and for each $z \in S$ let $l_z(t) := tz + 1 - t$, for $0 \le t \le 1$. Without assuming that a primitive of f exists, show that $F(z) := \int_{l_z} f$ defines a holomorphic function in S such that $F' = f$. Explain why the conclusion fails whenever f is not holomorphic.*

Key terms: Cauchy's Theorem in a Convex Set, complex differentiability.

Solution. This is problem is a specific example of the following more general result, whose proof follows the statement:

THEOREM 5.8.1 (Cauchy's Theorem in a Convex Set). *Let f be a function defined on a convex open subset Ω of $\mathbb{C}$. Show that $f \in H(\Omega)$ if and only if there exists $F \in H(\Omega)$ such that $F' = f$.*

PROOF. ($\Rightarrow$) Fix $a \in \Omega$. Since Ω is convex, it contains the line segment $[a, z]$, for every $z \in \Omega$. Define $F : \mathbb{C} \to \mathbb{C}$ by $F(z) := \int_{[a,z]} f(\xi)d\xi$. Now, fix $z_0 \in \Omega$. For any $z \in \Omega$, Cauchy's Theorem for a triangle gives

$$\begin{aligned} \int_{[z,z_0]} f(\xi)d\xi &= -\int_{[z_0,a]} f(\xi)d\xi - \int_{[a,z]} f(\xi)d\xi = \int_{[a,z_0]} f(\xi)d\xi - F(z) \\ &= F(z_0) - F(z). \end{aligned}$$

Hence, for $z \in \Omega \setminus \{z_0\}$, we have

$$\begin{aligned} \left| \frac{F(z_0) - F(z)}{z_0 - z} - f(z_0) \right| &= \left| \frac{1}{z_0 - z} \int_{[z,z_0]} f(\xi)d\xi - \frac{1}{z_0 - z} \int_{[z,z_0]} f(z_0)d\xi \right| \\ &\leq \frac{1}{|z_0 - z|} \int_{[z,z_0]} |f(\xi) - f(z_0)| \, d|\xi| \,. \end{aligned} \tag{5.8.1}$$

Let $\epsilon > 0$ be given. Using the continuity of f, choose $\delta > 0$ so small so that $|\xi - z_0| < \delta$ implies $\xi \in \Omega$ and $|f(\xi) - f(z_0)| < \epsilon$. Now, if $0 < |z - z_0| < \delta$, then $|\xi - z_0| < \delta$, for each $\xi \in [z, z_0]$. Thus, from (5.8.1) we may deduce that

$$\left| \frac{F(z_0) - F(z)}{z_0 - z} - f(z_0) \right| < \frac{1}{|z_0 - z|} \int_{[z,z_0]} \epsilon d\,|\xi| = \epsilon,$$

for all $z \in \Omega$ with $0 < |z - z_0| < \delta$. This proves that F is differentiable at z_0, and moreover, $F'(z_0) = f(z_0)$. As $z_0 \in \Omega$ is arbitrary, the proof of this implication is complete.

($\Leftarrow$) As holomorphic functions have complex derivatives of all orders, $F \in H(\Omega)$ with $F' = f$ implies that $f \in H(\Omega)$.

$\square$

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