On uniformly convex functions

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ON UNIFORMLY CONVEX FUNCTIONS

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We study the class UCV of uniformly convex functions which was introduced by A.W. Goodman. We introduce a class that unifies the UCV and the corresponding clas of starlike functions S_p introduced by Rϕnning. We prove that the new class is closed under convolution with prestarlike functions of order α. Some integral operators on S_p are discussed. A sufficient condition for functions to be uniformly convex is given.

1. INTRODUCTION

Let A denote the class of all analytic functions $f(z)$ defined on the unit disk $U = \{z; |z| < 1\}$ normalized by $f(0) = 0$, $f'(0) = 1$. The function $f \in A$ is uniformly convex (starlike) if for every circular arc γ contained in U with center $\zeta \in U$ the image arc $f(\gamma)$ is convex (starlike with respect to $f(\zeta)$). The class of all uniformly convex (starlike) functions is denoted by UCV (UST). Note that

(1.1) $$f \in \mathrm{UCV} \Leftrightarrow \mathrm{Re}\left\{1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right\} \geq 0, \quad z, \zeta \in U,$$

(1.2) $$f \in \mathrm{UST} \Leftrightarrow \mathrm{Re}\left\{\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right\} \geq 0, \quad z, \zeta \in U.$$

These classes were introduced and studied by A.W.Goodman [2, 3]. He remarked that the class UCV is preserved under the transformation $e^{-i\alpha} f(e^{i\alpha} z)$ and no other transformation seems to be available. Using the following one variable characterization of UCV, it is easy to obtain other transformation which preserve UCV.

Theorem 1.1. [4,7] Let $f \in A$. Then $f \in \mathrm{UCV}$ if and only if

(1.3) $$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U.$$

Since the Alexander type result $f \in \mathrm{UCV}$ if and only if $zf' \in \mathrm{UST}$ fails [9], the class

(1.4) $$S_p = \{f; f = zF', F \in \mathrm{UCV}\}$$

introduced by F. Rϕnning [7] became interesting.

Let f and F be analytic in U. Then f is subordinate to F (written $f \prec F$ or $f(z) \prec F(z)$ if $f(0) = F(0)$ and $f(U) \subseteq F(U)$. If $f(z), g(z) \in A$ and

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

then their convolution is the function $(f*g)(z) \in A$ given by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

We introduce below a class of functions that unifies several subclasses of starlike and convex functions.

Definition 1.2. Let h(z) be a convex univalent function with $h(0) = 1$, $\operatorname{Re} h(z) > \alpha > 0$. Let $g(z) \in A$ be a fixed function. Denote by $S_g(h)$ the class of all analytic functions $f \in A$ satisfying

$$\frac{(f*g)(z)}{z} \neq 0 \text{ and } \frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z). \tag{1.5}$$

If $g(z) = z/(1-z)^2$ and $h(z) = P(z)$, where P(z) is the Riemann mapping of U onto the parabolic region $\Omega = \{w; \operatorname{Re} w > |w-1|\}$, then $S_g(h) \equiv UCV$. Similarly, if $g(z) = z/1 - z$ and $h(z) = P(z)$, then $S_g(h) \equiv S_p$.

If (1.5) holds only for $|z| < r < 1$ then we say that $f \in S_g(h)$ for $|z| < r$. Note that the class R_α of pre–starlike functions of order $\alpha < 1$ consists of functions $f \in A$ satisfying

$$f * \frac{z}{(1-z)^{2-2\alpha}} \in S^*(\alpha) \qquad \text{for } \alpha < 1$$

$$\operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} \qquad \text{for } \alpha = 1$$

where $S^*(\alpha)$ denote the class of starlike functions of order α . If $g(z) = z/(1-z)^{2-2\alpha}$ and $h(z) = (1 + (1-2\alpha))\, z/(1-z)$, then $S_g(h) \equiv R_\alpha$ [11] .

Theorem 2.
If $f \in R_\alpha$ and $g \in S^*(\alpha)$, then for any analytic function H(z) in U, $\frac{f*(Hg)}{f*g}(U) \in \overline{Co}(H(U))$, where $\overline{Co}(H(U))$ denotes the closed convex hull of H(U).

In this paper, we prove that the class $S_g(h)$ is closed under convolution with prestarlike functions of order α. We discuss some integral operators on S_p. Also we give sufficient condition for a function to be uniformly convex.

2. MAIN RESULTS

Theorem 2.1. If $f \in S_g(h)$ and $\phi \in R_\alpha$, then $\phi * f \in S_g(h)$.

Proof. Since $f \in S_g(h)$, it follows that

$$\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z).$$

Clearly $(g * f)(z) \in S^*(\alpha)$. Let

$$H(z) = \frac{z(g*f)'(z)}{(g*f)(z)}.$$

Using Theorem 1.3, it follows that

$$\frac{[\phi * H(g*f)](z)}{[\phi*(g*f)](z)} \prec h(z).$$

Since

$$\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} = \frac{[\phi*z(g*f)'](z)}{[\phi*(g*f)](z)} = \frac{[\phi*H(g*f)](z)}{[\phi*(g*f)](z)},$$

we have

$$\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} \prec h(z),$$

which shows that $\phi * f \in S_g(h)$.

If g(z) is equal to $z/(1-z)$, $z/(1-z)^2$ respectively and $h(z) = (1+(1-2\alpha)z)/(1-z)$, we see that the classes of convex and starlike functions of order α are closed under convolution with prestarlike functions of order α. If $g(z) = z/(1-z)$ and

$$h(z) = 1 + \frac{2}{\pi^2}\left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2$$

then we have the following theorem of Rϕnning :

Corollary 2.1. [8] If $f(z)$ is on S_p and g(z) is starlike of order ½, then $(f*g)(z)$ is in S_p again.

Similarly, it follows that UCV is closed under convolution with starlike functions of order 1/2. Since $s^*_{1/2} = R_{1/2}$, we have the following

Corollary 2.2. The class $S_g(P)$ is closed under convolution with starlike functions of order 1/2.

Corollary 2.3. Let

$$\Gamma_1(f(z)) = zf'(z)$$

$$\Gamma_2(f(z)) = \frac{1}{2}[f(z) + zf'(z)]$$

$$\Gamma_3(f(z)) = \frac{k+1}{z^k}\int_0^z \zeta^{k-1} f(\zeta)d\zeta, \quad \text{Re } k > 0$$

$$\Gamma_4(f(z)) = \int_0^z \zeta^{k-1}\frac{f(\zeta)-f(\eta\zeta)}{\zeta-\eta\zeta}d\zeta, \quad |n| \le 1, \eta \ne 1.$$

Then $\Gamma_i(f(z)) \in S_g(P)$ in $|z| < r_i$ whenever $f(z) \in S_g(P)$, where $r_1=1/3$, $r_2=(\sqrt{17}-3)/2 \approx 0.56155$, $r_3 = r_4 = 1$.

Proof. For each i = 1, 2, 3, 4 we have $\Gamma_i = f(z) * h_i(z)$ where

$$h_1(z) = \frac{z}{(1-z)^2}$$

$$h_2(z) = \frac{z - z^2/2}{(1-z)^2}$$

$$h_3(z) = \sum_{n=1}^{\infty} \frac{k+1}{k+n} z^n$$

$$h_4(z) = \sum_{n=1}^{\infty} \frac{1-\eta^n}{(1-\eta)n} z^n.$$

Since the radius of starlike of order 1/2 for the functions h_1 and h_2 are 1/3 and $(\sqrt{17}-3)/2$ respectively, the results for i = 1, 2 follow. Since h_3 and h_4 are convex and hence starlike of order 1/2 , the result follows.

Corollary 2.4. $UCV \subset S_p$.

Proof. Let $f \in UCV$. Then, by Corollary 2.3, $g(z) = \int_0^z \frac{f(\zeta)}{\zeta}d\zeta$ is in UCV. Therefore, $f(z) = zg'(z) \in S_p$.

Theorem 2.2. If f and g are in S_p, then the function

$$H(z) = \left[\frac{\gamma+\alpha+1}{z^\gamma}\int_0^z \{f(t)\}^\alpha g(t) t^{\gamma-1}dt\right]^{\frac{1}{\alpha+1}}$$

$\gamma > 0$, $\alpha > 0$ is also in S_p.

Proof. Note that the class PAR defined by

$$PAR = \{p = 1 + cz + \dots ; |p - 1| < \text{Re}\, p\}$$

is a convex set in the sense that $\alpha_1\alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$, p_1, p_2 $\in$PAR implies $\alpha_1 p_1+\alpha_2 p_2 \in$PAR. Also a function $f \in S_p$ if and only if $zf'/f \in$PAR [7].

Let $$M(z) = z^{\gamma}\{f(z)\}^{\alpha} g(z) - \gamma \int_0^z \{f(t)\}^{\alpha} g(t) t^{\gamma-1} dt$$

$$N(z) = \int_0^z \{f(t)\}^{\alpha} g(t) t^{\gamma-1} dt.$$

Then straightforward computation shows that

$$\frac{zH'(z)}{H(z)} = \frac{M(z)}{(\alpha+1)N(z)}$$

and

$$\frac{M'(z)}{(\alpha+1)N'(z)} = \frac{1}{\alpha+1}\left[\alpha\frac{zf'(z)}{f(z)} + \frac{zg'(z)}{g(z)}\right].$$

Since $f, g \in S_p$ and PAR is convex, $\frac{M'(z)}{(\alpha+1)N'(z)} \in$ PAR. By Corollary 1 in [6], we see that $\frac{M(z)}{(\alpha+1)N(z)} \in$ PAR. This completes the proof.

Theorem 2.5. If $f \in A$ and $\left|\frac{z(f*g)'(z)}{(f*g)(z)} - 1\right| < \frac{1}{2}$, then $f \in S_p(P)$.

Proof. Let $p(z) = 1 + \frac{1}{2}z$. By hypotheses, $f \in S_p(p)$. Since $p(z) \prec P(z)$, we have $f \in S_g(P)$.

Corollary 2.5. Let $f \in A$.

(i) If $\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}$, then $f \in$ UCV.

(ii) If $\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{1}{2}$, then $f \in S_p$.

It should be noted that the above conditions are also necessary for a function f to be in UCV (or S_p) when $f(z)$ is a function with negative coefficients. Since the classes S_p and UCV are rotationally invariant, from Corollary 2.4 and these remarks, we get the following corollaries.

Corollary 2.6. [3] $f(z) = z - Az^2$ is in UCV if and only if $|A| \le 1/6$.

Corollary 2.7. [7] $f(z) = z - A_n z^n$ is (i) in S_p if if and only if $|A_n| \le 1/(2n-1)$.

(ii) in UCV if and only if $|A_n| \le \dfrac{1}{n(2n-1)}$.

Corollary 2.8. [3] $f(z) = z/(1 - Az)^2$ if and only if $|A| \le 1/3$.

Theorem 2.4. Let $f \in A, \varepsilon^n = 1, n \ge 1$. Let $f_n(z) = \dfrac{1}{n}\sum_0^{n-1} \varepsilon^{-k} f(\varepsilon^k z)$. Then

(i) If $\operatorname{Re}\left\{\dfrac{(z-\zeta)f'(z)}{f_n(z) - f_n(\zeta)}\right\} \ge 0$, then $f_n \in$ UST.

(ii) If $\operatorname{Re}\left\{1 + \dfrac{(z-\zeta)f''(z)}{f'_n(z)}\right\} \ge 0$, then $f_n \in$ UCV.

(iii) If $\left\{\dfrac{zf'(z)}{f_n(z)} - 1\right\} \le \operatorname{Re}\left\{\dfrac{zf'(z)}{f_n(z)}\right\}$, then $f_n \in S_p$.

Proof. We prove (i). Replacing z by $\varepsilon^k z$ and ζ by $\varepsilon^k \zeta$ in the given condition and then summing over k = 0, 1, 2, 3,..., n–1, we get the desired result upon simplification. The proof of (ii) and (iii) are similar.

Let $D_{\alpha,n}$, $0 \le \alpha < 1$ denote the class of analytic functions $q(z) = 1 + c_n z^n + \ldots$ and

$$\left|q(z) - \frac{1}{2\alpha}\right| \le \frac{1}{2\alpha}.$$

Note that for $\alpha = 0$, this class of functions with positive real part [12].

Then we have the following

Theorem 2.5. Let $f'(z) \in D_{\alpha,n}$, then $f \in$ UCV in $|z| < r_0$, where r_0 is given by

$$r_0 = 2[\sqrt{(2n+1) + (2n-1)c} - (2n+1) - (2n-1)\,c]^{-1/n}, \quad c = 1 - 2\alpha.$$

Proof. Let $q(z) = f'(z)$. Since $q(z) \in D_{\alpha,n}$, we have [12]

$$\left|\frac{zq'(z)}{q(z)}\right| \leq \frac{(1+c)n|z|^n}{(1-|z|^n(1+c|z|^n)}.$$

By Corollary 2.5, $f \in UCV$ if

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{2}.$$

Since $\frac{zq'(z)}{q(z)} = \frac{zf''(z)}{f'(z)}$, we have $f \in UCV$ if

$$\frac{(1+c)n|z|^n}{(1-|z|^n(1+c|z|^n)} < \frac{1}{2}$$

or

$$cr^{2n} + [(2n+1) + (2n-1)c]r^n - 1 \leq 0, \quad r = |z|$$

which is satisfied if $0 \leq r \leq r_0$. The result is sharp for the function

$$f(z) = \int_0^z \frac{1-\zeta^n}{1+c\zeta^n} d\zeta.$$

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