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RADII OF STARLIKENESS AND CONVEXITY OF ANALYTIC FUNCTIONS SATISFYING CERTAIN COEFFICIENT INEQUALITIES

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ABSTRACT. For $0 \leq \alpha < 1$, the sharp radii of starlikeness and convexity of order α for functions of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ whose Taylor coefficients a_n satisfy the conditions $|a_2| = 2b$, $0 \leq b \leq 1$, and $|a_n| \leq n$, M or M/n $(M > 0)$ for $n \geq 3$ are obtained. Also a class of functions related to Carathéodory functions is considered.

1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ with Taylor series expansion $f(z) = z + \sum_{2}^{\infty} a_n z^n$. For functions belonging to the subclass $\mathcal{S}$ of $\mathcal{A}$ consisting of univalent functions, it is well-known that $|a_n| \leq n$ for $n \geq 2$. A function f whose coefficients satisfy the inequality $|a_n| \leq n$ for $n \geq 2$ are analytic in $\mathbb{D}$ (by the usual comparison test) and hence they are members of $\mathcal{A}$. However, they need not be univalent. For example, the function

$$f(z) = z - 2z^2 - 3z^3 - 4z^4 - \cdots = 2z - \frac{z}{(1-z)^2}$$

satisfies the inequality $|a_n| \leq n$ but its derivative vanishes inside $\mathbb{D}$ and therefore the function f is not univalent in $\mathbb{D}$. In 1970, Gavrilov [5] showed that the radius of univalence of functions satisfying the inequality $|a_n| \leq n$ is the real root of the equation $2(1-r)^3 - (1+r) = 0$ while, for the functions whose coefficients satisfy $|a_n| \leq M$, the radius of univalence is $1 - \sqrt{M/(1+M)}$. Later, in 1982,

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Yamashita showed that the radius of univalence obtained by Gavrilov is also the same as the radius of starlikeness of the corresponding functions. He also found lower bounds for the radii of convexity for these functions. Recently, in 2006, Graham *et al.* [7: Theorem 4.2, Lemma 5.6] considered the corresponding radius problems for holomorphic mappings on the unit ball in $\mathbb{C}^n$. Kalaj, Ponnusamy, and Vuorinen [4] have investigated related problems for harmonic functions. In this paper, several related radius problems for the following classes of functions will be investigated.

For $0 \leq \alpha < 1$, let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ be subclasses of $\mathcal{S}$ consisting of starlike functions of order α and convex functions of order α, respectively defined analytically by the following equalities:

$$\mathcal{S}^*(\alpha) := \left\{ f \in \mathcal{S} :\ \operatorname{Re}\left(\tfrac{zf'(z)}{f(z)}\right) > \alpha \right\},$$

and

$$\mathcal{C}(\alpha) := \left\{ f \in \mathcal{S} :\ \operatorname{Re}\left(1 + \tfrac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{C} := \mathcal{C}(0)$ are the familiar classes of starlike and convex functions respectively. Closely related are the following classes of functions:

$$\mathcal{S}^*_\alpha := \left\{ f \in \mathcal{S} :\ \left|\tfrac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \right\},$$

and

$$\mathcal{C}_\alpha := \left\{ f \in \mathcal{S} :\ \left|\tfrac{zf''(z)}{f'(z)}\right| < 1 - \alpha \right\}.$$

Note that $\mathcal{S}^*_\alpha \subseteq \mathcal{S}^*(\alpha)$ and $\mathcal{C}_\alpha \subseteq \mathcal{C}(\alpha)$.

A function $f \in \mathcal{S}$ is *uniformly convex* if f maps every circular arc γ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ onto a convex arc. The class of all uniformly convex functions, introduced by Goodman [6], is denoted by $\mathcal{UCV}$. Rønning [9: Theorem 1, p. 190], and Ma and Minda [8: Theorem 2, p. 162], independently showed that $f \in \mathcal{S}$ is uniformly convex if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \qquad (z \in \mathbb{D}).$$

Rønning [9] also considered the class $\mathcal{S}_P$ of parabolic starlike functions consisting of functions $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{D}).$$

In other words, the class $\mathcal{S}_P$ consists of function $f = zF'$ where $F \in \mathcal{UCV}$. For a recent survey on uniformly convex functions, see [2].

For a fixed b with $0 \leq b \leq 1$, let $\mathcal{A}_b$ denote the class of all analytic functions f of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \qquad (|a_2| = 2b, \quad z \in \mathbb{D}).$$

The second coefficient of univalent functions determines important properties such as growth and distortion estimates. For recent investigation of functions with fixed second coefficients, see [1, 3]. For $0 \leq \alpha < 1$, the sharp radii of starlikeness and convexity of order α are obtained for functions $f \in \mathcal{A}_b$ satisfying the condition $|a_n| \leq n$ or $|a_n| \leq M$ $(M > 0)$ for $n \geq 3$. Special case $(\alpha = 0)$ of the results shows that the lower bounds for the radii of convexity obtained by Yamashita [10] are indeed sharp. The coefficient inequalities are natural in the sense that the inequality $|a_n| \leq n$ is satisfied by univalent functions and while the inequality $|a_n| \leq M$ is satisfied by functions which are bounded by M. For a function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with positive real part, it is well-known that $|c_n| \leq 2$ and so if $f \in \mathcal{A}$ and $\operatorname{Re} f'(z) > 0$, then $|a_n| \leq 2/n$. In view of this, the determination of the radius of starlikeness and the radius of convexity of functions whose coefficients satisfy the inequality $|a_n| \leq M/n$ is also investigated. A corresponding radius problem for certain function $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with coefficients satisfying the conditions $|c_1| = 2b$, $0 \leq b \leq 1$ and $|c_n| \leq 2M$ $(M > 0)$ is also investigated.

2. Radii of starlikeness of order α and parabolic starlikeness

In this section, the sharp $\mathcal{S}^*(\alpha)$-radius and the sharp $\mathcal{S}^*_\alpha$-radius for $0 \leq \alpha < 1$ as well as the sharp $\mathcal{S}_P$-radius are obtained for functions $f \in \mathcal{A}_b$ satisfying one of the conditions $|a_n| \leq n$, $|a_n| \leq M$ or $|a_n| \leq M/n$ $(M > 0)$ for $n \geq 3$.

Theorem 2.1. *Let $f \in \mathcal{A}_b$ and $|a_n| \leq n$ for $n \geq 3$. Then f satisfies the inequality*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \alpha \qquad (|z| \leq r_0) \tag{2.1}$$

where $r_0 = r_0(\alpha)$ is the smallest root in $(0, 1)$ of the equation

$$1 - \alpha + (1 + \alpha) r = 2\big(1 - \alpha + (2 - \alpha)(1 - b) r\big)(1 - r)^3. \tag{2.2}$$

The number $r_0(\alpha)$ is also the radius of starlikeness of order α. The number $r_0(1/2)$ is the radius of parabolic starlikeness of the given functions. The results are all sharp.

P r o o f. If

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n|r_0^{n-1} \leq 1-\alpha, \tag{2.3}$$

then the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies, on $|z| = r_0$,

$$\begin{aligned}
&|zf'(z) - f(z)| - (1-\alpha)|f(z)| \\
&\quad \leq \sum_{n=2}^{\infty}(n-1)|a_n||z|^n - (1-\alpha)(|z| - \sum_{n=2}^{\infty}|a_n|\,|z|^n) \\
&\quad = -(1-\alpha)|z| + \sum_{n=2}^{\infty}(n-\alpha)|a_n||z|^n \\
&\quad \leq r_0\left(-(1-\alpha) + \sum_{n=2}^{\infty}(n-\alpha)|a_n|r_0^{n-1}\right) \\
&\quad \leq 0.
\end{aligned}$$

This shows that the condition (2.3) is a sufficient condition for the inequality (2.1) to hold. Using $|a_2| = 2b$ for the function $f \in \mathcal{A}_b$, and the inequality $|a_n| \leq n$ for $n \geq 3$, it follows that, for $|z| \leq r_0$,

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-\alpha)|a_n|\,|z|^{n-1} \\
&\quad \leq \sum_{n=2}^{\infty}(n-\alpha)|a_n|r_0^{n-1} \\
&\quad \leq 2(2-\alpha)br_0 + \sum_{n=3}^{\infty} n^2 r_0^{n-1} - \alpha\sum_{n=3}^{\infty} n r_0^{n-1} \\
&\quad = 2(2-\alpha)br_0 + \frac{1+r_0}{(1-r_0)^3} - 1 - 4r_0 - \alpha\left(\frac{1}{(1-r_0)^2} - 1 - 2r_0\right) \\
&\quad = \alpha - 1 - 2(2-\alpha)(1-b)r_0 + \frac{(1+r_0) - \alpha(1-r_0)}{(1-r_0)^3} \\
&\quad = 1-\alpha
\end{aligned}$$

provided r_0 is the root of the Equation (2.2) in the hypothesis of the theorem. The intermediate value theorem shows that the Equation (2.2) has a root in the interval (0,1).

The function f_0 given by

$$f_0(z) = 2z + 2(1-b)z^2 - \frac{z}{(1-z)^2} = z - 2bz^2 - 3z^3 - 4z^4 - \cdots \tag{2.4}$$

satisfies the hypothesis of the theorem and, for this function, we have

$$\frac{zf_0'(z)}{f_0(z)} - 1 = \frac{2(1-b)z(1-z)^3 - 2z}{(2+2(1-b)z)(1-z)^3 - (1-z)}.$$

For $z = r_0$, we have

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \frac{2r_0 - 2(1-b)r_0(1-r_0)^3}{(2+2(1-b)r_0)(1-r_0)^3 - (1-r_0)} = 1 - \alpha. \tag{2.5}$$

This shows that the radius r_0 of functions to satisfy (2.1) is sharp. The numerator of the rational function in the middle of (2.5) is positive as $0 \leq 1-b \leq 1$ and $0 \leq (1-r_0) < 1$ which shows that $(1-b)(1-r_0)^3 < 1$. The denominator expression is also positive as

$$(2+2(1-b)r_0)(1-r_0)^2 \geq 2(1-r_0)^2 > 1.$$

The inequality $2(1-r_0)^2 > 1$ is in fact equivalent to $r_0 < 1 - 1/\sqrt{2} = 0.292893$. This inequality holds as $r_0 = r_0(\alpha) \leq r_0(0) = 0.1648776$.

Since the functions satisfying (2.1) are starlike of order α, the radius of starlikeness is at least $r_0(\alpha)$. However, this radius is also sharp for the same function f_0 as

$$\operatorname{Re}\left(\frac{zf_0'(z)}{f_0(z)}\right) = \alpha \qquad (z = r_0). \tag{2.6}$$

The inequality

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{1}{2}$$

is sufficient (see [2]) for the function to be parabolic starlike and hence the radius of parabolic starlikeness is at least $r_0(1/2)$. The Equations (2.5) and (2.6) with $\alpha = 1/2$ shows that

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \frac{1}{2} = \operatorname{Re}\left(\frac{zf_0'(z)}{f_0(z)}\right) \qquad (z = r_0), \tag{2.7}$$

and hence the radius of parabolic starlikeness is sharp. □

Corollary 2.1.1. *The radius of starlikeness of order α of functions whose coefficients satisfy $|a_n| \leq n$ for all $n \geq 2$ is the smallest real root in $(0,1)$ of the equation*

$$2(1-\alpha)(1-r)^3 = 1 - \alpha + (1+\alpha)r.$$

In particular, the radius of starlikeness is given by

$$r_0(0) = 1 + \frac{1}{6^{2/3}}\left(\left(\sqrt{330} - 18\right)^{1/3} - \left(\sqrt{330} + 18\right)^{1/3}\right) \approx 0.164878.$$

The radius of starlikeness of order $1/2$ *is the same as the radius of parabolic starlikeness and it is given by*

$$r_0(1/2) = 1 + \frac{1}{\sqrt{2}}\left(\left(3-2\sqrt{2}\right)^{1/3} - \left(3+2\sqrt{2}\right)^{1/3}\right) \approx 0.120385.$$

The results are sharp.

Corollary 2.1.2. *The radius of starlikeness of order* α *of functions whose coefficients satisfy* $a_2 = 0$ *and* $|a_n| \leq n$ *for all* $n \geq 3$ *is the smallest real root in* $(0,1)$ *of the equation*

$$2\big(1-\alpha+(2-\alpha)r\big)(1-r)^3 = 1-\alpha+(1+\alpha)r.$$

In particular, the radius of starlikeness is the root $r_0 \approx 0.253571$ *of the equation*

$$2(1+2r)(1-r)^3 = 1+r.$$

The radius of starlikeness of order $1/2$ *which is the same as the radius of parabolic starlikeness is* $r_0 = 1 - \sqrt[3]{1/2} \approx 0.206299$. *The results are sharp.*

Remark 1. It is clear from Corollaries 2.1.1 and 2.1.2 that the various radii are improved if the second coefficient of the function vanishes.

Theorem 2.2. *Let* $f \in \mathcal{A}_b$ *and* $|a_n| \leq M$ *for* $n \geq 3$. *Then* f *satisfies the condition* (2.1) *where* $r_0 = r_0(\alpha)$ *is the smallest real root in* $(0,1)$ *of the equation*

$$M(1-\alpha+\alpha r) = \big((1+M)(1-\alpha) - (2-\alpha)(2b-M)r\big)(1-r)^2.$$

The number $r_0(\alpha)$ *is also the radius of starlikeness of order* α. *The number* $r_0(1/2)$ *is the radius of parabolic starlikeness of the given functions. The results are all sharp.*

P r o o f. Using $|a_2| = 2b$ for the function $f \in \mathcal{A}_b$, and the inequality $|a_n| \leq M$ for $n \geq 3$, a calculation shows that, for $|z| \leq r_0$,

$$\begin{aligned}
&\sum_{n=2}^{\infty}(n-\alpha)|a_n|\,|z|^{n-1} \\
&\quad \leq \sum_{n=2}^{\infty}(n-\alpha)|a_n| r_0^{n-1} \\
&\quad \leq 2(2-\alpha)br_0 + M\left(\sum_{n=3}^{\infty} n r_0^{n-1} - \alpha\sum_{n=3}^{\infty} r_0^{n-1}\right) \\
&\quad = 2(2-\alpha)br_0 + M\left(\frac{1}{(1-r_0)^2} - 1 - 2r_0 - \alpha\left(\frac{1}{1-r_0} - 1 - r_0\right)\right) \\
&\quad = (2-\alpha)(2b-M)r_0 - M(1-\alpha) + M\frac{1-\alpha+\alpha r_0}{(1-r_0)^2} \\
&\quad = 1-\alpha
\end{aligned}$$

where r_0 is as stated in the hypothesis of the theorem. Thus, the function f satisfies the condition (2.1). The other two results follow easily.

The results are sharp for the function f_0 given by

$$f_0(z) = z - 2bz^2 - M(z^3 + z^4 + \cdots) = z - 2bz^2 - \frac{Mz^3}{1-z}. \tag{2.8}$$

A calculation shows that

$$\frac{zf_0'(z)}{f_0(z)} - 1 = -\frac{2bz + \frac{2Mz^2}{1-z} + \frac{Mz^3}{(1-z)^2}}{1 - 2bz - \frac{Mz^2}{1-z}}.$$

At the point $z = r_0$, the function f_0 satisfies

$$\operatorname{Re}\left(\frac{zf_0'(z)}{f_0(z)}\right) = 1 - \frac{2br_0 + \frac{2Mr_0^2}{1-r_0} + \frac{Mr_0^3}{(1-r_0)^2}}{1 - 2br_0 - \frac{Mr_0^2}{1-r_0}} = \alpha.$$

Since $\alpha < 1$, the last equation shows that the denominator of the rational expression in the middle is positive. This leads to the following equality:

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \frac{2br_0 + \frac{2Mr_0^2}{1-r_0} + \frac{Mr_0^3}{(1-r_0)^2}}{1 - 2br_0 - \frac{Mr_0^2}{1-r_0}} = 1 - \alpha.$$

Also the Equation (2.7) holds. This proves the sharpness of the results. □

Corollary 2.2.1. *Let $f \in \mathcal{A}$ and $|a_n| \le M$ for $n \ge 2$. Then f satisfies the condition* (2.1) *where $r_0(\alpha)$ is the real root in* $(0,1)$ *of the equation*

$$M(1 - \alpha + \alpha r) = (1 + M)(1 - \alpha)(1 - r)^2.$$

The number $r_0(\alpha)$ is also the radius of starlikeness of order α. The number $r_0(1/2)$ is the radius of parabolic starlikeness of the given functions. The results are all sharp.

Remark 2. The radius of starlikeness of the functions f with $|a_n| \le M$ given by $r_0 = 1 - \sqrt{M/(1+M)}$ is the root in $(0,1)$ of the equation

$$M = (1 + M)(1 - r)^2.$$

When the second coefficient $a_2 = 0$, the radius of starlikeness r_1 is the root in $(0,1)$ of the equation

$$M = (1 + M + 2Mr)(1 - r)^2.$$

Clearly, $r_1 > r_0$.

Theorem 2.3. *Let $f \in \mathcal{A}_b$ and $|a_n| \leq M/n$ for $n \geq 3$. Then f satisfies the condition* (2.1) *where $r_0 = r_0(\alpha)$ is the smallest real root in $(0,1)$ of the equation*

$$2M\big(1 + \alpha(1-r)\log(1-r)/r\big) = \big(2(1+M)(1-\alpha) + (2-\alpha)(M-4b)r\big)(1-r).$$

The number $r_0(\alpha)$ is also the radius of starlikeness of order α. The number $r_0(1/2)$ is the radius of parabolic starlikeness of the given functions. The results are all sharp for the function f_0 given by

$$f_0(z) := (1+M)z + (M/2 - 2b)z^2 + M\log(1-z).$$

The logarithm in the above equation is the branch that takes the value 1 at $z = 0$. Proof of this theorem is omitted as it is similar to those of Theorems 2.1 and 2.2.

3. Radii of convexity and uniform convexity

In this section, the sharp $\mathcal{C}(\alpha)$-radius and the sharp $\mathcal{C}_\alpha$-radius for $0 \leq \alpha < 1$ as well as the sharp $\mathcal{UCV}$-radius for functions $f \in \mathcal{A}_b$ satisfying the condition $|a_n| \leq n$ or $|a_n| \leq M$ $(M > 0)$ for $n \geq 3$ are obtained.

Theorem 3.1. *Let $f \in \mathcal{A}_b$ and $|a_n| \leq n$ for $n \geq 3$. Then f satisfies the condition*

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq 1-\alpha \qquad (|z| \leq r_0) \tag{3.1}$$

where $r_0 = r_0(\alpha)$ is the smallest real root in $(0,1)$ of the equation

$$2\big(1-\alpha+2(2-\alpha)(1-b)r\big)(1-r)^4 = 1-\alpha+4r+(1+\alpha)r^2. \tag{3.2}$$

The number $r_0(\alpha)$ is also the radius of convexity of order α. The number $r_0(1/2)$ is the radius of uniform convexity of the given functions. The results are all sharp.

P r o o f. A function f satisfies (3.1) if and only if zf' satisfies (2.1). In view of this and the inequality (2.3), the inequality

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n||z|^{n-1} \leq 1-\alpha, \qquad (|z| \leq r_0) \tag{3.3}$$

is sufficient for function f to satisfy (3.1). Let φ be defined by

$$\varphi(r) := 2\big(1-\alpha+2(2-\alpha)(1-b)r\big)(1-r)^4 - (1-\alpha) - 4r - (1+\alpha)r^2.$$

Since $\varphi(0) = 1-\alpha > 0$ and $\varphi(1) = -6 < 0$, the intermediate value theorem shows that the Equation (3.2) has a root in the interval (0,1). Let r_0 be the smallest root in $(0,1)$ of the Equation (3.2). Now, for $|z| \le r_0$,

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-\alpha)|a_n||z|^{n-1} \\
&\quad \le \sum_{n=2}^{\infty} n(n-\alpha)|a_n| r_0^{n-1} \\
&\quad \le 4(2-\alpha)br_0 + \sum_{n=3}^{\infty} (n-\alpha)n^2 r_0^{n-1} \\
&\quad = 4(2-\alpha)br_0 + \left(\frac{1+4r_0+r_0^2}{(1-r_0)^4} - 1 - 8r_0\right) - \alpha\left(\frac{1+r_0}{(1-r_0)^3} - 1 - 4r_0\right) \\
&\quad = -\big(1-\alpha+4(2-\alpha)(1-b)r_0\big) + \frac{1-\alpha+4r_0+(1+\alpha)r_0^2}{(1-r_0)^4} \\
&\quad = 1-\alpha.
\end{aligned}$$

To prove the sharpness, consider the function f_0 defined by (2.4). For this function, a calculation shows that

$$\frac{zf_0''(z)}{f_0'(z)} = \frac{4(1-b)z - \frac{4z}{(1-z)^3} - \frac{6z^2}{(1-z)^4}}{2+4(1-b)z - \frac{1}{(1-z)^2} - \frac{2z}{(1-z)^3}}.$$

If r_0 is the root of the equation (3.2), then, at the point $z = r_0$,

$$\operatorname{Re}\left(\frac{zf_0''(z)}{f_0'(z)}\right) = \frac{4(1-b)r_0 - \frac{4r_0}{(1-r_0)^3} - \frac{6r_0^2}{(1-r_0)^4}}{2+4(1-b)r_0 - \frac{1}{(1-r_0)^2} - \frac{2r_0}{(1-r_0)^3}} = \alpha - 1.$$

The denominator of the rational function in the middle of the equation above is positive while the numerator is negative. Noting this, it also follows that, at the point $z = r_0$,

$$\left|\frac{zf_0''(z)}{f_0'(z)}\right| = \frac{-4(1-b)r_0 + \frac{4r_0}{(1-r_0)^3} + \frac{6r_0^2}{(1-r_0)^4}}{2+4(1-b)r_0 - \frac{1}{(1-r_0)^2} - \frac{2r_0}{(1-r_0)^3}} = 1-\alpha.$$

In the case of $\alpha = 1/2$, the equation (2.7) also holds. □

The special case where $b = 1$ is important and it is stated as a corollary below.

Corollary 3.1.1. *Let $f \in \mathcal{A}$ and $|a_n| \le n$ for $n \ge 2$. Then f satisfies the condition (3.1) where $r_0 = r_0(\alpha)$ is the real root in $(0,1)$ of the equation*

$$2(1-\alpha)(1-r)^4 = 1-\alpha+4r+(1+\alpha)r^2 \tag{3.4}$$

The number $r_0(\alpha)$ is also the radius of convexity of order α. The number $r_0(1/2) \approx 0.064723$ is the radius of uniform convexity of the given functions. The results are all sharp.

Remark 3. For $\alpha = 0$, the Equation (3.4) reduces to

$$2(1-r)^4 = (1+4r+r^2).$$

The root of this equation in $(0,1)$ is approximately 0.09033. Our result shows that radius of convexity obtained by Yamashita [10: Theorem 2] is sharp.

Corollary 3.1.2. *Let $f \in \mathcal{A}$, $a_2 = 0$ and $|a_n| \le n$ for $n \ge 3$. Then f satisfies the condition (3.1) holds where $r_0 = r_0(\alpha)$ is the real root in $(0,1)$ of the equation*

$$2(1-\alpha+2(2-\alpha)r)(1-r)^4 = 1-\alpha+4r+(1+\alpha)r^2 \tag{3.5}$$

The number $r_0(\alpha)$ is also the radius of convexity of order α. The number $r_0(1/2) \approx 0.125429$ is the radius of uniform convexity of the given functions. The results are all sharp.

Remark 4. It is easy to see from Corollaries 3.1.1 and 3.1.2 that the radius of convexity of order α improves when $a_2 = 0$. In the particular case $\alpha = 0$, the root of the Equation (3.4) is $r_0(0) \approx 0.0903331$ while the Equation (3.5) has the root $r_0(0) \approx 0.155972$.

Theorem 3.2. *Let $f \in \mathcal{A}_b$ and $|a_n| \le M$ for $n \ge 3$. Then f satisfies the condition (3.1) where $r_0 = r_0(\alpha)$ is the smallest real root in $(0,1)$ of the equation*

$$\big((1-\alpha)(1+M) - 2(2-\alpha)(2b-M)r\big)(1-r)^3 = M\big(1-\alpha+(1+\alpha)r\big). \tag{3.6}$$

The number $r_0(\alpha)$ is also the radius of convexity of order α. The number $r_0(1/2)$ is the radius of uniform convexity of the given functions. The results are all sharp.

Proof. Let φ be defined by

$$\varphi(r) := \big((1-\alpha)(1+M) - 2(2-\alpha)(2b-M)r\big)(1-r)^3 - M\big(1-\alpha+(1+\alpha)r\big).$$

Since $\varphi(0) = 1-\alpha > 0$ and $\varphi(1) = -2M < 0$, the intermediate value theorem shows that the Equation (3.6) has a root in the interval (0,1). Let r_0 be the smallest root in $(0,1)$ of the Equation (3.6). Using $|a_2| = 2b$ for the function $f \in \mathcal{A}_b$, and the inequality $|a_n| \le M$ for $n \ge 3$, a calculation shows that, for $|z| \le r_0$,

$$\begin{aligned}
&\sum_{n=2}^{\infty} n(n-\alpha)|a_n|\,|z|^{n-1} \le \sum_{n=2}^{\infty} n(n-\alpha)|a_n| r_0^{n-1} \\
&\le 4(2-\alpha)br_0 + M\left(\sum_{n=3}^{\infty} n^2 r_0^{n-1} - \alpha \sum_{n=3}^{\infty} n r_0^{n-1}\right) \\
&= 4(2-\alpha)br_0 + M\left(\frac{1+r_0}{(1-r_0)^3} - 1 - 4r_0 - \alpha\left(\frac{1}{(1-r_0)^2} - 1 - 2r_0\right)\right) \\
&= -M(1-\alpha) + 2(2-\alpha)(2b-M)r_0 + M\left(\frac{1-\alpha+(1+\alpha)r_0}{(1-r_0)^3}\right) = 1-\alpha.
\end{aligned}$$

Thus, the function f satisfies the condition (3.1). The other two results follow easily. The results are sharp for the function f_0 given by (2.8). □

Corollary 3.2.1. *Let $f \in \mathcal{A}$ and $|a_n| \leq M$ for $n \geq 2$. Then f satisfies the condition (3.1) where $r_0 = r_0(\alpha)$ is the real root in $(0,1)$ of the equation*

$$(1-\alpha)(1+M)(1-r)^3 = M\big(1-\alpha+(1+\alpha)r\big). \tag{3.7}$$

The number $r_0(\alpha)$ is also the radius of convexity of order α. The number $r_0(1/2)$ is the radius of uniform convexity of the given functions. The results are all sharp.

Remark 5. For $\alpha = 0$, the Equation (3.7) reduces to

$$(1+M^{-1})(1-r)^3 = 1+r.$$

Our result again shows that radius of convexity obtained by Yamashita [10: Theorem 2] is sharp.

Remark 6. The problem of determining the radius of convexity of functions satisfying $|a_n| \leq M/n$ is the same as the determination of radius of starlikeness of functions satisfying the inequality $|a_n| \leq M$. The latter problem is investigated in Theorem 2.3.

4. Carathéodory functions

An analytic function p of the form $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is called a Carathéodory function if $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. The class of all such functions is denoted by $\mathcal{P}$. For such functions $p \in \mathcal{P}$, it is well-known that $|c_n| \leq 2$. Denote the class of all Carathéodory functions satisfying the inequality $\operatorname{Re} p(z) > \alpha$ for some $0 \leq \alpha < 1$ by $\mathcal{P}(\alpha)$. It is easy to see that $|c_n| \leq 2(1-\alpha)$ for $p \in \mathcal{P}(\alpha)$. In this section, we determine $\mathcal{P}(\alpha)$-radius of functions satisfying the inequality $|c_n| \leq 2M$ for $n \geq 3$ with $|c_2| = 2b$ fixed. The proof of the following result is straightforward and the details are omitted.

Theorem 4.1. *Let p be an analytic function of the form $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ with $|c_2| = 2b$ and $|c_n| \leq 2M$ for $n \geq 3$. Then*

$$|p(z) - 1| \leq 1 - \alpha \qquad (|z| \leq r_0)$$

where

$$r_0 = r_0(\alpha) = \frac{2(1-\alpha)}{1-\alpha+2b+\sqrt{(1-\alpha+2b)^2+8(1-\alpha)(M-b)}}.$$

Also $\operatorname{Re} p(z) > \alpha$ for $|z| \leq r_0(\alpha)$. These results are sharp for the function p_0 given by

$$p_0(z) = 1 - 2bz - 2M\frac{z^2}{1-z}.$$

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