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Some classes of infinite series associated with the Riemann Zeta and Polygamma functions and generalized harmonic numbers

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Abstract

The authors aim at presenting a systematic investigation of several families of infinite series which are associated with the Riemann Zeta function, the Digamma (and Polygamma) functions, the harmonic (and generalized harmonic) numbers, and the Stirling numbers of the first kind. Relevant connections of the results derived here with those considered in many earlier works on this subject are also indicated.

Keywords: Riemann Zeta function; Polygamma functions; Generalized harmonic numbers; Fractional calculus; Gauss summation theorem; Stirling numbers of the first kind

1. Introduction, definitions, and preliminaries

Many recent works (or serendipities) involving fractional calculus, especially in the area of closed-form summation of certain classes of infinite series, revived (as illustrations emphasizing the usefulness of the underlying fractional calculus techniques) various special cases and consequences of the following

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well-known (rather *classical*) result in the theory of the Psi (or Digamma) function $\psi(z) := \psi^{(0)}(z)$:

$$\sum_{n=1}^{\infty} \frac{(b)_n}{n(c)_n} = \psi(c) - \psi(c-b)$$
$$(\Re(c-b) > 0;\; c \notin \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}), \tag{1.1}$$

where, in general, $\psi^{(p)}(z)$ denotes the Polygamma functions defined by (cf., e.g., [11])

$$\psi^{(p)}(z) := \frac{\mathrm{d}^{p+1}}{\mathrm{d}z^{p+1}}\{\log \Gamma(z)\}$$
$$= \frac{\mathrm{d}^{p}}{\mathrm{d}z^{p}}\{\psi(z)\} \quad (p \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\; \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{1.2}$$

and $(\lambda)_n$ denotes the Pochhammer symbol (or the *shifted* factorial, since $(1)_n = n!$) defined, in terms of the familiar Gamma function $\Gamma(z)$, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases} \tag{1.3}$$

For a reasonably detailed *historical* account of the summation formula (1.1), and indeed also of its numerous consequences and generalizations, we refer the interested reader to the work on the subject by Nishimoto and Srivastava [14], who also provided a number of relevant earlier references on summation of infinite series by means of fractional calculus. Many *further* developments on this subject are reported by (among others) Srivastava [19], Al-Saqabi et al. [1], Aular de Durán et al. [2], Wu et al. [21], and Chen and Srivastava [5]. Each of these recent works contains citations of many other *earlier* investigations on the subject.

Yet another *classical* result, which is related rather closely to the summation formula (1.1), is the *celebrated* Gauss summation theorem:

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (\Re(c-a-b) > 0;\; c \notin \mathbb{Z}_0^-), \tag{1.4}$$

which, in view of the *obvious* derivative formulas:

$$\frac{\partial}{\partial a}\left\{\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}\right\} = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} [\psi(a+n) - \psi(a)] \frac{z^n}{n!}$$
$$(|z| < 1;\; |z| = 1 \text{ when } \Re(c-a-b) > 0;\; c \notin \mathbb{Z}_0^-), \tag{1.5}$$

$$\frac{\partial}{\partial b}\left\{\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}\right\} = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} [\psi(b+n) - \psi(b)] \frac{z^n}{n!}$$
$$(|z| < 1;\; |z| = 1 \text{ when } \Re(c-a-b) > 0;\; c \in \mathbb{Z}_0^-), \tag{1.6}$$

and

$$\frac{\partial}{\partial c}\left\{\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\right\} = -\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n}[\psi(c+n)-\psi(c)]\frac{z^n}{n!}$$
$$(|z|<1;\ |z|=1 \text{ when } \Re(c-a-b)>0;\ c\in\mathbb{Z}_0^-), \tag{1.7}$$

yields the following summation identities:

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}H_n(a;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}[\psi(c-a)-\psi(c-a-b)]$$
$$(\Re(c-a-b)>0;\ c\notin\mathbb{Z}_0^-), \tag{1.8}$$

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}H_n(b;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}[\psi(c-b)-\psi(c-a-b)]$$
$$(\Re(c-a-b)>0;\ c\notin\mathbb{Z}_0^-), \tag{1.9}$$

and (cf., e.g., [5, p. 380, Eq. (2.5)])

$$\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}H_n(c;1)$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}[\psi(c-a)+\psi(c-b)-\psi(c)-\psi(c-a-b)]$$
$$(\Re(c-a-b)>0;\ c\notin\mathbb{Z}_0^-), \tag{1.10}$$

where, for convenience, $H_n(z;\kappa)$ denotes the generalized harmonic numbers defined by (cf. [8] for the *special* case $z=1$)

$$H_n(z;\kappa) := \sum_{k=1}^{n}\frac{1}{(z+k-1)^{\kappa}} \quad (n\in\mathbb{N};\ \kappa\in\mathbb{C};\ z\in\mathbb{C}\setminus\mathbb{Z}_0^-), \tag{1.11}$$

so that we readily have

$$H_n(z;1) = \psi(z+n)-\psi(z) \quad (n\in\mathbb{N}), \tag{1.12}$$

since [7, p. 16, Eq. 1.7.1(10)]

$$\psi(z+n) = \psi(z)+\sum_{k=1}^{n}\frac{1}{z+k-1} \quad (n\in\mathbb{N}). \tag{1.13}$$

More generally, in view of the definitions (1.2) and (1.11), it is easily observed from (1.13) that

$$H_n(z;p+1) = (-1)^p[\psi^{(p)}(z+n)-\psi^{(p)}(z)] \quad (p\in\mathbb{N}_0;\ n\in\mathbb{N}). \tag{1.14}$$

We recall also the following *well-known* series representation for $\psi(z)$ (cf. [12, p. 13]):

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right), \tag{1.15}$$

where γ denotes the Euler–Mascheroni constant given by

$$\gamma := -\psi(1) \cong 0.577215664901532860606512\ldots$$

The main object of this paper is to consider and investigate several classes of infinite series associated *especially* with the harmonic numbers and Digamma functions. Some of the summation formulas, which are considered in this paper, involve the Riemann Zeta function $\zeta(s)$ and the Polygamma functions $\psi^{(p)}(z)$ $(p \in \mathbb{N})$ as well. We also point out relevant connections of the results presented here with those discussed in many recent works on this subject.

2. Some interesting deductions

First of all, in view of the series representation (1.15), we can write

$$\psi(\lambda) - \psi(\mu) = \sum_{n=0}^{\infty} \left(\frac{1}{n+\mu} - \frac{1}{n+\lambda} \right), \tag{2.1}$$

so that the well-known (rather *classical*) result (1.1) can immediately be rewritten in its *equivalent* form:

$$\sum_{n=1}^{\infty} \frac{(b)_n}{n(c)_n} = \sum_{n=0}^{\infty} \left(\frac{1}{n+c-b} - \frac{1}{n+c} \right) \quad (\Re(c-b) > 0;\ c \notin \mathbb{Z}_0^-), \tag{2.2}$$

which happens to be one of the *main* results proven recently by Shen [18, p. 1397, Proposition 2(30)].

Next we recall here the other *main* result of Shen [18, p. 1397, Proposition 2(31)]:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(b)_n}{(c)_n} \left(\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \right) &= \frac{1}{2} \left\{ \sum_{n=0}^{\infty} \left(\frac{1}{n+c-b} - \frac{1}{n+c} \right)^2 + \sum_{n=0}^{\infty} \left(\frac{1}{(n+c-b)^2} - \frac{1}{(n+c)^2} \right) \right\} \\ &\quad (\Re(c-b) > 0;\ c \notin \mathbb{Z}_0^-),\end{aligned} \tag{2.3}$$

which can immediately be rewritten in its *equivalent* form:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(b)_n}{(c)_n} \left(\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \right) &= \sum_{n=0}^{\infty} \frac{1}{(n+c-b)^2} - \frac{1}{b} \sum_{n=0}^{\infty} \left(\frac{1}{n+c-b} - \frac{1}{n+c} \right) \\ &\quad (\Re(c-b) > 0;\ c \notin \mathbb{Z}_0^-).\end{aligned} \tag{2.4}$$

The summation formulas (2.3) and (2.4) do not appear to hold true as claimed by Shen [18]. Indeed, upon differentiating both sides of the classical result (1.1) or (2.2) partially with respect to the parameter b, if we apply the definition (1.11), we find that

$$\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}H_n(b;1)=\sum_{n=0}^{\infty}\frac{1}{(n+c-b)^2}\quad(\Re(c-b)>0;\ c\notin\mathbb{Z}_0^-).\tag{2.5}$$

Now let $\Omega(b,c)$ denote the second member of the *equivalent* formulas (2.3) and (2.4). Then, making use of (2.2) and (2.5), we obtain

$$\begin{aligned}\Omega(b,c)&:=\sum_{n=0}^{\infty}\frac{1}{(n+c-b)^2}-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{1}{n+c-b}-\frac{1}{n+c}\right)\\&=\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}H_n(b;1)-\frac{1}{b}\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}=\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}\left[H_n(b;1)-\frac{1}{b}\right]\\&=\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}\sum_{k=1}^{n-1}\frac{1}{b+k},\end{aligned}$$

where we have also applied the relationships (1.12) and (1.13). Therefore, judging it by its claimed right-hand side, Shen's assertion (2.3) may be *corrected* to read as follows (see also Eq. (2.17) below for *another* corrected version when it is judged by its claimed left-hand side):

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(b)_n}{(c)_n}\left(\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{b+k}\right)\\&\quad=\frac{1}{2}\left\{\sum_{n=0}^{\infty}\left(\frac{1}{n+c-b}-\frac{1}{n+c}\right)^2+\sum_{n=0}^{\infty}\left(\frac{1}{(n+c-b)^2}-\frac{1}{(n+c)^2}\right)\right\}\\&\qquad(\Re(c-b)>0;\ c\notin\mathbb{Z}_0^-)\end{aligned}\tag{2.6}$$

or, equivalently,

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(b)_n}{(c)_n}\left(\frac{1}{n}\sum_{k=1}^{n-1}\frac{1}{b+k}\right)=\sum_{n=0}^{\infty}\frac{1}{(n+c-b)^2}-\frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{1}{n+c-b}-\frac{1}{n+c}\right)\\&\quad(\Re(c-b)>0;\ c\notin\mathbb{Z}_0^-).\end{aligned}\tag{2.7}$$

Evidently, both (2.6) and (2.7) are rather simple consequences of the *classical* result (1.2) which, in view of the relationship (1.14), also yields the following *further* summation formulas:

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}\left[\{H_n(b;1)\}^2-H_n(b;2)\right]=-\psi^{(2)}(c-b)\\&\quad(\Re(c-b)>0;\ c\notin\mathbb{Z}_0^-),\end{aligned}\tag{2.8}$$

$$\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}[\{H_n(b;1)\}^3-3H_n(b;1)H_n(b;2)+H_n(b;3)]$$
$$=\psi^{(3)}(c-b)\quad(\Re(c-b)>0;\ c\notin\mathbb{Z}_0^-),\tag{2.9}$$

and (in general)

$$\sum_{j=0}^{l}(-1)^j\binom{l}{j}\sum_{n=1}^{\infty}\frac{H_n(b;l-j+1)}{n(c)_n}\frac{\partial^j}{\partial b^j}\{(b)_n\}$$
$$=\psi^{(l+1)}(c-b)\quad(l\in\mathbb{N}_0;\ \Re(c-b)>0;\ c\notin\mathbb{Z}_0^-),\tag{2.10}$$

where we have also used the following immediate consequence of the series representation (1.15):

$$\psi^{(p)}(z):=\frac{\mathrm{d}^p}{\mathrm{d}z^p}\{\psi(z)\}=(-1)^{p-1}\sum_{n=0}^{\infty}\frac{p!}{(n+z)^{p+1}}\quad(p\in\mathbb{N}).\tag{2.11}$$

We remark in passing that, by differentiating both sides of the classical result (1.1) partially with respect to the parameter c, one readily gets

$$\sum_{n=1}^{\infty}\frac{(b)_n}{n(c)_n}H_n(c;1)=\psi^{(1)}(c-b)-\psi^{(1)}(c)$$
$$(\Re(c-b)>0;\ c\notin\mathbb{Z}_0^-)\tag{2.12}$$

or, more generally,

$$\sum_{j=0}^{l}(-1)^{l-j}\binom{l}{j}\sum_{n=1}^{\infty}\frac{(b)_n}{n}H_n(c;l-j+1)\frac{\partial^j}{\partial c^j}\left\{\frac{1}{(c)_n}\right\}$$
$$=\psi^{(l+1)}(c-b)-\psi^{(l+1)}(c)\quad(l\in\mathbb{N}_0;\ \Re(c-b)>0;\ c\notin\mathbb{Z}_0^-),\tag{2.13}$$

which may be compared with the summation formula (2.10).

Finally, by setting $a=1$ in (1.8) or $b=1$ in (1.9), we obtain a summation formula of the form:

$$\sum_{n=1}^{\infty}\frac{(b)_n}{(c)_n}\sum_{k=1}^{n}\frac{1}{k}=\frac{c-1}{c-b-1}[\psi(c-1)-\psi(c-b-1)]$$
$$(\Re(c-b)>1;\ c\notin\mathbb{Z}_0^-),\tag{2.14}$$

which, for $b=1$, immediately yields

$$\sum_{n=1}^{\infty}\frac{n!}{(c)_n}\sum_{k=1}^{n}\frac{1}{k}=\frac{c-1}{(c-2)^2}\quad(\Re(c)>2)\tag{2.15}$$

or, equivalently,

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(c)_n} \sum_{k=1}^{n-1} \frac{1}{k} = \frac{1}{(c-1)^2} \quad (\Re(c) > 1). \tag{2.16}$$

The summation formula (2.16) was derived recently by Mavromatis [13, p. 294, Eq. (3.20)] using standard nondegenerate perturbation theory. In fact, (2.16) is the *obvious* special case $b = 1$ of the following *corrected* version of Shen's assertion (2.3) when it is judged by its claimed left-hand side (see also Eq. (2.6) above):

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(b)_n}{(c)_n} \left(\frac{1}{n} \sum_{k=1}^{n-1} \frac{1}{k} \right) \\ &\quad = \frac{1}{2} \left[\left\{ \sum_{n=0}^{\infty} \left(\frac{1}{n+c-b} - \frac{1}{n+c} \right) \right\}^2 + \sum_{n=0}^{\infty} \left(\frac{1}{(n+c-b)^2} - \frac{1}{(n+c)^2} \right) \right] \\ &\quad (\Re(c-b) > 0;\ c \in \mathbb{Z}_0^-),\end{aligned} \tag{2.17}$$

as pointed out by Srivastava and Choi [20, p. 253, Eq. 3.5(17)].

By letting $c = 1$ in (1.10), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(n!)^2} \sum_{k=1}^{n} \frac{1}{k} \\ &\quad = \frac{\Gamma(1-a-b)}{\Gamma(1-a)\Gamma(1-b)} [\gamma + \psi(1-a) + \psi(1-b) - \psi(1-a-b)] \\ &\quad (\Re(a+b) < 1),\end{aligned} \tag{2.18}$$

which, for $a = -b = m$ $(m \in \mathbb{N})$, is recorded by (for example) Hansen [9, p. 362, Entry (55.4.7)]. Several other interesting special cases of the summation formulas (1.8), (1.9), and (1.10) are also recorded by Hansen [9, p. 362].

3. General summation formulas

The following unification (and generalization) of many (known or new) summation formulas, which extend the classical result (1.1), was given by Srivastava (cf. [19, p. 80, Eq. (1.3)]):

$$\begin{aligned}&\sum_{n=1}^{m} \frac{a+2n}{n(a+n)} \frac{(b)_n (c)_n (1+2a-b-c+m)_n (-m)_n}{(1+a-b)_n (1+a-c)_n (b+c-a-m)_n (1+a+m)_n} \\ &= \psi(1+a-b) + \psi(1+a-c) + \psi(1+a+m) \\ &\quad + \psi(1+a-b-c+m) - \psi(1+a) - \psi(1+a-b-c) \\ &\quad - \psi(1+a-b+m) - \psi(1+a-c+m) \quad (m \in \mathbb{N}_0),\end{aligned} \tag{3.1}$$

where an empty sum is interpreted (as usual) to be nil. In this section, we consider some of the applications of Srivastava's general result (3.1), which are relevant to our present investigation, in place of those of its very specialized form (1.1).

Upon differentiating each member of (3.1) partially with respect to the parameter b, if we make use of the relationship (1.14), we obtain the general summation formula:

$$\begin{aligned}\sum_{n=1}^{m} \frac{a+2n}{n(a+n)} &\frac{(b)_n(c)_n(1+2a-b-c+m)_n(-m)_n}{(1+a-b)_n(1+a-c)_n(b+c-a-m)_n(1+a+m)_n} \\ &\cdot [H_n(b;1) - H_n(1+2a-b-c+m;1) + H_n(1+a-b;1) \\ &- H_n(b+c-a-m;1)] \\ = H_m(&1+a-b-c;2) - H_m(1+a-b;2) \quad (m \in \mathbb{N}_0). \end{aligned} \tag{3.2}$$

If, in the summation formula (3.2), we let $m \to \infty$ and apply (2.11) in conjunction with the definition (1.11), we find that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a+2n}{n(a+n)} &\frac{(b)_n(c)_n}{(1+a-b)_n(1+a-c)_n} [H_n(b;1) + H_n(1+a-b;1)] \\ = \psi^{(1)}&(1+a-b-c) - \psi^{(1)}(1+a-b) \quad (\Re(a-b-c) > -1). \end{aligned} \tag{3.3}$$

Next, if we replace c on both sides of (3.2) by $1+a-c$, and then let $a \to \infty$, we shall get

$$\begin{aligned}\sum_{n=1}^{m} &\frac{(b)_n(-m)_n}{n(c)_n(1+b-c-m)_n} [H_n(b;1) - H_n(1+b-c-m;1)] \\ &= H_m(c-b;2) \quad (m \in \mathbb{N}_0), \end{aligned} \tag{3.4}$$

which, for $m \to \infty$, immediately yields the summation formula (2.5).

In its limit case when $c \to -\infty$, the general summation formula (3.2) reduces to the form:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a+2n}{n(a+n)} &\frac{(b)_n(-m)_n}{(1+a-b)_n(1+a+m)_n} [H_n(b;1) + H_n(1+a-b;1)] \\ &= -H_m(1+a-b;2) \quad (m \in \mathbb{N}_0), \end{aligned} \tag{3.5}$$

which, upon letting $m \to \infty$, yields

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{a+2n}{n(a+n)} &\frac{(-1)^n(b)_n}{(1+a-b)_n} [H_n(b;1) + H_n(1+a-b;1)] \\ &= -\psi^{(1)}(1+a-b) \quad (\Re(a-2b) > -2). \end{aligned} \tag{3.6}$$

Several *further* consequences of the general summation formula (3.2) can be deduced by suitably specializing the various parameters involved.

4. Remarks and observations

By setting $c = 1$ and $b = z$ in the classical result (1.1), and expanding each side of the resulting equation in power series about $z = 0$, it is easily seen for the Riemann Zeta function $\zeta(s)$ that

$$\zeta(k+1) = \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{n \cdot n!} s(n,k), \tag{4.1}$$

where $s(n,k)$ denotes the Stirling numbers of the first kind, defined by [16, p. 90]

$$z(z-1)\cdots(z-n+1) = (-1)^n(-z)_n = \sum_{k=0}^{n} s(n,k)z^k, \tag{4.2}$$

so that

$$s(n,1) = (-1)^{n+1}(n-1)!, \tag{4.3}$$

$$s(n,2) = (-1)^n(n-1)! \sum_{k=1}^{n-1} \frac{1}{k}, \tag{4.4}$$

$$s(n,3) = \frac{(-1)^{n+1}}{2}(n-1)! \left\{ \left(\sum_{k=1}^{n-1} \frac{1}{k} \right)^2 - \sum_{k=1}^{n-1} \frac{1}{k^2} \right\}, \tag{4.5}$$

$$s(n,4) = \frac{(-1)^n}{6}(n-1)! \left\{ \left(\sum_{k=1}^{n-1} \frac{1}{k} \right)^3 - 3\left(\sum_{k=1}^{n-1} \frac{1}{k} \right)\left(\sum_{k=1}^{n-1} \frac{1}{k^2} \right) + 2\sum_{k=1}^{n-1} \frac{1}{k^3} \right\}, \tag{4.6}$$

and so on.

The summation formula (4.1), derived recently by Shen [18, p. 1397, Eq. (32)] from the classical result (1.1) in the aforementioned manner, is actually a *special* case of the following known result (cf. [10, p. 343, Eq. (14)]; see also [9, p. 348, Entry (52.1.16)]):

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+k)(a)_n} s(n+k,k) = -\frac{(1-a)_k}{k!} \psi^{(k)}(a-k) \tag{4.7}$$

when $a = k+1$ (and $n \mapsto n-k$), since [cf. Eq. (2.11)]

$$\psi^{(p)}(1) = (-1)^{p-1} p!\, \zeta(p+1) \quad (p \in \mathbb{N}). \tag{4.8}$$

Indeed the special case (4.1) of (4.7) when $a = k+1$ is recorded by (for example) Jordan [10, p. 166, Eq. (6); p. 194, Eq. (11); p. 339]. It is the summation formula (4.1) that was actually used by James Stirling (1692–1770) in 1730 for the determination of sums of reciprocal power series (cf., e.g., [10, p. 195]). Several interesting companions of the summation formula (4.1) are also known (see Eqs. (4.22), (4.25), (4.26), and (4.27) below).

For the *simple* harmonic numbers H_n defined by [cf. Eq. (1.11) above]

$$H_n := H_n(1;1) = \sum_{k=1}^{n} \frac{1}{k}, \tag{4.9}$$

(4.1) with $k = 2$ immediately yields the following summation formula:

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^2} = \zeta(3), \tag{4.10}$$

which was given by Leonhard Euler (1707–1783) in 1775 (cf., e.g., [3, p. 252 *et seq*.] for a detailed historical account of Euler's formula (4.10) above). In fact, Euler's formula (4.10) also follows as the special case $k = 2$ of another known result [9, p. 362, Entry (55.5.1)]:

$$\sum_{n=1}^{\infty} \frac{n!\, H_n}{(n+1)(k)_n} = -\frac{k-1}{2} \psi^{(2)}(k-1). \tag{4.11}$$

In his 1775 paper (referred to by Berndt [3, p. 252]), Euler not only discovered (4.10) or its *obvious* variant:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \tag{4.12}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{5}{4}\zeta(4), \tag{4.13}$$

but also gave the following general summation formula:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^m} = \frac{1}{2}(m+2)\zeta(m+1) - \sum_{n=1}^{m-2} \zeta(m-n)\zeta(n+1) \quad (m \in \mathbb{N} \setminus \{1\}). \tag{4.14}$$

Since

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^{\kappa}} = \sum_{n=1}^{\infty} \frac{H_n}{n^{\kappa}} - \sum_{n=1}^{\infty} \frac{1}{n^{\kappa+1}} \quad (\Re(\kappa) > 1), \tag{4.15}$$

by choosing $\kappa = 3$ and applying Euler's formula (4.13), we readily arrive at the known summation formula (cf., e.g., [9, p. 25, Entry (5.5.10); p. 361, Entry (55.2.5)] and [15, p. 695, Entry 5.1.32.7]):

$$\sum_{n=1}^{\infty} \frac{H_n}{(n+1)^3} = \frac{1}{4}\zeta(4), \tag{4.16}$$

which was derived markedly differently by Shen [18, p. 1398] (see also [6, p. 127, Eq. (7)]).

Another interesting known result in this context is the summation formula (cf., e.g., [17, p. 267]; see also [9, p. 366, Entry (55.8.2)]):

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n}\right)^2=\frac{17}{4}\zeta(4), \tag{4.17}$$

which was rediscovered recently by (for example) Borwein and Borwein [4, p. 1192, Eq. (3)]. Since

$$H_n-\frac{1}{n}=H_{n-1},$$

by squaring both sides of this identity and applying the known results (4.13) and (4.17), we are led fairly easily to the following *essentially equivalent* form of (4.17):

$$\sum_{n=1}^{\infty}\left(\frac{H_n}{n+1}\right)^2=\frac{11}{4}\zeta(4), \tag{4.18}$$

which can be found in the works of (for example) de Doelder [6, p. 129, Eq. (9)] (see also [4, p. 1192, Eq. (2)]).

In view of the explicit representation (4.5), we find from (4.1) *with* $k=3$ that

$$\begin{aligned}\zeta(4)&=\frac{1}{2}\sum_{n=3}^{\infty}\frac{1}{n^2}\left\{\left(\sum_{k=1}^{n-1}\frac{1}{k}\right)^2-\sum_{k=1}^{n-1}\frac{1}{k^2}\right\}\\&=\frac{1}{2}\left\{\sum_{n=2}^{\infty}\left(\frac{H_n}{n+1}\right)^2-\sum_{n=2}^{\infty}\frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{k^2}\right\}\\&=\frac{1}{2}\left\{\sum_{n=1}^{\infty}\left(\frac{H_n}{n+1}\right)^2-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{k^2}\right\},\end{aligned}$$

which, in conjunction with the known result (4.18), immediately yields (cf., e.g., [18, p. 1396, Eq. (25)])

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{3}{4}\zeta(4), \tag{4.19}$$

which can at once be rewritten in the (relatively simpler) form:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{k^2}=\{\zeta(2)\}^2-\frac{3}{4}\zeta(4). \tag{4.20}$$

It should be remarked in passing that the general problem of evaluating double sums of the type:

$$\mathfrak{S}_{p,q} := \sum_{n=1}^{\infty} \frac{1}{n^p} \sum_{k=1}^{n} \frac{1}{k^q} \quad (p \in \mathbb{N} \setminus \{1\};\ q \in \mathbb{N}), \tag{4.21}$$

which corresponds to (4.20) for $p = q = 2$, was first proposed in a letter from Christian Goldbach (1690–1764) to Euler in 1742, and that Euler was successful in obtaining closed-form sums in several cases (cf., e.g., [3, p. 253]; see also [20, p. 138, Problem 37; p. 157, Proposition 3.7] for some recent developments involving the general sum $\mathfrak{S}_{p,q}$).

Next we turn once again to the general formula (4.7) which, for $a = k + 2$, yields the following companion of (4.1) (cf. [10, p. 339, Eq. (2)]):

$$\zeta(k+1) = 1 + \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{n \cdot (n+1)!} s(n,k). \tag{4.22}$$

Two further results, analogous to (4.7), include the following known summation formulas recorded by (for example) Hansen [9, p. 348, Entries (52.1.17) and (52.1.20)] (see also [10, p. 343, Eq. (14); p. 339]):

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+k-1)(a)_n} s(n+k,k) = \frac{(-a)_k}{a \cdot (k-1)!} \psi^{(k-1)}(a-k) \tag{4.23}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+k-1)(n+k-2)(k)_n} s(n+k,k) = (k-1)!\zeta(k-1). \tag{4.24}$$

In particular, for $a = k + 1$ and $a = k + 2$, we find from (4.23) that (cf., e.g., [10, p. 195, Eq. (12); p. 339, Eq. (2)]; see also Eq. (4.1) above)

$$\zeta(k) = \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{(n-1) \cdot n!} s(n,k) \tag{4.25}$$

and

$$\zeta(k) = 1 + 2\sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{(n-1) \cdot (n+1)!} s(n,k). \tag{4.26}$$

On the other hand, (4.24) can easily be rewritten in the form (cf. [10, p. 339]):

$$\zeta(k-1) = \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{(n-1)(n-2) \cdot (n-1)!} s(n,k). \tag{4.27}$$

Just as the summation formula (4.1), each of the results (4.22), (4.25), (4.26), and (4.27) is capable of yielding sums involving harmonic numbers by means of such explicit representations for the Stirling numbers $s(n,k)$ as (4.4) and (4.5), and possibly also (4.6). The details involved in these derivations are being omitted here.

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