

An Introduction to Inversion in an Ellipse

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Abstract

In this paper we study the inversion in an ellipse and some properties, which generalizes the classical inversion with respect to a circle. We also study the inversion in an ellipse of lines, ellipses and other curves. Finally, we generalize the Pappus Chain with respect to ellipses and the Pappus Chain Theorem.

Keywords: Inversion, elliptic inversion, elliptic inversion of curves, Elliptic Pappus Chain.

1 Introduction

In this paper we study the elliptic inversion, which was introduced in [2], and some related properties to the distance of elliptic inverse points, cross ratio, harmonic conjugates and the elliptic inversion of different curves. Elliptic inversion generalizes the classical inversion, which has a lot of properties and applications, see [1, 5, 6].

The outline of this paper is as follow. In Section 2 we define the inversion respect to an ellipse. In Section 3 we study some basic properties of the inversion in an ellipse and its relations with the cross ratio and the harmonic conjugates. We also study the cartesian coordinates of elliptic points. In Section 4 we describe the inversion in an ellipse of lines and conics. Finally, in Section 5 we introduce the Elliptic Pappus Chain and we apply the inversion in an ellipse to proof the generalize Pappus Chain Theorem.

2 Elliptic Inversion

Definition 1. Let E be an ellipse centered at a point O with focus F_1 and F_2 in \mathbb{R}^2 . The inversion in the ellipse E or *Elliptic Inversion* respect to E is the mapping $\psi : \mathbb{R}^2 \setminus \{O\} \mapsto \mathbb{R}^2 \setminus \{O\}$ defined by $\psi(P) = P'$, where P' lies on the ray \overrightarrow{OP} and $OP \cdot OP' = (OQ)^2$, where Q is the point of intersection of the ray \overrightarrow{OP} and the ellipse E .

The point P' is said to be the *elliptic inverse* of P in the ellipse E , or with respect to the ellipse E , E is called the *ellipse of inversion*, O is called the *center of inversion*, and the number $OQ = w$ is called the *radius of inversion*, see Figure 1. The inversion with respect to the ellipse E , center of inversion O and radius of inversion $w > 0$ is denoted by $\mathcal{E}(O, w)$. Unlike the classical case, here the radius is not constant.

The elliptic inversion is an involutive mapping, i.e., $\psi(\psi(P)) = P$. The fixed points are the points on the ellipse E . Indeed, if F is a fixed point, $\psi(F) = F$, then $OF \cdot OF = (OF)^2 = (OQ)^2$, then $OF = OQ$ and as Q lies on the ray \overrightarrow{OF} , then $F = Q$.

Proposition 1. If P is in the exterior of E then P' is interior to E , and conversely.

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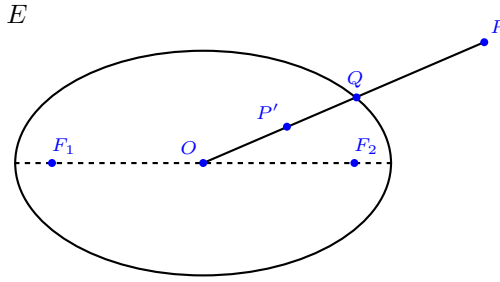


Figure 1: Inversion in an Ellipse.

Proof. Let P be an exterior point of $\mathcal{E}(O, w)$, then $w < OP$. If P' is the elliptic inverse of P , then $OP \cdot OP' = w^2$. Hence $w^2 = OP \cdot OP' > w \cdot OP'$ and $OP' < w$. \square

Inversion in an ellipse inversion does not hold for the center of inversion O , as in the usual definition. However, we can add to the Euclidean plane a single point at infinite O_∞ , which is the inverse of the center of any elliptic inversion. This plane is denoted by \mathbb{R}_∞^2 . We now have a one-to-one map of our extended plane.

Definition 2. Let E be an ellipse centered at a point O in \mathbb{R}_∞^2 , the elliptic inversion in this ellipse is the mapping $\psi : \mathbb{R}_\infty^2 \mapsto \mathbb{R}_\infty^2$ defined by $\psi(P) = P'$, where P' lies on the ray \overrightarrow{OP} and $(OP)(OP') = (OQ)^2$, where Q is the point of intersection of the ray \overrightarrow{OP} and the ellipse E , $\psi(O_\infty) = O$ and $\psi(O) = O_\infty$.

3 Basic Properties

Theorem 1. Let P and T be different points. Let P' and T' their respective elliptic inverse points respect to $\mathcal{E}(O, w)$ and $\mathcal{E}(O, u)$. Then

i. If P, T and O are not collinear, then

$$P'T' = \frac{\sqrt{(w^2 - u^2)(w^2(OT)^2 - u^2(OP)^2) + w^2u^2(PT)^2}}{OP \cdot OT}.$$

ii. If P, T and O are collinear, then

$$P'T' = \frac{w^2 PT}{OP \cdot OT}.$$

Proof. i. If P, T and O are not collinear. Then P', T' and O are not also collinear, see Figure 2.

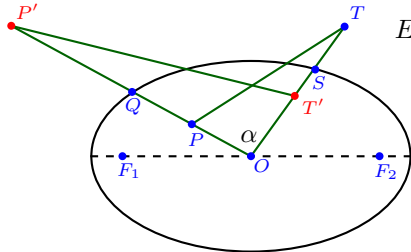


Figure 2: Distance and Inverse Points.

Let α be the measure of the angle $\angle P'OT'$, then by law of cosines

$$(P'T')^2 = (OP')^2 + (OT')^2 - 2 \cdot OP' \cdot OT' \cdot \cos \alpha \quad (1)$$

From $OP \cdot OP' = (OQ)^2 = w^2$ and $OT \cdot OT' = (OS)^2 = u^2$, we have $OP' = \frac{w^2}{OP}$ and $OT' = \frac{u^2}{OT}$, where Q and S are respectively the points of intersection of rays \overrightarrow{OP} and \overrightarrow{OT} with E , see Figure 2.

Replacing these values in (1):

$$(P'T')^2 = \frac{w^4}{(OP)^2} + \frac{u^4}{(OT)^2} - 2 \frac{w^2 u^2}{OP \cdot OT} \cos \alpha \quad (2)$$

As α is also the measure of the angle $\angle POT$, then by law of cosines

$$(PT)^2 = (OP)^2 + (OT)^2 - 2 \cdot OP \cdot OT \cdot \cos \alpha$$

$$2 \cos \alpha = \frac{(OP)^2 + (OT)^2 - (PT)^2}{OP \cdot OT}$$

Replacing in (2):

$$\begin{aligned} (P'T')^2 &= \frac{w^4}{(OP)^2} + \frac{u^4}{(OT)^2} - \frac{w^2 u^2}{OP \cdot OT} \left(\frac{(OP)^2 + (OT)^2 - (PT)^2}{OP \cdot OT} \right) \\ &= \frac{w^2 (OT)^2 (w^2 - u^2) - u^2 (OP)^2 (w^2 - u^2) + w^2 u^2 (PT)^2}{(OP)^2 (OT)^2} \\ &= \frac{(w^2 - u^2) (w^2 (OT)^2 - u^2 (OP)^2) + w^2 u^2 (PT)^2}{(OP)^2 (OT)^2} \end{aligned}$$

Hence

$$P'T' = \frac{\sqrt{(w^2 - u^2) (w^2 (OT)^2 - u^2 (OP)^2) + w^2 u^2 (PT)^2}}{OP \cdot OT}$$

ii. When P, Q are O collinear, then $OQ = w = u = OS$. Therefore

$$P'T' = \frac{w^2 \cdot PT}{OP \cdot OT} \quad \square$$

Note that if E is a circumference, then $OQ = w = u = OS$. Hence

$$\begin{aligned} P'T' &= \frac{\sqrt{(w^2 - w^2) (w^2 (OT)^2 - w^2 (OP)^2) + w^2 w^2 (PT)^2}}{(OP)(OT)} \\ &= \frac{\sqrt{w^4 (PT)^2}}{OP \cdot OT} \\ &= \frac{w^2 \cdot PT}{OP \cdot OT} \end{aligned}$$

where w is the radius of the circumference.

3.1 Inversion in an Ellipse and Cross Ratio

Suppose that A, B, C and D are four distinct points on a line l ; we define their *cross ratio* $\{AB, CD\}$ by

$$\{AB, CD\} = \frac{\overrightarrow{AC} \cdot \overrightarrow{BD}}{\overrightarrow{AD} \cdot \overrightarrow{BC}}$$

where \overrightarrow{AB} denote the signed distance from A to B . The cross ratio is an invariant under inversion in a circle whose center is not any of the four points A, B, C or D , see [1]. However, the inversion in an ellipse does not preserve the cross ratio, for example see Figure 3.

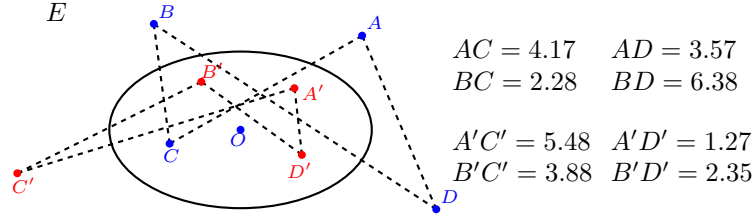


Figure 3: Elliptic Inversion and Cross Ratio.

$$\{AB, CD\} = \frac{AC \cdot BD}{AD \cdot BC} = \frac{4.17 \cdot 2.28}{3.57 \cdot 2.28} \approx 1.168,$$

$$\{A'B', C'D'\} = \frac{A'C' \cdot B'D'}{A'D' \cdot B'C'} = \frac{5.48 \cdot 2.35}{1.27 \cdot 3.88} \approx 2.613.$$

3.2 Inversion in an Ellipse and Harmonic Conjugates

If A and B are two points on a line l , any pair of points P and Q on l for which

$$\frac{AP}{PB} = \frac{AQ}{QB},$$

are said to divide \overline{AB} *harmonically*. The points P and Q are called *harmonic conjugates with respect to A and B* . It is clear that two distinct points P and Q are harmonic conjugates with respect to A and B if and only if $\{AB, PQ\} = 1$.

Theorem 2. *Let E be an ellipse with center O , and $\overline{Q_1Q_2}$ a diameter of E . Let P and P' be distinct points of the ray $\overrightarrow{OQ_1}$, which divide the segment $\overline{Q_1Q_2}$ internally and externally. Then P and P' are harmonic conjugates with respect to Q_1 and Q_2 if and only if P and P' are elliptic inverse points with respect to E .*

Proof. Suppose that P and P' are harmonic points with respect to Q_1 and Q_2 . Then

$$\{Q_1Q_2, PP'\} = 1,$$

$$\frac{Q_1P \cdot Q_2P'}{Q_1P' \cdot Q_2P} = 1.$$

Note that if P divide the segment $\overline{Q_1Q_2}$ internally and $P \in \overrightarrow{OQ_1}$. Then $Q_1P = OQ_1 - OP = w - OP$ and $Q_2P = OQ_2 + OP = w + OP$. Moreover, P' divide the segment $\overline{Q_1Q_2}$ externally and $P' \in \overrightarrow{OQ_1}$. Then $Q_1P' = OP' - OQ_1 = OP' - w$ and $Q_2P' = OQ_2 + OP' = w + OP'$. Hence

$$\frac{(w - OP)(k + OP')}{(OP' - w)(w + OP)} = 1,$$

$$(w - OP)(w + OP') = (OP' - w)(k + OP).$$

Simplifying this equation, we have $OP \cdot OP' = w^2$. Therefore P and P' are elliptic inverse points with respect to E .

Conversely, if P and P' are elliptic inverse points with respect to $\mathcal{E}(O, w)$, the proof is similar. \square

3.3 Inversion in an Ellipse and Cartesian Coordinates

Theorem 3. *Let E be an ellipse with center O and equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, a and b are respectively the semi-major axis and semi-minor axis. Let $P = (u, v)$ and $P' = (x, y)$ be a pair of elliptic*

points with respect to E . Then

$$x = \frac{a^2 b^2 u}{b^2 u^2 + a^2 v^2}, \quad (3)$$

$$y = \frac{a^2 b^2 v}{b^2 u^2 + a^2 v^2}. \quad (4)$$

Proof. Let E be an ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Suppose that $P = (u, v)$ is an exterior point to E . Let $T = (x_1, y_1)$ and $M = (x_2, y_2)$ be the points of contact of the tangent lines to E from P , see Figure 4. Then the tangent lines \overleftrightarrow{PT} and \overleftrightarrow{PM} have the following equations [3, p. 186]:

$$b^2 x_1 x + a^2 y_1 y = a^2 b^2, \quad (5)$$

$$b^2 x_2 x + a^2 y_2 y = a^2 b^2. \quad (6)$$

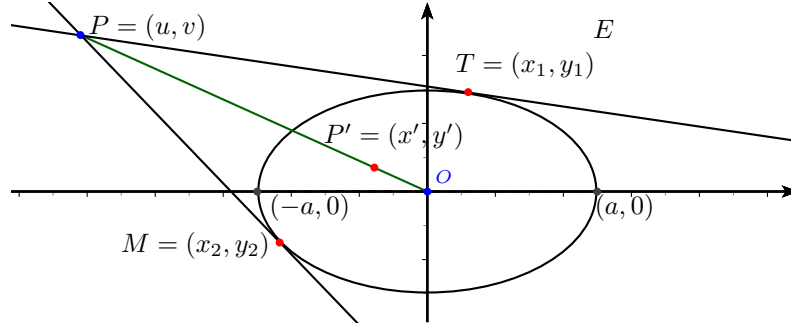


Figure 4: Inversion in an Ellipse and Cartesian Coordinates.

Particularly $P = (u, v)$ satisfies these equations. Hence

$$b^2 x_1 u + a^2 y_1 v = a^2 b^2, \quad (7)$$

$$b^2 x_2 u + a^2 y_2 v = a^2 b^2. \quad (8)$$

Equating equations (7) and (8)

$$b^2 x_1 u + a^2 y_1 v = b^2 x_2 u + a^2 y_2 v,$$

$$-\frac{b^2 u}{a^2 v} = \frac{y_1 - y_2}{x_1 - x_2}.$$

Then the line \overleftrightarrow{TM} has slope $-\frac{b^2 u}{a^2 v}$. Therefore, \overleftrightarrow{TM} has the following equation

$$y - y_1 = -\frac{b^2 u}{a^2 v} (x - x_1), \quad (9)$$

$$a^2 v y - a^2 v y_1 = -b^2 u x + b^2 u x_1, \quad (10)$$

$$a^2 v y + b^2 u x = b^2 u x_1 + a^2 v y_1. \quad (11)$$

Replacing (7) in (11), we have

$$a^2 v y + b^2 u x = a^2 b^2, \quad (12)$$

i.e., (12) is the equation of the line \overleftrightarrow{TM} . On the other hand, the line \overleftrightarrow{OP} has slope $\frac{v}{u}$, then its equation is $y = \frac{v}{u}x$. As P' is the meeting point of the lines \overleftrightarrow{TM} and \overleftrightarrow{OP} , then

$$\begin{aligned} a^2v\left(\frac{v}{u}x\right) + b^2ux &= a^2b^2 \\ (a^2v^2 + b^2u^2)x &= ua^2b^2 \\ x &= \frac{ua^2b^2}{a^2v^2 + b^2u^2} \end{aligned}$$

and

$$y = \frac{va^2b^2}{a^2v^2 + b^2u^2}$$

When P is an interior point of E , the proof is analogous. \square

When $a = b = 1$, i.e., when E is a circle, we obtain

$$\psi : (u, v) \mapsto \left(\frac{u}{v^2 + u^2}, \frac{v}{v^2 + u^2} \right)$$

4 Elliptic Inversion of Curves

In this section we study the inversion in an ellipse of lines, ellipses and other curves. If a point P moves on a curve \mathcal{C} , and P' , the elliptic inverse of P with respect to the E moves on a curve \mathcal{C}' , the curve \mathcal{C}' is called the elliptic inverse of \mathcal{C} . It is evident that \mathcal{C} is the elliptic inverse of \mathcal{C}' in E .

Theorem 4. *i. The elliptic inverse of a line l which pass through the center of the elliptic inversion is the line itself.*

ii. The elliptic inverse of a line l which does not pass through the center of the elliptic inversion is an ellipse which pass through the center of inversion, see Figure 5.

Proof. *i.* Let E be an ellipse of inversion with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and l a line with equation $Mx + Ny = 0$. Applying ψ to $Mx + Ny = 0$ gives $Mx + Ny = 0$. Indeed

$$\begin{aligned} Mx + Ny &= 0 \\ M\left(\frac{a^2b^2x}{b^2x^2 + a^2y^2}\right) + N\left(\frac{a^2b^2y}{b^2x^2 + a^2y^2}\right) &= 0 \\ Ma^2b^2x + Na^2b^2y &= 0 \\ Mx + Ny &= 0 \end{aligned}$$

ii. Let E be an ellipse of inversion with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and l a line with equation $Mx + Ny + P = 0$ ($P \neq 0$). Applying ψ to $Mx + Ny + P = 0$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{M}{P}x + \frac{N}{P}y = 0$. Indeed

$$\begin{aligned} Mx + Ny + P &= 0 \\ M\left(\frac{a^2b^2x}{b^2x^2 + a^2y^2}\right) + N\left(\frac{a^2b^2y}{b^2x^2 + a^2y^2}\right) + P &= 0 \\ M(a^2b^2x) + N(a^2b^2y) + (b^2x^2 + a^2y^2)P &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{M}{P}x + \frac{N}{P}y &= 0 \end{aligned}$$

Moreover, it is clear that the ellipse passing through the center of inversion. \square

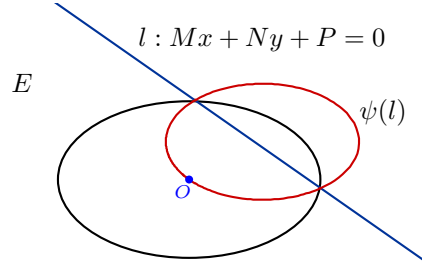


Figure 5: Elliptic Inversion of the line l .

Corollary 1. Let l_1 and l_2 be perpendicular lines intersecting at point P . Then

- i. If $P \neq O$, then $\psi(l_1)$ and $\psi(l_2)$ are orthogonal ellipses (their tangents at the points of intersection are perpendicular), which pass through P' and O .
- ii. If $P = O$, then $\psi(l_1)$ and $\psi(l_2)$ are perpendicular lines.
- iii. If l_1 through O but l_2 not through O , then $\psi(l_1)$ is an ellipse and $\psi(l_2)$ is a line which passes through O and it is orthogonal to $\psi(l_1)$ in O .

Proof. i. Let E be an ellipse of inversion with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let l and m be two perpendicular lines intersecting at P , ($P \neq O$), with respectively equations $Mx + Ny + P = 0$ ($P \neq 0$) and $My - Nx + D = 0$ ($D \neq 0$), see Figure 6.

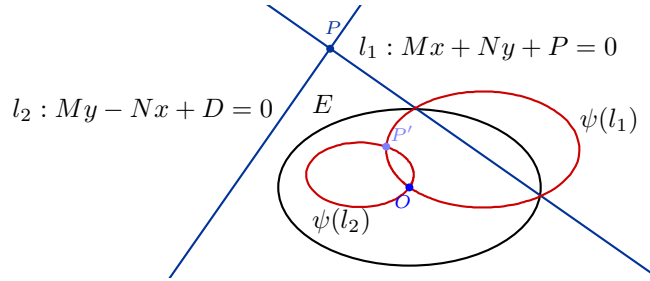


Figure 6: Inversion in an Ellipse of Perpendicular Lines.

By Theorem 4, $\psi(l_1) = l'_1$ and $\psi(l_2) = l'_2$ are ellipses pass through O and their equations are

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{M}{P}x + \frac{N}{P}y &= 0, \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{N}{D}x + \frac{M}{D}y &= 0. \end{aligned}$$

The equations of the tangent lines to these ellipses at O , are:

$$\begin{aligned} \frac{Ma^2b^2}{2}x + \frac{Na^2b^2}{2}y &= 0, \\ -\frac{Na^2b^2}{2}x + \frac{Ma^2b^2}{2}y &= 0. \end{aligned}$$

Simplifying

$$\begin{aligned} Mx + Ny &= 0, \\ -Nx + My &= 0. \end{aligned}$$

Therefore the lines are perpendicular and hence the ellipses are orthogonal.

ii. It is clear by Theorem 4.

iii. It is similar to the part 1. □

Corollary 2. *The inversion in an ellipse of a system of concurrent lines for a point H , distinct of the center of inversion is a set of coaxial system of circles with two common points H' and the center of inversion, see Figure 7.*

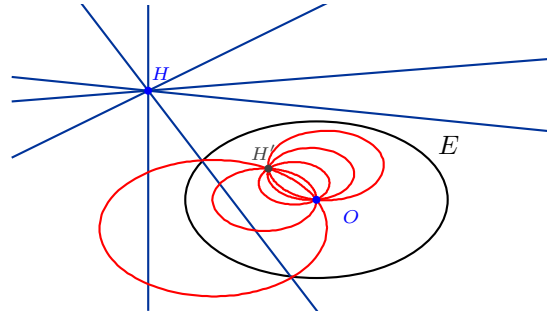


Figure 7: Inversion in an Ellipse of a System of Concurrent Lines.

Corollary 3. *The inversion in an ellipse of a system of parallel lines which does not pass through of the center of inversion is a set of tangent ellipses at the center of inversion, see Figure 8.*

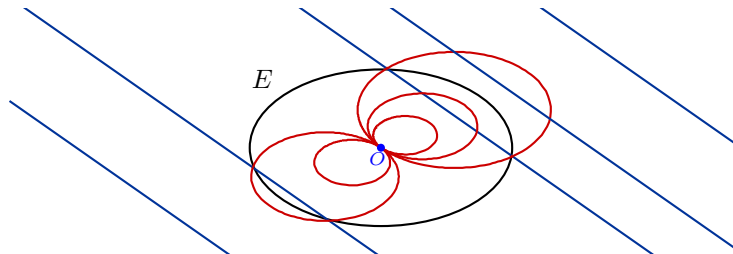


Figure 8: Inversion in an Ellipse of a system of parallel lines.

4.1 Elliptic Inversion of Ellipses

Definition 3. *If two ellipses E_1 and E_2 have parallel axes and have equal eccentricities, then they are said to be of the same semi-form. If in addition the principal axes are parallel, then they are called homothetic and it is denoted by $E_1 \sim E_2$.*

Theorem 5. *Let χ and χ' be an ellipse and its elliptic inverse curve with respect to the ellipse E . Let χ and E be homothetic curves ($\chi \sim E$), then*

- i. *If χ not passing through the center of inversion, then χ' is an ellipse not passing through the center of inversion and $\chi' \sim E$, see Figure 9.*
- ii. *If χ passing through the center of inversion, then χ' is a line, see Figure 10.*
- iii. *If χ is orthogonal to E , then χ' is the ellipse itself.*

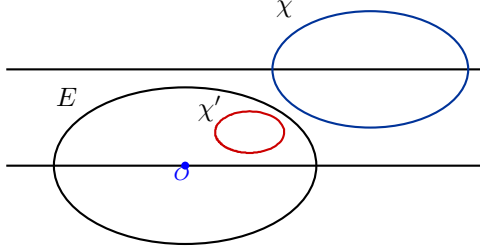


Figure 9: Theorem 5, Case *i*.

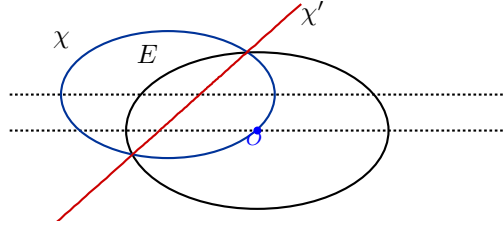


Figure 10: Theorem 5, Case *ii*.

Proof. *i.* Let χ be the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} + Dx + Ey + F = 0$ ($F \neq 0$). Applying ψ to this equation gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{D}{F}x + \frac{E}{F}y + \frac{1}{F} = 0$. Indeed

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + Dx + Ey + F &= 0 \\ \left(\frac{\frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2}}{a^2}\right)^2 + \left(\frac{\frac{a^2 b^2 y}{b^2 x^2 + a^2 y^2}}{b^2}\right)^2 + D \left(\frac{a^2 b^2 x}{b^2 x^2 + a^2 y^2}\right) + E \left(\frac{a^2 b^2 y}{b^2 x^2 + a^2 y^2}\right) + F &= 0 \\ a^2 b^4 x^2 + a^4 b^2 y^2 + D a^2 b^2 x (b^2 x^2 + a^2 y^2) + E a^2 b^2 y (b^2 x^2 + a^2 y^2) + F (b^2 x^2 + a^2 y^2)^2 &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + Dx \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + Ey \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + F \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 &= 0 \\ \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \left(1 + Dx + Ey + F \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\right) &= 0 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{D}{F}x + \frac{E}{F}y + \frac{1}{F} &= 0 \end{aligned}$$

ii and *iii* proof run like in *i*. □

4.2 Elliptic Inversion of Other Curves

Theorem 6. *The inverse of any conic not of the same semi-form as the central conic of inversion and passing through the center of inversion is a cubic curve.*

Proof. Let χ be the conic $Ax^2 + Bxy + Cy^2 + Dx + Ey = 0$, ($A = 1/a^2$, $B = 0$ and $C = 1/b^2$ cannot hold simultaneously). Applying ψ to this equation, we have

$$Aa^4 b^4 x^2 + Ba^4 b^4 xy + Ca^4 b^4 y^2 + Db^2 x^3 + Da^2 xy^2 + Eb^2 x^2 y + Ea^2 y^3 = 0$$

□

Theorem 7. *The inverse of any conic not of the same semi-form as the central conic of inversion and not passing through the center of inversion is a curve of the fourth degree.*

Proof. Similar to Theorem 6.

□

Example 1. In Figure 11, we show the elliptic inverse of a circumference χ .

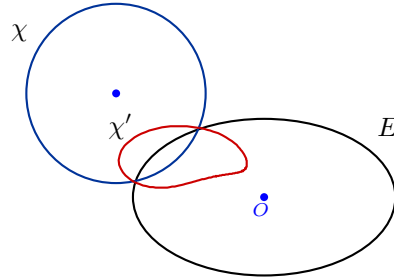


Figure 11: Inversion in an Ellipse of a Circumference.

Example 2. In Figure 12, we show the elliptic inverse of a parabola χ .

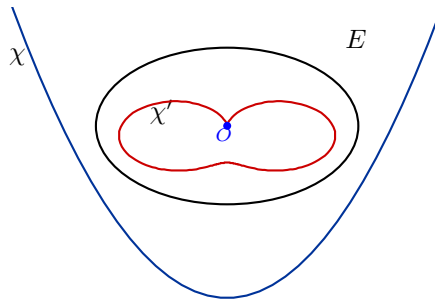


Figure 12: Inversion in an Ellipse of a Parabola.

Example 3. In Figure 13, we show the elliptic inverse of an hyperbola χ .

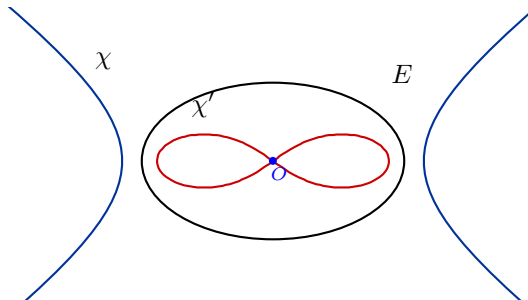


Figure 13: Inversion in an Ellipse of a Hyperbola.

Note that the inversion in an ellipse is not conformal.

5 Pappus Elliptic Chain

The classical inversion has a lot of applications, such as the Pappus Chain Theorem, Feuerbach's Theorem, Steiner Porism, the problem of Apollonius, among others [1, 5, 6]. In this section, we generalize The Pappus Chain Theorem with respect to ellipses.

Theorem 8. Let E be a semiellipse with principal diameter \overline{AB} , and E' and E_0 semiellipses on the same side of \overline{AB} with principal diameters \overline{AC} and \overline{CD} respectively, and $E \sim E_0, E_0 \sim E'$, see Figure 14. Let E_1, E_2, \dots be a sequence of ellipses tangent to E and E' , such that E_n is tangent to E_{n-1} and $E_n \sim E_{n-1}$ for all $n \geq 1$. Let r_n be the semi-minor axis of E_n and h_n the distance of the center of E_n from \overline{AB} . Then $h_n = 2nr_n$.

Proof. Let ψ_i the elliptic inversion such that $\psi(E_i) = E_i$, (in Figure 14 we select $i = 2$), i.e., $\psi_i = \mathcal{E}(B, t_i)$, where t_i is the length of the tangent segment to the Ellipse E from the point B .

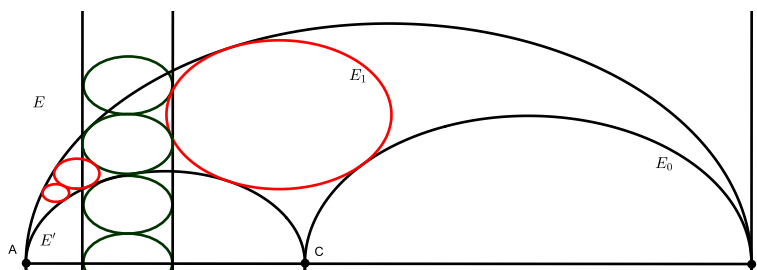


Figure 14: Elliptic Pappus Chain.

By Theorem 5, $\psi_i(E)$ and $\psi_i(E_0)$ are perpendicular lines to the line \overleftrightarrow{AB} and tangents to the ellipse E_i . Hence, ellipses $\psi_i(E_1), \psi_i(E_2), \dots$ will also invert to tangent ellipses to parallel lines $\psi_i(E)$ and $\psi_i(E_0)$. Whence $h_i = 2ir_i$. \square

6 Concluding remarks

The study of elliptic inversion suggests interesting and challenging problems. For example, generalized the Steiner Porism or Apollonius Problems with respect to ellipses.

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