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Some properties of an extended Wigner function☆

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Abstract

A covariant generalisation of Wigner function proposed by us some years ago is reviewed, and its remarkable and useful properties are elucidated; its being a natural solution to a relativistically covariant Liouville equation is also demonstrated.

Keywords: Wigner function; Liouville equation

1. Introduction

In a remarkable paper Wigner [5] introduced a distribution function, which made the averages (expectation values) in quantum mechanical observables look like the averages in statistical mechanics. The function is so structured as to satisfy the Liouville equation automatically. This function is interesting from both the applicational and epistemological view points [1].

The original Wigner function is of the form

$$\rho(x;p) = \pi^{-3} \int \mathrm{d}^3 q \psi^*(\bar{p}+\bar{q})\psi(\bar{p}-\bar{q})\mathrm{e}^{\mathrm{i}\bar{x}\cdot\bar{q}} \tag{1}$$

with the following properties:

$$\int \rho(x;p)\,\mathrm{d}^3 p = \rho(x), \qquad \int \rho(x;p)\,\mathrm{d}^3 x = \tilde{\rho}(p)$$

and the average of a quantity $f(\bar{x}) + g(\bar{p})$ given by

$$\langle f(\bar{x}) + g(\bar{p})\rangle = \int (f(x) + g(p))\rho(x;p)\,\mathrm{d}^2 x\,\mathrm{d}^3 p. \tag{2}$$

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The ψ and ψ^* in (1) are, respectively, the Schrödinger wave function and its complex conjugate and x, p are the position and momenta. It is readily verified that for a free particle (1) satisfies the Liouville equation

$$\frac{\partial\rho(\bar{x};\bar{p})}{\partial t} + \vec{v}\cdot\vec{\nabla}\rho(\bar{x};\bar{p}) = 0. \tag{3}$$

The Wigner function has proved to be a useful tool for easy evaluation of quantum averages in analogy with the averages in statistical mechanics the past several decades, and has also been widely studied in the past [1].

In this paper, I will briefly outline a very economical procedure for constructing covariant Wigner functions for the Dirac, Klein–Gordon (K–G) and spin-1 fields as already elucidated earlier [3]. Finally, I shall list the useful field theoretic structure for the generalised Wigner function for the Dirac field which can be extended to the case of spin-0 and -1 case by identical procedure.

2. The extended Wigner function

In what follows, I shall adopt the notation and conventions of Roman [4] and use the Euclidean metric $g_{\mu\nu}$ with $(++++)$ entries along the diagonal and μ, ν taking values 1,2,3 and 4.

First, I define a four-vector quantity

$$V_\mu(x,p) = \pi^{-4}\int \mathrm{d}^4q\,\mathrm{e}^{2\mathrm{i}q\cdot p}\bar{\psi}(x+q)\mathrm{i}\gamma_\mu\psi(x-q), \tag{4}$$

where ψ and $\bar{\psi}$ are free Dirac fields satisfying

$$(\gamma_\mu\partial_\mu + m)\psi = 0 \quad \text{and} \quad \bar{\psi}(m - \partial_\mu\gamma_\mu) = 0, \tag{5}$$

where $\bar{\psi} = \psi^+\gamma_4$, ψ^+ being the Hermitian conjugate of ψ.

It is readily seen that the fourth component of $V_\mu(x;p)$ is the covariant Wigner function for the Dirac field,

$$\rho(x,p) = V_4(x,p) = \pi^{-4}\int \mathrm{d}^4q\,\mathrm{e}^{2\mathrm{i}q\cdot p}\psi^+(x+q)\psi(x-q). \tag{6}$$

It is also easy to see that by virtue of (5) the following conservation equation holds:

$$\partial_\mu^{(x)}V_\mu(x,p) = 0. \tag{7}$$

The usual Dirac current is also recovered by noticing that $J_\mu(x) = \int \mathrm{d}^4p\,V_\mu(x;p) = \bar{\psi}\mathrm{i}\gamma_\mu\psi$, which also satisfies the current conservation

$$\partial_\mu J_\mu(x) = 0. \tag{8}$$

Recalling that the Dirac matrices γ_μ satisfy the algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu} \tag{9}$$

and the corresponding anti-symmetric spin tensor is defined as

$$\sigma_{\mu\nu} = \mathrm{i}[\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu], \tag{10}$$

it is easy to verify the Gordan identity

$$\bar{\psi}[\mathrm{i}\gamma_\mu]\psi = \bar{\psi}\left[\frac{p_\mu}{m} + \frac{1}{2m}\,\partial_v^{(x)}\sigma_{\mu v}\right]\psi\,. \tag{11}$$

Now using (11) in (7) with the use of (4), we can split the conservation equation into the convective and spin parts

$$V_\mu^{\mathrm{c}}(x,p) = \pi^{-4}\int \mathrm{d}^4q\mathrm{e}^{2\mathrm{i}q\cdot p}\bar{\psi}(x+q)\,\frac{p_\mu}{m}\psi(x-q) \tag{12}$$

and

$$V_\mu^{\mathrm{s}}(x,p) = \pi^{-4}\partial_v^{(x)}\int \mathrm{d}^4q\mathrm{e}^{2\mathrm{i}q\cdot p}\bar{\psi}(x+q)\,\frac{\sigma_{\mu v}}{m}\psi(x-q). \tag{13}$$

Remembering the total anti-symmetry of the spin part, it is readily seen that (7) implies the separate conservation of the convective and spin parts of $V_\mu(x,p)$:

$$\partial_\mu V_\mu^{\mathrm{c}}(x,p) = \partial_\mu V_\mu^{\mathrm{s}}(x,p) = 0. \tag{14}$$

Now it is apparent that the conservation of the convective part (14) implies the Liouville equation for the Dirac–Wigner function

$$\partial_t\rho(x,p) + v_i\partial_i\rho(x,p) = 0. \tag{15}$$

So, the conservation of the convective part of V_μ guarantees that the Wigner function for the Dirac field satisfies the Liouville equation. In arriving at (15), we have used the kinematical equality $v_i = p_i/p_4$, the velocity associated with the Liouville flow.

2.1. Wigner function for the K–G and Spin-1 fields

To repeat the procedure adopted for the free Dirac fields for the case of the spin-0 K–G fields and the spin-1 vector fields, we resort to the use of the Duffin–Kemmer equation [4], which recasts the usual second-order differential equation for the integral spin fields into first-order differential equation akin to the Dirac equation by introducing multi-component fields for the spin-0 and -1 fields with appropriate identifications for the components, to relate with the usual second-order integral spin equation. Using the β-matrix notation of Roman [4] we have

$$(\beta_\mu\partial_\mu + m)\psi(x) = 0, \qquad \bar{\psi}(m - \partial_\mu\beta_\mu) = 0, \tag{16}$$

where $\psi(x)$ is a multi-component field, each of whose components ψ_i satisfies the K–G equation, and $\bar{\psi} = \psi^+\eta_4$ and $\eta_4 = 2\beta_4^2 - 1$. The spin-0 and -1 cases correspond to 5- and 10-component irreducible representations, respectively. The components are as follows:

$$\text{Spin-0: } \psi_i(i=1\text{–}4) = -\partial_\mu\phi(x)/m, \quad \psi_5 = \phi,$$

$$\text{Spin-1: } \psi_i(i=1\text{–}6) = -F_\mu v/m, \quad \psi_i(i=7\text{–}10) = \phi_\mu, \tag{17}$$

where ϕ and ϕ_μ are the usual K–G and spin-1 fields, respectively. The β_μ matrices in (16) are, therefore, the 5×5 irreducible representations for the spin-0 fields and 10×10 for the spin-1 fields.

The algebra of the β matrices are given by [4]

$$\beta_\lambda\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\lambda = \delta_{\lambda\mu}\beta_\nu + \delta_{\mu\nu}\beta_\lambda. \tag{18}$$

The ‘spin’ operator is given by

$$\mathrm{i}S_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu. \tag{19}$$

To write the counterpart of the Gordan identity for the β-matrices analogous to relation (11), we need the further equation

$$(\partial_\mu - \beta_\nu\beta_\mu\partial_\nu)\psi(x) = \bar{\psi}(x)(\partial_\mu - \beta_\nu\beta_\mu\partial_\nu) = 0. \tag{20}$$

From (18), (19) and (15) it is readily seen that

$$\bar{\psi}(\mathrm{i}\beta_\mu)\psi = \bar{\psi}\left(\frac{p_\mu}{m} + \partial_\nu\frac{S_{\mu\nu}}{m}\right)\psi\,. \tag{21}$$

Thus, with the Duffin–Kemmer fields replacing the Dirac fields in (4) we can readily extract the Wigner function for the spin-0 and -1 fields by proceeding along the same arguments and noting that the convective part of conservation equation (14) with the replacements mentioned above leads to the corresponding Wigner function satisfying Liouville equation (15). As only the 4, 5 components of ψ contribute to the phase density for the spin-0 fields, the corresponding Wigner function works out as

$$\rho(x,p) = \frac{1}{m}\pi^{-4}\mathrm{i}\int \mathrm{d}^4q\,\mathrm{e}^{2\mathrm{i}p\cdot q}[\partial_t\phi^*(x+q)\phi(x-q) - \phi^*(x+q)\partial_t\phi(x-q)]. \tag{22}$$

2.2. Explicit forms of the extended Wigner function

It is now straightforward to use the usual free field expansions of the Dirac, spin-0 and -1 fields, respectively, to compute the corresponding Wigner function in terms of the Fock-space (number operators) of the fields [2].

I shall just present the outline of the calculation for the Dirac field which can be repeated for the spin-0 and -1 fields as well.

The Fourier representation for the Dirac field [2] is

$$\psi(x_1) = (2\pi)^{-3/2}\int \mathrm{d}^3p_1(m/E_1)^{1/2}\{p_{p_1r_1}u_{r1}(p_1)\mathrm{e}^{\mathrm{i}x_1p_1} + d^*_{p_1r_1}v^{r_1}(p_1)\mathrm{e}^{-\mathrm{i}x_1p_1}\}, \tag{23}$$

$$\psi(x_2) = (2\pi)^{-3/2}\int \mathrm{d}^3p_2(m/E_2)^{1/2}\{p_{p_2r_2}\bar{u}_{r2}(p_2)\mathrm{e}^{\mathrm{i}x_2p_2} + d_1\bar{v}^{r_2}(p_2)\mathrm{e}^{-\mathrm{i}x_2p_2}\}, \tag{24}$$

where b and d are (c-number) electron and positron amplitudes, respectively, and u and v are the corresponding positive and negative energy spinors. The latter are more conveniently written in terms of the Lorentz boost operators as follows [2]:

$$u_{r_1}(p_1) = [(m - \mathrm{i}rp_1)/2\sqrt{m(m+E_1)}]u^{r_1}(0), \tag{25}$$

$$v_{r_1}(p_1) = [(m + \mathrm{i}rp_1)/2\sqrt{m(m+E_1)}]v^{r_1}(0). \tag{26}$$

Putting all these ingredients into the expression for the Wigner function and doing the covariant integration, after the algebraic simplifications, which are tedious but straightforward, we arrive at the form

$$\rho(x, p) = \pi^{-3} \int \mathrm{d}^4 k \delta(p^2 + m^2 + k^2/4)\delta(2p\cdot k)$$
$$\times[b^*(\vec{p} + \vec{k}/2)b(\vec{p} - \vec{k}/2)\mathrm{e}^{\mathrm{i}k\cdot x} + d^*(\vec{p} + \vec{k}/2)d(\vec{p} - \vec{k}/2)$$
$$\times\mathrm{e}^{-\mathrm{i}k\cdot x}](2mp_0 + 2\mathrm{i}\hat{n}\cdot(\vec{p} \times \vec{k}))\sqrt{\frac{E_1 E_2}{(m + E_1)(m + E_2)}}, \quad (27)$$

where $\hat{n}$ is a unit vector normal to the plane of $\vec{p}$ and $\vec{k}$.

Now the Wigner function of form (27) can be readily used in any field theoretic computation involving free Dirac fields. The same strategy may be adopted to compute the Wigner function for the spin-0 and -1 fields in terms of 'c-number' amplitudes.

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