Applied Mathematical Sciences, Vol. 9, 2015, no. 47, 2299 - 2308
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2015.53211

Initial Coefficient Estimates for Certain Subclasses of Bi-Univalent Functions of Ma-Minda Type

C. Ramachandran[†], R. Ambrose Prabhu[‡] and N. Magesh[*]

[†]Department of Mathematics
University College of Engineering Villupuram
Villupuram 605 602,
Tamilnadu, India

[‡]Department of Mathematics
Anna University Chennai-600 025
Tamilnadu, India

[*]Post-Graduate and Research
Department of Mathematics
Government Arts College for Men
Krishnagiri 635 001
Tamilnadu, India

Abstract

In the present work, we propose to investigate the coefficient estimates for certain subclasses of bi-univalent functions of Ma-Minda type. Some interesting applications of the results presented here are also discussed.

Mathematics Subject Classification: 30C45

Keywords: Bi-univalent functions, Ma-Minda starlike and Ma-Minda convex functions, bi-starlike of Ma-Minda type and bi-convex of Ma-Minda type.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U})$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in \mathbb{U})$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U})\,.$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\; r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). Many interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties and characteristics associated with the bi-univalent function class Σ (including also several open problems and conjectures involving estimates on the TaylorMaclaurin coefficients of functions in Σ), can be found in recent literatures [1, 3, 5, 6, 8] and [11]-[16].

Let φ be an analytic and univalent function with positive real part in $\mathbb{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, and symmetric with respect to the real axis. The Taylor's series expansion of such function is of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots \quad \text{with } B_1 > 0. \tag{1.3}$$

Throughout this paper we assume that the function φ satisfies the above conditions one or otherwise stated.

By $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ we denote the following classes of functions

$$\mathcal{S}^*(\varphi) := \left\{ f : f \in \mathcal{S} \text{ and } \frac{zf'(z)}{f(z)} \prec \varphi(z);\; z \in \mathbb{U} \right\} \tag{1.4}$$

and

$$\mathcal{K}(\varphi) := \left\{ f : f \in \mathcal{S} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z);\; z \in \mathbb{U} \right\}. \tag{1.5}$$

The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are the extensions of a classical sets of a starlike and convex functions and in a such form were defined and studied by Ma and Minda [7]. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{K}_\Sigma(\varphi)$ (see [1]).

Similarly, the familiar classes $\mathcal{S}^*(\gamma, \varphi)$ and $\mathcal{K}(\gamma, \varphi)$ of Ma-Minda starlike and convex of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$), which are respectively, characterized by

$$\mathcal{S}^*(\gamma, \varphi) := \left\{ f : f \in \mathcal{S} \text{ and } 1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z);\; z \in \mathbb{U} \right\} \tag{1.6}$$

and

$$\mathcal{K}(\gamma, \varphi) := \left\{ f : f \in \mathcal{S} \text{ and } 1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right) \prec \varphi(z);\; z \in \mathbb{U} \right\}. \tag{1.7}$$

The classes $\mathcal{S}^*(\gamma, \varphi)$ and $\mathcal{K}(\gamma, \varphi)$ were introduced and studied by Ravichandran et al. [10]. Also, a function f is bi-starlike and bi-convex of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$) of Ma-Minda type if both f and f^{-1} are, respectively, Ma-Minda starlike and Ma-Minda convex of complex order γ ($\gamma \in \mathbb{C}\backslash\{0\}$). These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\gamma, \varphi)$ and $\mathcal{K}_\Sigma(\gamma, \varphi)$.

In this paper, estimates on the initial coefficients for bi-univalent functions of Ma-Minda type are obtained. Several related classes are also considered, and a connection to earlier known results are made.

2 Coefficient Estimates for the Function Class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \varphi)$,

In order to derive our results, we shall need the following lemma.

Lemma 2.1. (see [9]) *If $p \in \mathcal{P}$, then $|p_i| \leq 2$ for each i, where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which*

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}),$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$

Next, a function $f \in \Sigma$ is said to be in the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \varphi)$, $\gamma \in \mathbb{C}\backslash\{0\}$, $\alpha \geq 0$, $\lambda \geq 0$, if the following subordinations hold:

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{f(z)}{z} + (\alpha - 2\lambda)f'(z) + \lambda z f''(z) - 1\right) \prec \varphi(z) \qquad (2.1)$$

and

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{g(w)}{w} + (\alpha - 2\lambda)g'(w) + \lambda w g''(w) - 1\right) \prec \varphi(w), \qquad (2.2)$$

and $g(w) = f^{-1}(w)$. The class introduced in this paper is motivated by the corresponding class investigated in [2, 4].

It is interesting to note that the special values of α, γ, λ and φ lead the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \varphi)$ to various subclasses, we illustrate the following subclasses:

1. For $\alpha = 1 + 2\lambda$ the class $\mathcal{W}_\Sigma(\gamma, \lambda, 1 + 2\lambda, \varphi) \equiv \mathcal{R}_\Sigma(\gamma, \lambda, \varphi)$ is

$$1 + \frac{1}{\gamma}\left(f'(z) + \lambda z f''(z) - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left(g'(w) + \lambda w g''(w) - 1\right) \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$. The class $\mathcal{R}_\Sigma(\gamma, \lambda, \varphi)$ was independently introduced and studied by Tudor [14] and Deniz [3].

2. For $\lambda = 0$ the class $\mathcal{W}_\Sigma(\gamma, 0, \alpha, \varphi) \equiv \mathcal{B}_\Sigma(\gamma, \alpha, \varphi)$ is

$$1 + \frac{1}{\gamma}\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left((1-\alpha)\frac{g(w)}{w} + \alpha g'(w) - 1\right) \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$.

Remark 2.2. *For $\gamma = 1$ the class $\mathcal{B}_\Sigma(1,\alpha,\varphi) \equiv \mathcal{B}_\Sigma(\alpha,\varphi)$ was introduced and studied by Sivaprasad Kumar et al. [11] (see [16]). For $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \le \beta < 1$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\eta$, $0 < \eta \le 1$, the classes $\mathcal{B}_\Sigma(\alpha, \frac{1+(1-2\beta)z}{1-z}) \equiv \mathcal{B}_\Sigma(\alpha,\beta)$ and $\mathcal{B}_\Sigma(\alpha, \left(\frac{1+z}{1-z}\right)^\eta) \equiv \mathcal{B}^\eta_\Sigma(\alpha)$ were introduced and studied by Frasin and Aouf [5].*

3. For $\lambda = 0$ and $\alpha = 1$ the class $\mathcal{W}_\Sigma(\gamma, 0, 1, \varphi) \equiv \mathcal{P}_\Sigma(\gamma,\varphi)$ is

$$1 + \frac{1}{\gamma}\left(f'(z) - 1\right) \prec \varphi(z)$$

and

$$1 + \frac{1}{\gamma}\left(g'(w) - 1\right) \prec \varphi(w),$$

where $g(w) = f^{-1}(w)$.

Remark 2.3. *For $\gamma = 1$ the class $\mathcal{P}_\Sigma(1,\varphi) \equiv \mathcal{P}_\Sigma(\varphi)$ was introduced and studied by Ali et al. [1] (see [15]). For $\varphi(z) = \frac{1+(1-2\beta)z}{1-z}$, $0 \le \beta < 1$ and $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\eta$, $0 < \eta \le 1$, the classes $\mathcal{P}_\Sigma(\frac{1+(1-2\beta)z}{1-z}) \equiv \mathcal{P}_\Sigma(\beta)$ and $\mathcal{P}_\Sigma(\left(\frac{1+z}{1-z}\right)^\eta) \equiv \mathcal{P}^\eta_\Sigma$ were introduced and studied by Srivastava et al. [12] (see [6]).*

For $f \in \mathcal{W}_\Sigma(\gamma,\lambda,\alpha,\varphi)$, the following coefficient estimation holds.

Theorem 2.4. *If $f \in \mathcal{W}_\Sigma(\gamma,\lambda,\alpha,\varphi)$, then*

$$|a_2| \le \frac{|\gamma| B_1 \sqrt{B_1}}{\sqrt{|\gamma(1+2\alpha+2\lambda)B_1^2 + (1+\alpha)^2(B_1 - B_2)|}} \tag{2.3}$$

and

$$|a_3| \le \frac{|\gamma| B_1}{1+2\alpha+2\lambda} + \frac{|\gamma|^2 B_1^2}{(1+\alpha)^2}. \tag{2.4}$$

Proof. Since $f \in \mathcal{W}_\Sigma(\gamma,\lambda,\alpha,\varphi)$, there exists two analytic functions $r, s : \mathbb{U} \to \mathbb{U}$, with $r(0) = 0 = s(0)$, such that

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{f(z)}{z} + (\alpha - 2\lambda)f'(z) + \lambda z f''(z) - 1\right) = \varphi(r(z)) \tag{2.5}$$

and

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{g(w)}{w} + (\alpha - 2\lambda)g'(w) + \lambda w g''(w) - 1\right) = \varphi(s(z)). \tag{2.6}$$

Define the functions p and q by

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{2.7}$$

$$q(z) = \frac{1+s(z)}{1-s(z)} = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \dots \tag{2.8}$$

or equivalently,

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \left(p_3 + \frac{p_1}{2}\left(\frac{p_1^2}{2} - p_2\right) - \frac{p_1 p_2}{2}\right) z^3 + \dots\right) \tag{2.9}$$

$$s(z) = \frac{q(z)-1}{q(z)+1} = \frac{1}{2}\left(q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \left(q_3 + \frac{q_1}{2}\left(\frac{q_1^2}{2} - q_2\right) - \frac{q_1 q_2}{2}\right) z^3 + \dots\right). \tag{2.10}$$

It is clear that p and q are analytic in $\mathbb{U}$ and $p(0) = 1 = q(0)$. Also p and q have positive real part in $\mathbb{U}$, and hence $|p_i| \leq 2$ and $|q_i| \leq 2$. In the view of (2.5), (2.6), (2.9) and (2.10), clearly

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{f(z)}{z} + (\alpha - 2\lambda) f'(z) + \lambda z f''(z) - 1\right) = \varphi\left(\frac{p(z)-1}{p(z)+1}\right) \tag{2.11}$$

and

$$1 + \frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{g(w)}{w} + (\alpha - 2\lambda) g'(w) + \lambda w g''(w) - 1\right) = \varphi\left(\frac{q(w)-1}{q(w)+1}\right). \tag{2.12}$$

Using (2.9) and (2.10) together with (1.3), it is evident that

$$\varphi\left(\frac{p(z)-1}{p(z)+1}\right) = 1 + \frac{1}{2}B_1 p_1 z + \left(\frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2\right) z^2 + \dots \tag{2.13}$$

$$\varphi\left(\frac{q(w)-1}{q(w)+1}\right) = 1 + \frac{1}{2}B_1 q_1 w + \left(\frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) + \frac{1}{4}B_2 q_1^2\right) w^2 + \dots. \tag{2.14}$$

Since $f \in \Sigma$ is of the form (1.1), a computation shows that its inverse $g = f^{-1}$ has the expression given by (1.2).

It follows from (2.11), (2.12), (2.13) and (2.14) that

$$\frac{1}{\gamma}(1+\alpha) a_2 = \frac{1}{2}B_1 p_1 \tag{2.15}$$

$$\frac{a_3}{\gamma}(1 + 2\alpha + 2\lambda) = \frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2 p_1^2 \tag{2.16}$$

$$-\frac{1}{\gamma}(1+\alpha) a_2 = \frac{1}{2}B_1 q_1 \tag{2.17}$$

and

$$\frac{(1+2\alpha+2\lambda)}{\gamma}(2a_2^2-a_3)=\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right)+\frac{1}{4}B_2q_1^2. \tag{2.18}$$

From (2.15) and (2.17), it follows that

$$p_1=-q_1 \tag{2.19}$$

and

$$\frac{4}{\gamma^2}(1+\alpha)^2a_2^2=B_1^2(p_1^2+q_1^2). \tag{2.20}$$

Now, (2.16), (2.18) and (2.20) yield

$$a_2^2=\frac{\gamma^2B_1^3(p_2+q_2)}{4\gamma(1+2\alpha+2\lambda)B_1^2+4(1+\alpha)^2(B_1-B_2)]}. \tag{2.21}$$

Thus the desired estimate on $|a_2|$ as asserted in (2.3), follows using the Lemma 2.1 that $|p_2|\leq 2$ and $|q_2|\leq 2$.

By subtracting (2.16) from (2.18) and a computation using (2.15) finally lead to

$$a_3=\frac{\gamma B_1(p_2-q_2)}{4+8\alpha+8\lambda}+\frac{\gamma^2B_1^2p_1^2}{4(1+\alpha)^2} \tag{2.22}$$

Applying Lemma 2.1 once again, we readily get the estimate given in (2.4). □

Remark 2.5. *Taking $\alpha=1+2\lambda$ in Theorem 2.4, we obtain the corresponding result given earlier by Deniz [3, Theorem 2.6, p.56] and further a result obtained in the Theorem 2.4 corrects the result of Tudor [14, Theorem 2.1, p.79]. For $\gamma=1$ and $\alpha=1+2\lambda$ in Theorem 2.4 we have a result of Kumar et al. [11, Theorem 2.1, p.5]. For $\gamma=1$, $\lambda=0$ and $\alpha=1$ Theorem 2.4 reduces to a result of Ali et al. [1, Theorem 2.1, p.345].*

If we set $\varphi(z)=\dfrac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in the class $\mathcal{W}_\Sigma(\gamma,\lambda,\alpha,\varphi)$, we have $\mathcal{W}_\Sigma(\gamma,\lambda,\alpha,\frac{1+Az}{1+Bz})$ and defined as

$$1+\frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{f(z)}{z}+(\alpha-2\lambda)f'(z)+\lambda zf''(z)-1\right)\prec\frac{1+Az}{1+Bz},\quad z\in\mathbb{U}$$

and

$$1+\frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{g(w)}{w}+(\alpha-2\lambda)g'(w)+\lambda wg''(w)-1\right)\prec\frac{1+Aw}{1+Bw},\quad w\in\mathbb{U},$$

where $g(w)=f^{-1}(w)$.

Corollary 2.6. *If $f \in \mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \frac{1+Az}{1+Bz})$, then*

$$|a_2| \leq \frac{|\gamma|(A-B)}{\sqrt{|\gamma(1+2\alpha+2\lambda)(A-B)+(1+\alpha)^2(1+B)|}}$$

and

$$|a_3| \leq \frac{|\gamma|(A-B)}{1+2\alpha+2\lambda} + \frac{|\gamma|^2(A-B)^2}{(1+\alpha)^2}.$$

Taking $\varphi(z) = \dfrac{1+(1-2\beta)z}{1-z}$, $0 \leq \beta < 1$ in the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \varphi)$, we have $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \beta)$ and $f \in \mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \beta)$ if the following conditions are satisfied:

$$\Re\left(1+\frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{f(z)}{z}+(\alpha-2\lambda)f'(z)+\lambda z f''(z)-1\right)\right) > \beta,\ z \in \mathbb{U}$$

and

$$\Re\left(1+\frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{g(w)}{w}+(\alpha-2\lambda)g'(w)+\lambda w g''(w)-1\right)\right) > \beta,\ w \in \mathbb{U},$$

where $g(w) = f^{-1}(w)$.

Corollary 2.7. *If $f \in \mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \beta)$, then*

$$|a_2| \leq |\gamma|\sqrt{\frac{2(1-\beta)}{|\gamma(1+2\alpha+2\lambda)|}} \text{ and } |a_3| \leq \frac{2|\gamma|(1-\beta)}{1+2\alpha+2\lambda} + \frac{4|\gamma|^2(1-\beta)^2}{(1+\alpha)^2}.$$

Remark 2.8. *Taking $\gamma = 1$, $\lambda = 0$ in Corollary 2.7 our result coincides with a result of Frasin and Aouf [5, Theorem 3.2, p.1572], $\gamma = 1$, $\lambda = 0$ and $\alpha = 1$ in Corollary 2.7 we obtain a result of Srivastava et al. [12, Theorem 2, p.1191]*

Taking $\varphi(z) = \left(\frac{1+z}{1-z}\right)^\eta$, $0 < \eta \leq 1$ in the class $\mathcal{W}_\Sigma(\gamma, \lambda, \alpha, \varphi)$, we have $\mathcal{W}^\eta_\Sigma(\gamma, \lambda, \alpha)$ and $f \in \mathcal{W}^\eta_\Sigma(\gamma, \lambda, \alpha)$ if the following conditions are satisfied:

$$\arg\left|1+\frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{f(z)}{z}+(\alpha-2\lambda)f'(z)+\lambda z f''(z)-1\right)\right| < \eta,\ z \in \mathbb{U}$$

and

$$\arg\left|1+\frac{1}{\gamma}\left((1-\alpha+2\lambda)\frac{g(w)}{w}+(\alpha-2\lambda)g'(w)+\lambda w g''(w)-1\right)\right| < \eta,\ w \in \mathbb{U},$$

where $g(w) = f^{-1}(w)$.

Corollary 2.9. *If* $f \in \mathcal{W}_{\Sigma}^{\eta}(\gamma, \lambda, \alpha)$, *then*

$$|a_2| \leq \frac{2|\gamma|\eta}{\sqrt{|2\gamma(1+2\alpha+2\lambda)\eta + (1+\alpha)^2(1-\eta)|}} \text{ and } |a_3| \leq \frac{2|\gamma|\eta}{1+2\alpha+2\lambda} + \frac{4|\gamma|^2\eta^2}{(1+\alpha)^2}.$$

Remark 2.10. *Taking* $\gamma = 1$, $\lambda = 0$ *in Corollary 2.9 our result coincides with a result of Frasin and Aouf [5, Theorem 2.2, p.1570],* $\gamma = 1$, $\lambda = 0$ *and* $\alpha = 1$ *in Corollary 2.9 we obtain a result of Srivastava et al. [12, Theorem 1, p.1190]*

Acknowledgement : The work is supported by UGC, under the grant F.MRP-3977/11 (MRP/UGC-SERO) of the third author.

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Received: December 15, 2014; Published: March 21, 2015