Research Journal of Pure Algebra -2(10), 2012, Page: 305-311
Available online through www.rjpa.info ISSN 2248–9037

SOME GENERALIZATION OF ENESTROM KAKEYA THEOREM

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(Received on: 09-10-12; Revised & Accepted on: 31-10-12)

ABSTRACT

In this paper we obtain some extensions and generalizations of a well known theorem due to Enestrom and Kakeya. We obtain all the zeros of polynomial $P(z)=\sum_0^n a_j z^j$ satisfying certain restrictions on real as well as imaginary coefficients of complex number $a_j=(\alpha_j, \beta_j)$ lying within the disk $R^{\lambda\mu} \leq |z-z_{\lambda\mu}| \leq R_{\lambda\mu}$, $z_{\lambda\mu}$ (an arbitrary point) is the centre of the disk in the complex plane.

Mathematics Subjects Classification: *26C10, 30C10, 30C15*

Keywords: *Polynomials, Zeros, Enestrom - Kakeya theorem, sharper bounds.*

INTRODUCTION

The following result due to Enestrom&Kakeya [12] is well known in the theory of distribution of zeros of polynomials.

Theorem A (1): If $P(z)=\sum_0^n a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots\ldots \geq a_1 \geq a_0 > 0, \quad a_j \in R \tag{1}$$

Then P(z) does not vanish in $|z|>1$

This is a very elegant result but it is equally limited in scope as the hypothesis is very restrictive.

In the literature [1-10], [13-15], diverse attempts have been made for generalizing the Enestrom-Kakeya theorem to polynomials and analytic functions.

A. Joyal et al [11] extended this theorem to the polynomials whose coefficient are monotonic but not necessarily non negative and proved the following:

Theorem A (2):

If $P(z)=\sum_0^n a_j z^j$ be a polynomial of degree n such that
$$a_n \geq a_{n-1} \geq a_{n-2} \geq \ldots\ldots \geq a_1 \geq a_0, \quad a_j \in R$$

Then all the zeros of P(z) lie in

$$|z| \leq (a_n - a_0 + |a_0|) \div |a_n|. \tag{2}$$

This was further improved upon by Dewan & Govil[7].

Aziz and Zargar[1] relaxed the hypothesis of Theorem A(1) and proved the following result.

Theorem B: Let $P(z)=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients such that for some $k\geq 1$,
$$ka_n \geq a_{n-1} \geq \ldots\ldots a_1 \geq a_0 > 0$$
then all the zeros of P(z) lie in $|z+k-1| \leq k$ (3)

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Shah & Liman [15] also proved the following extensions of Enestrom-Kakeya theorem

Theorem C: Let P (z) $=\sum_0^n a_j\, z^j$ be a polynomial of degree n with complex coefficients. If

$Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. such that for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq \alpha_{n-2} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0 \quad ,$$

$$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \ldots\ldots \geq \beta_1 \geq \beta_0 > 0$$

Then all the zeroes of P(z) lie in

$$|z + \frac{\alpha_n}{a_n}(\lambda - 1)| \leq [\lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\ ] \div |a_n| \tag{4}$$

Theorem D: Let P(z) $=\sum_0^n a_j\, z^j$ be a polynomial of degree n with complex coefficients. If Re $(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. such that for some $k \geq 1$,

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \ldots\ldots \leq \alpha_{p+1} \leq \alpha_p \geq \alpha_{p-1} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0$$

$$\beta_n \geq \beta_{n-1} \geq \beta_{n-2} \geq \ldots\ldots \geq \beta_1 \geq \beta_0 > 0$$

where $0 \leq p \leq n-1$, then all the zeros of P(z) lie in

$$|z + \frac{\alpha_n}{a_n}(\lambda - 1)| \leq [2\alpha_p - \lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n\ ] \div |a_n| \tag{5}$$

Recently, Choo[5] has proved the following theorem

Theorem E: Let P(z) $=\sum_0^n a_j\, z^j$ be a polynomial of degree n with complex coefficients. If Re $(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. such that for some p and r and for some λ , $\mu > 0$

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \ldots\ldots \leq \alpha_{p+1} \leq \alpha_p \geq \alpha_{p-1} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0$$

$$\mu\,\beta_n \leq \beta_{n-1} \leq \ldots\ldots \leq \beta_{r+1} \leq \beta_r \geq \beta_{r-1} \geq \ldots\ldots \geq \beta_1 \geq \beta_0$$

then P(z) has all its zeros in $R_1 \leq |z| \leq R_2$ where

$R_1 = \frac{|a_0|}{M_1}$ and $R_2 = \frac{M_2}{|a_n|}$

with
$M_1 = |a_n| + |(\lambda - 1)\alpha_n| + |(\mu - 1)\beta_n| + 2(\alpha_p + \beta_r) - (\lambda\alpha_n + \mu\beta_n) - (\alpha_0 + \beta_0)$

and

$M_2 = |(\lambda - 1)\alpha_n| + |(\mu - 1)\beta_n| + 2(\alpha_p + \beta_r) - (\lambda\alpha_n + \mu\beta_n) - (\alpha_0 + \beta_0) + |a_0|$

Here we notice that the annulus $R_1 \leq |z| \leq R_2$ is expressed in terms of λ and μ as associated to the coefficients α_n and β_n in the given constraint in Theorem E . In our investigation we are able to associate these parameters λ and μ to the centre of the disk and obtain sharper bound in the general standard form as given below:

Theorem 1: Let P(z) $=\sum_0^n a_j\, z^j$ be a polynomial of degree n with complex coefficients. If Re $(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$, for $j = 0,1,2$-----n. such that for some λ , $\mu \geq 1$,

$$\lambda\alpha_n \leq \alpha_{n-1} \leq \ldots\ldots \leq \alpha_{p+1} \leq \alpha_p \geq \alpha_{p-1} \geq \ldots\ldots \geq \alpha_1 \geq \alpha_0$$

$$\mu\beta_n \leq \beta_{n-1} \leq \ldots\ldots \leq \beta_{q+1} \leq \beta_q \geq \beta_{q-1} \geq \ldots\ldots \geq \beta_1 \geq \beta_0$$

(6)

where $0 \leq p, q \leq n-1$, then all the zeros of P(z) lie in the disk

$$R^{\lambda\mu} \leq |z - z_{\lambda\mu}| \leq R_{\lambda\mu} \quad , \tag{7}$$

where

$$z_{\lambda\mu} = -\left[\frac{(\lambda-1)\alpha_n}{a_n} + i\,\frac{(\mu-1)\beta_n}{a_n}\right] , \tag{8a}$$

$$R_{\lambda\mu} = \frac{1}{|a_n|}[2(\alpha_p + \beta_q)-(\lambda\,\alpha_n + \mu\beta_n)- (\alpha_0 + \beta_0)+|a_0|] \tag{8b}$$

$$R^{\lambda\mu} = \frac{|a_0|}{|a_n|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|+2(\alpha_p+\beta_q)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0)} - \frac{1}{|a_n|}[(\lambda-1)^2\alpha_n^{\,2}+(\mu-1)^2\beta_n^{\,2}]^{1/2} \tag{8c}$$

Proof: Consider the polynomial

$F(z) = (1-z)P(z) = -a_n z^{n+1} +(a_n - a_{n-1})z^n +\text{-------}+(a_1-a_0)+a_0$

$= [-\alpha_n z^{n+1} +(\alpha_n - \alpha_{n-1})z^n +\ldots+(\alpha_1- \alpha_0)z+ \alpha_0] + i[-\beta_n z^{n+1} +(\beta_n - \beta_{n-1})z^n +\ldots\ldots+ (\beta_1- \beta_0)z+\beta_0]$

$= [-\alpha_n z^{n+1} +[(\alpha_n -\lambda\alpha_n)+(\lambda\alpha_n - \alpha_{n-1})]z^n +(\alpha_{n-1}- \alpha_{n-2})z^{n-1} +\ldots\ldots+ (\alpha_1- \alpha_0)z+ \alpha_0]$
$+i[-\beta_n z^{n+1} +[(\beta_n -\mu\beta_n) +(\mu\beta_n - \beta_{n-1})]z^n +(\beta_{n-1}- \beta_{n-2})z^{n-1} +\ldots\ldots+ (\beta_1- \beta_0)z+\beta_0]$

$= -z^n\{(\alpha_n +i\beta_n)z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n\}+[(\lambda\alpha_n - \alpha_{n-1})z^n +(\alpha_{n-1}- \alpha_{n-2})z^{n-1} +\text{-------}+ (\alpha_1- \alpha_0)z +i[(\mu\beta_n - \beta_{n-1})z^n$
$+ (\beta_{n-1}- \beta_{n-2})z^{n-1} +\text{----}+(\beta_1- \beta_0)z] + (\alpha_0 +i\beta_0)$

Now if $|z|>1$, $\frac{1}{|z|^{n-j}} < 1$, j= 0,1,2---n-1

Therefore,

$|F(z)| \geq |-z|^n \{|a_n z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n|\} - \{|\lambda\alpha_n - \alpha_{n-1}| + \frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|} +\text{-----}+ \frac{|\alpha_{p+1}-\alpha_p|}{|z|^{n-p-1}} + \frac{|\alpha_p-\alpha_{p-1}|}{|z|^{n-p}}+ \frac{|\alpha_{p-1}-\alpha_{p-2}|}{|z|^{n-p+1}}+\ldots\ldots+$
$\frac{|\alpha_1-\alpha_0|}{|z|^{n-1}} +|\mu\beta_n - \beta_{n-1}| + \frac{|\beta_{n-1}-\beta_{n-2}|}{|z|} +\ldots\ldots+ \frac{|\beta_{q+1}-\beta_q|}{|z|^{n-q-1}} + \frac{|\beta_q-\beta_{q-1}|}{|z|^{n-q}} + \frac{|\beta_{q-1}-\beta_{q-2}|}{|z|^{n-q+1}}+\ldots\ldots+ \frac{|\beta_1-\beta_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \}$

$= |z|^n \{|a_n z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n|\} - \{2\alpha_p - \lambda\alpha_n - \alpha_0+|a_0|+2\beta_q-\mu\beta_n - \beta_0\}$

>0, if

$\left| z + \frac{(\lambda-1)\alpha_n}{a_n} + i\frac{(\mu-1)\beta_n}{a_n} \right| > \frac{1}{|a_n|} \{2(\alpha_p + \beta_q)-(\lambda\alpha_n+ \mu\beta_n)-(\alpha_0+\beta_0) +|a_0| \}$

This shows that the zeros of F (z) having modulus greater than 1 lie in

$$\left|z + \frac{(\lambda-1)\alpha_n+i(\mu-1)\beta_n}{a_n}\right| \leq \frac{1}{|a_n|}\{2(\alpha_p + \beta_q)-(\lambda\alpha_n + \mu\beta_n)-(\alpha_0+\beta_0)+|a_0| \} \tag{9}$$

Since all the zeros of P (z) with modulus greater than 1 lie in the disc given by equ. (9), it can be shown that $R_{\lambda\mu} \geq 1$.

Consequently the zeros of P (z) with modulus less than or equal to one are already contained in the disk

$$|z-z_{\lambda\mu}| \leq R_{\lambda\mu} \tag{10}$$

In order to prove the lower bound $R^{\lambda\mu} \leq |z-z_{\lambda\mu}|$ we first prove the following lemma.

Lemma: Let P (z) $=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients. Then for $|z|<1$,

We show that $|z| \leq \frac{|a_0|}{M_2} = \frac{|a_0|}{|a_n|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|+2(\alpha_p+\beta_q)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0)}$

Proof: Let $|z|<1$.
Consider $F(z) = (1-z)P(z) = -a_n z^{n+1} +(a_n - a_{n-1})z^n +\text{-------}+(a_1-a_0)+a_0$

$= [-\alpha_n z^{n+1} +(\alpha_n - \alpha_{n-1})z^n +\text{-------}+(\alpha_1- \alpha_0)z+ \alpha_0] + i[-\beta_n z^{n+1} +(\beta_n - \beta_{n-1})z^n +\ldots\ldots+ (\beta_1- \beta_0)z+\beta_0]$

$= -z^n\{(\alpha_n +i\beta_n)z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n\}+[(\lambda\alpha_n - \alpha_{n-1})z^n +(\alpha_{n-1}- \alpha_{n-2})z^{n-1} +\ldots\ldots+ (\alpha_1- \alpha_0)z]$
$+i[(\mu\beta_n - \beta_{n-1})z^n + (\beta_{n-1}- \beta_{n-2})z^{n-1} +\ldots\ldots+(\beta_1- \beta_0)z] + (\alpha_0 +i\beta_0)$

$$= \Psi(z) + a_0, \tag{11}$$

where

$\Psi(z) = a_n z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n + [(\lambda\alpha_n - \alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+\ldots\ldots+(\alpha_1-\alpha_0)z+i[(\mu\beta_n-\beta_{n-1})z^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}+\ldots\ldots+(\beta_1-\beta_0)z]$

$\therefore |\Psi(z)| = |a_n z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n +[(\lambda\alpha_n - \alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+\ldots\ldots+(\alpha_1-\alpha_0)z+i[(\mu\beta_n-\beta_{n-1})z^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}+\ldots\ldots+(\beta_1-\beta_0)z]|$

$\leq |a_n z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n| +|[(\lambda\alpha_n - \alpha_{n-1})+(\alpha_{n-1}-\alpha_{n-2})+\ldots\ldots+(\alpha_1-\alpha_0)]|+|[(\mu\beta_n-\beta_{n-1})+(\beta_{n-1}-\beta_{n-2})+\ldots\ldots+(\beta_1-\beta_0)]|$

$\leq |a_n z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n|+\{2\alpha_p-\lambda\alpha_n-\alpha_0+2\beta_q-\mu\beta_n-\beta_0\}$

$\leq |a_n z+(\lambda-1)\alpha_n+i(\mu-1)\beta_n| +M_1,$

$\leq |a_n z|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n| +|M_1|$

where $M_1 = 2\alpha_p-\lambda\alpha_n-\alpha_0+2\beta_q-\mu\beta_n-\beta_0$ (12)

Since $\Psi(0)=0$, it follows by Schwarz lemma that

$|\Psi(z)| \leq M_1|z|$ for $|z|<1$

Therefore for $|z|<1$,

$|F(z)| = |\Psi(z)+a_0| \geq |a_0|-|\Psi(z)|= |a_0|-|a_n z|-|(\lambda-1)\alpha_n| - |(\mu-1)\beta_n| - M_1|z|$

> 0, if

$|a_0| \geq |a_n z|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n| + M_1|z|$

$\geq |z|[\,|a_n|+M_1+\frac{|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|}{|z|}]$

$\geq |z|(|a_n|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|+M_1)$

$> |z|M_2,$

where $M_2 = (|a_n|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|+2(\alpha_p+\beta_q)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0))$ (13)

Thus, $|z| \leq \frac{|a_0|}{M_2} = \frac{|a_0|}{|a_n|+|(\lambda-1)\alpha_n|+|(\mu-1)\beta_n|+2(\alpha_p+\beta_q)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0)}$ (14)

Hence P(z) does not vanish in $|z| < \frac{|a_0|}{M_2}$. It can be shown that $M_2 \leq |a_0|$ so that $|z|\leq 1$. Hence P(z) has all its zeros in $\frac{|a_0|}{M_2} \leq |z|$. (15)

Now we prove the second part of the main theorem (1)

Since $|z - z_{\lambda\mu}| \geq |z| - |z_{\lambda\mu}|$, (16)

then using eq(15) of above lemma in eq(16), we have

$|z - z_{\lambda\mu}| \geq |z| - |z_{\lambda\mu}| \geq \frac{|a_0|}{M_2} - |z_{\lambda\mu}|$

This implies $\frac{|a_0|}{M_2} - |z_{\lambda\mu}| \leq |z - z_{\lambda\mu}|$

$\frac{|a_0|}{M_2} - \left|\frac{(\lambda-1)\alpha_n}{a_n} + i\,\frac{(\mu-1)\beta_n}{a_n}\right| \leq |z - z_{\lambda\mu}|$ (17)

From above eq(17) we obtain $R^{\lambda\mu} \leq |z-z_{\lambda\mu}|$, (18)
where $R^{\lambda\mu}$ is given in equation 8(c)

On combining equ. (10) and equ.(18) the above theorem is completely proved.

Remark 1: The bound given by eq(7) shows that the arbitrary constants λ and μ associated to coefficients α_n and β_n have the dependence on the centre (arbitrary) of the disc. We note that this bound coincides with the annulus corresponding to $\lambda = \mu = 1$ given in Theorem E. However concentric circles in Theorem E, centred at the origin do not have the dependence on λ and μ.

Remark 2: Further we note with regard to the upper bound of above Theorem 1 given as $|z-z_{\lambda\mu}| \le R_{\lambda\mu}$,

where

$z_{\lambda\mu} = -\frac{(\lambda-1)\alpha_n}{a_n} - i\,\frac{(\mu-1)\beta_n}{a_n}$ = A+iB, where $A = -\frac{(\lambda-1)\alpha_n}{a_n}$ and $B = -\frac{(\mu-1)\beta_n}{a_n}$

and $R_{\lambda\mu} = \frac{1}{|a_n|}[2(\alpha_p + \beta_q)-(\lambda\,\alpha_n + \mu\beta_n)-(\alpha_0+\beta_0)+|a_0|$

and that if we transfer the centre of the above disk to the disk at the origin so that eq(9) can be written as:

$$|z| = |\overline{z - z_{\lambda\mu}} + z_{\lambda\mu}| \le |z-z_{\lambda\mu}| + |z_{\lambda\mu}|$$

$$\le R_{\lambda\mu} + |z_{\lambda\mu}|$$

$$\le \frac{1}{|a_n|}[2(\alpha_p+\beta_q)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0)+|a_0|] + \sqrt{A^2+B^2} \quad (19)$$

Comparing this bound with upper bound of Theorem E given by:

$|z| \le R_2 = \frac{M_2}{|a_n|}$

$$\le \frac{1}{|a_n|}\{|(\lambda-1)\alpha_n| + |(\mu-1)\beta_n| + 2(\alpha_p+\beta_r) - (\lambda\alpha_n+\mu\beta_n) - (\alpha_0+\beta_0) + |a_0|\}$$

$$\le \frac{1}{|a_n|}[2(\alpha_p+\beta_q)-(\lambda\alpha_n+\mu\beta_n)-(\alpha_0+\beta_0)+|a_0|] + |A| + |B| \quad (20)$$

We here find that the present bound given by (19) is sharper than (20) of Choo[5], in view of $\sqrt{A^2+B^2}$ < A+B.

Theorem 2: Let $P(z) = \sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients such that for some $\lambda \ge 1$, $0<\tau\le 1$

$$\lambda a_n \le a_{n-1} \le \text{------} \le a_{p+1} \le a_p \ge a_{p-1} \ge \text{----} \ge a_1 \ge \tau a_0$$

where $0 \le p \le n-1$, then all the zeros of P(z) lie in

$$|z+\lambda-1| \le \frac{2a_p - \lambda a_n + 2|a_0| - \tau(a_0+|a_0|)}{|a_n|} \quad (21)$$

Proof: Consider a polynomial

$F(z) = (1-z)P(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \text{------} + (a_1-a_0)z + a_0$

Let |z|>1 so that $\frac{1}{|z|^{n-j}} < 1$, j = 0,1,---n-1

$$\therefore |F(z)| = |-a_n z^{n+1} + (a_n - a_{n-1})z^n + \text{------} + (a_1-a_0)z + a_0|$$

$$= |-a_n z^{n+1} + [(a_n - \lambda a_n) + (\lambda a_n - a_{n-1})]z^n + \text{------} + [(a_1 - \tau a_0) + (\tau a_0 - a_0)]z + a_0|$$

$$\ge |-z|^n \{|a_n z + (\lambda-1)a_n| - \{|\lambda a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \text{---} + \frac{|a_{p+1} - a_p|}{|z|^{n-p-1}} + \frac{|a_p - a_{p-1}|}{|z|^{n-p}} + \text{-----} + \frac{|a_1 - \tau a_0|}{|z|^{n-1}} + \frac{|\tau a_0 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}$$

$$\ge |z|^n \{|a_n z + (\lambda-1)a_n| - \{|\lambda a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \ldots + \frac{|a_{p+1} - a_p|}{|z|^{n-p-1}} + \frac{|a_p - a_{p-1}|}{|z|^{n-p}} + \ldots + \frac{|a_1 - \tau a_0|}{|z|^{n-1}} + \frac{|1-\tau||a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}$$

$$\therefore |F(z)| \ge |z|^n \{|a_n||z+\lambda-1| - \{a_{n-1} - \lambda a_n + a_{n-2} - a_{n-1} + \ldots + a_p - a_{p+1} + a_p - a_{p-1} + \cdots + a_1 - \tau a_0 + (1-\tau)|a_0| + |a_0|\}$$

$$\ge |z|^n \{|a_n||z+\lambda-1| - \{-\lambda a_n + 2a_p - \tau a_0 + (1-\tau)|a_0| + |a_0|\}$$

$$\ge |z|^n \{|a_n||z+\lambda-1| - \{-\lambda a_n + 2a_p - \tau(a_0 + |a_0|) + 2|a_0|\} > 0,$$

If, $|z+\lambda-1| > \frac{2a_p-\lambda a_n-\tau(a_0+|a_0|)+2|a_0|}{|a_n|}$

This shows that the zeros of F(z) having modulus greater than 1 lie in the disk

$$|z+\lambda-1| \leq \frac{2a_p-\lambda a_n-\tau(a_0+|a_0|)+2|a_0|}{|a_n|}$$

But the zeros of F(z) of modulus not greater than 1 already satisfy (21) and therefore all the zeros of F(z) lie in the disk $|z+\lambda-1| \leq \frac{2a_p-\lambda a_n-\tau(a_0+|a_0|)+2|a_0|}{|a_n|}$. Since the zeros of P (z) are also the zeros of F(z), Theorem 2 is proved completely.

Note: Here when $\tau=1$ and $\lambda=\frac{a_{n-1}}{a_n}$, we notice that Theorem 4 of Aziz & Zargar[1] turns out to be a special case of the bound given by eq(21).

Corollory 1. Let $P(z)=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients such that for some $0<\lambda\leq 1$, $0<\tau\leq 1$

$$\lambda a_n \leq a_{n-1} \leq \ldots\ldots \leq a_{p+1} \leq a_p \geq a_{p-1} \geq \ldots\ldots \geq a_1 \geq \tau a_0$$

where $0 \leq p \leq n-1$, then all the zeros of P(z) lie in

$$|z-(1-\lambda)| \leq \frac{2a_p-\lambda a_n+2|a_0|-\tau(a_0+|a_0|)}{|a_n|} \qquad (22)$$

Here we omit the proof of the above cor.1 since it is on the similar lines as given by Theorem 2.

We notice here that if $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta j=0$, then the result given by Theorem 2 in Gulzar[6] is a particular case of the general bound given by eq(22).

Theorem 3: Let $P(z)=\sum_0^n a_j z^j$ be a polynomial of degree n with complex coefficients such that for some m $\lambda\geq 1$, $0<\tau\leq 1$

$$\lambda a_n \geq a_{n-1} \geq \ldots\ldots \geq a_{p+1} \geq a_p \geq a_{p-1} \geq \ldots\ldots \geq a_1 \geq \tau a_0$$

then all the zeros of P(z) lie in

$$|z+\lambda-1| \leq \frac{\lambda a_n-\tau(a_0+|a_0|)+2|a_0|}{|a_n|} \qquad (23)$$

Proof: Consider a polynomial

$F(z) = (1-z)P(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + \text{-----} + (a_1-a_0)z + a_0$

Let $|z|>1$ so that $\frac{1}{|z|^{n-j}} < 1$, $j = 0,1,\text{---}n-1$

$\therefore |F(z)| = |-a_n z^{n+1} + (a_n - a_{n-1})z^n + \text{-----} + (a_1-a_0)z + a_0|$

$= |-a_n z^{n+1} + [(a_n - \lambda a_n)+(\lambda a_n - a_{n-1})]z^n + \text{-----} + [(a_1-\tau a_0)+(\tau a_0 - a_0)]z + a_0|$

$\geq -z|^n \{|a_n z+(\lambda-1)a_n| - \{|\lambda a_n - a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + \ldots + \frac{|a_{p+1}-a_p|}{|z|^{n-p-1}} + \frac{|a_p-a_{p-1}|}{|z|^{n-p}} + \ldots + \frac{|a_1-\tau a_0|}{|z|^{n-1}} + \frac{|1-\tau||a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}$

$\geq |z|^n \{|a_n||z+\lambda-1| - \{\lambda a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_{p+1} - a_p + a_p - a_{p-1} + \cdots + a_1 - \tau a_0 + (1-\tau)|a_0| + |a_0|\}$

$\geq |z|^n \{|a_n||z+\lambda-1| - \{\lambda a_n - \tau a_0 + (1-\tau)|a_0| + |a_0|\}$

> 0, if

$|z+\lambda-1| > \frac{\lambda a_n-\tau(a_0+|a_0|)+2|a_0|}{|a_n|}$

This shows that the zeros of F(z) having modulus greater than 1 lie in the disk

$$|z+\lambda-1| \leq \frac{\lambda a_n-\tau(a_0+|a_0|)+2|a_0|}{|a_n|}$$

It can also be verified that the zeros of F(z) whose modulus is less than or equal to one also lie in the disk defined by equation(23) and therefore all the zeros of P(z) lying in the disc given by equation(23)

Hence above theorem is proved.

Corollary:

(i) If $\tau = 1$, we get $|z+\lambda-1| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}$ which coincides with the result given by Aziz & Zargar [1]

(ii) If $\tau = 1$ and if all a_i's>0, then $|z+\lambda-1| \leq \lambda$ which coincides with the result Aziz & Zargar [1].

(iii) If $\tau = \lambda = 1$ and if all a_i's >0, then $|z| \leq 1$ which coincides with Theorem A.

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Source of support: Nil, Conflict of interest: None Declared