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Sharper Bounds for the Zeros of Polynomials Using Enestrom-Kakeya Theorem

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Abstract

A classical result of Enestrom & Kakeya if $a_n \geq a_{n-1} \geq ....a_0 > 0,$ then for $|z|>1, \sum_{j=0}^{n} a_j.z^j \neq 0, a_j \in R$ (the set of real number) is extended to the polynomials where coefficient a_j are complex such that

$\text{Re}(a_j) = \alpha_j \; and \; \text{Im}(a_j) = \beta_j, \; j = 0,1,.........n$ and

$\lambda\alpha_n \geq \alpha_{n-1} \geq ...... \geq \alpha_1 \geq \alpha_0,$

$\mu\beta_n \geq \beta_{n-1} \geq .... \geq \beta_1 \geq \beta_0 \; for \; \lambda, \mu \in R \, where \, \lambda, \mu \geq 1.$

Introducing a positive real number k associated to the mth mean of λ & μ and $m(\neq 0)$ and defining $k = (\lambda + \mu)/m,$ m>0, we obtain scaled and translated version of a circle with sharper bounds containing all the zeros of a polynomial.

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1. Introduction

The following result due to Enestrom & Kakeya[8] is well known in the theory of distribution of zeros of polynomials.

Theorem 1(a): If $P(z) = \sum_{j=0}^{n} a_j.z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_0 > 0 \qquad 1(a)$$

then P(z) does not vanish in |z|>1.

A. Joyal et al [7] extended theorem to the polynomials whose coefficient are monotonic but not necessary non-negative and proved the following:

Theorem 1(b): If P(z) = $\sum_{j=0}^{n} a_j.z^j$ is a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq .... \geq a_1 \geq a_{0,} \qquad a_j \in R, \qquad 1(b)$$

Then all the zeros of P(z) lie in

$$|z| \leq (a_n - a_0 + |a_0|) \div |a_n| .$$

Theorem 1(c): If $P(z) = \sum_{j=0}^{n} a_j.z^j$ is a polynomial of degree n such that for some $\lambda \geq 1.$

$$\lambda a_n \geq a_{n-1} \geq ....> a_1 \geq a_0, \qquad \lambda, \; a_j \in R,$$

then all the zeros of P(z) lie in

$$|z + \lambda - 1| \leq \left(\lambda a_n - a_0 + |a_0|\right) \div |a_0| \qquad 1(c)$$

Among other authors besides Joyal et al [7], Dewan & Govil [3] & Aziz & Zarger [1] also extended

Theorem 1 (a) to the polynomials whose coefficients are monotonic but not necessarily non-negative.

2. The polynomials with complex coefficients

Recently Govil & Mc Tume[6] extended the results of Aziz & Zarger[1] to the polynomials with complex coefficients given by:

Theorem 2: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be the polynomials of degree n with $\operatorname{Re}(a_j) = \alpha_j$ *and* $\operatorname{Im}(a_j)=\beta_j$, for

j = 0,1,......... .., n. *If for some* $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq .... \geq \alpha_1 \geq \alpha_0,$$

Then all the zeros of P(z) lie in

$$|z+\lambda-1| \leq \left(\lambda\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j| \right) \div |a_n|. \tag{2}$$

More recently Rather & Shakeel [9] on the lines of Govil & Mc Tume [6] obtained the following :

$$\left| z + (\lambda-1)\frac{\alpha_n}{|a_n|} \right| \leq \left(\lambda\alpha_n - \alpha_0 + |\alpha_0| + 2\sum_{j=0}^{n} |\beta_j| \right) \div |a_n|. \tag{3}$$

Generalizing the above result given by (2), they have also proved the following result:

Theorem 3*:* Let $P(z)=\sum^{n} a_j z^j$ be a polynomial of degree n with Re $(a_j)=\alpha_j$ *and* $\operatorname{Im}(a_j)=\beta_j$, for

j = 0,1,………, n. If for some $\lambda \geq 1$,

$$\lambda\alpha_n \geq \alpha_{n-1} \geq .... \geq \alpha_1 \geq \alpha_{0,}$$

and $$\lambda\beta_n \geq \beta_{n-1} \geq .... \geq \beta_1 \geq \beta_0,$$

then all the zeroes of P(z) lie in

$$|z+\lambda-1| \leq \left[\lambda(\alpha_n+\beta_n) - (\alpha_0+\beta_0) + |a_0|\right] \div |a_n| \tag{4}$$

In this paper we consider the generalization of the above theorem given by:

Theorem 4*:* Let $P(z)=\sum_{j=1}^{n} a_j z^j$ be a polynomial of degree n with $\operatorname{Re}:(a_j) = \alpha_j$ *and* $\operatorname{Im}(a_j) = \beta_j$ for $j = 0,1..........n$ if m[th] mean is associated to some λ & μ *each* ≥ 1, such that

$$\begin{aligned} &\lambda\alpha_n \geq \alpha_{n-1}.................. \geq \alpha_1 \geq \alpha_0\ , \\ &\mu\beta_n \geq \beta_{n-1}.................... \geq \beta_1 \geq \beta_0\ , \end{aligned} \tag{5}$$

and if we define $k = \dfrac{\lambda + \mu}{m}$ *for* $m \in R^+$, (the set of positive real numbers),
then all the zeros of P(z) lie in

$$|z + k - 1| \le \{k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|\} \div |a_n| \qquad (6)$$

Proof : Consider the polynomial

$$F(z) = (1 - z)P(z)$$

$$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \sum_{j=1}^{n}(a_j - a_{j-1})z^j + a_0$$

$$= -a_n z^n\left[z + \left(\frac{\lambda+\mu}{m}\right) - 1\right] + z^n\left\{\left(\frac{\lambda+\mu}{m}\right)a_n - a_{n-1}\right\} + \sum_{j=1}^{n-1}(\alpha_j - \alpha_{j-1})z^j + i\sum_{j=1}^{n-1}(\beta_j - \beta_{j-1})z^j + a_0,$$

Let $|z| > 1,$ then

$$|F(z)| \ge |a_n||z + (k-1)||z^n| - \Bigg| \{k\alpha_n - \alpha_{n-1}\}|z|^n + |k\beta_n - \beta_{n-1}||z|^n$$
$$+ \sum_{j=1}^{n-1}|\alpha_j - \alpha_{j-1}||z|^j + |i|\sum_{j=1}^{n-1}|\beta_j - \beta_{j-1}||z|^j + |a_0|\Bigg]$$

(By T. inequality & by above def. of k)

$$|F(z)| \ge |z|^n\Bigg[\left(|a_n||z + k - 1|\right) - \left\{|k\alpha_n - \alpha_{n-1}| + |k\beta_n - \beta_{n-1}|\right\} +$$
$$- \sum_{j=1}^{n-1}\left\{\frac{|\alpha_j - \alpha_{j-1}|}{|z|^{n-j}} + \frac{|\beta_j - \beta_{j-1}|}{|z|^{n-j}} + \frac{|a_0|}{|z|^n}\right\}\Bigg]$$

$$\ge |z|^n\Bigg[|a_n||z + k - 1| - \Bigg\{(k\alpha_n - \alpha_{n-1}) + (k\beta_n - \beta_{n-1}) +$$
$$- \sum_{j=1}^{n-1}\left\{(\alpha_j - \alpha_{j-1}) + (\beta_j - \beta_{j-1})\right\} + |a_0|\Bigg\}\Bigg]$$

$$\ge |z|^n\left[|a_n||z + k - 1| - \left\{k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|\right\}\right] \qquad > 0, \text{ if}$$

$$|z + k - 1| > \left\{k(\alpha_n + \beta_n) - (\alpha_0 + \beta_0) + |a_0|\right\} \div |a_n|$$

This shows that the zeros of F(z) whose modulus is greater than unity lie in the disc given by eq. (6). Since it can easily be verified that those zeros of F(z) whose modulus is less than or equal to one also lie in the disc as defined by the eq. (6), thus it follows that all the zeros of F(z) and hence of P(z) lie in the disc given by eq. (6). Hence theorem 4 is proved.

Cor. Let $P(z)=\sum_{j=0}^{n} a_j.z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and Im $(a_j)=\beta_j$ for

j = 0, 1,……….., n. If for some λ & $\mu \geq 1$ and $k=\dfrac{\lambda+\mu}{m}, m \neq 0$ such that

$$\lambda\alpha_n \geq \alpha_{n-1} .... \geq \alpha_1 \geq \alpha_0 > 0,$$
$$and \quad \mu\beta_n \geq \beta_{n-1} .... \geq \beta_1 \geq \beta_0 > 0,$$

Then all the zeros of P(z) lie in

$$|z+k-1| \leq k\sqrt{2} \tag{7}$$

Here we note that the above result directly follows from Theorem 4 together with the equations:

$$|a_0|=|\alpha_0+i\beta_0|=\sqrt{\alpha_0^2+\beta_0^2} \leq \alpha_0+\beta_0 \forall \alpha_0, \beta_0 \geq 0 \text{ and}$$
$$(\alpha_n-\beta_n)^2 \geq 0 \Rightarrow \sqrt{2(\alpha_n^2+\beta_n^2)} \geq \alpha_n+\beta_n.$$

We further note that if we choose $\mu=\lambda$ & $m=2$ so that k is defined as the mean of the multiples of leading coefficients α_n & β_n then the above results given by eq. (6) & (7) turn out to be the same as derived by Rather & Skakeel [9]. Also if we put $\beta_j=0\forall,\ j=0,1,......\textit{such that}\ \mu=0$ and choosing m=1, eq. (6) takes the form :

$$|z+\lambda-1| \leq \left(\lambda\alpha_n-\alpha_0+|\alpha_0|\right) \div |a_n|, \tag{7a}$$

coinciding with the result given by Aziz & Zarger [1]. Also in view of the $\beta_j=0,$ if we choose $\mu=0$, and m= k =1 then from $\lambda+\mu=mk\ \textit{we get}\ \lambda=1$ and then Eq. (6) concides with joyal et al [7].

We now prove

Theorem 5: Let $P(z)=\sum_{j=0}^{n} a_j.z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j$ and Im $(a_j)=\beta_j$ for j = 0, 1,……..,n and if for some $\lambda \geq 1$ where

$$\lambda\alpha_n \geq \alpha_{n-1} .... \geq \alpha_1 \geq \alpha_0, \tag{8}$$

then all the zeros of P(z) lie in the disc:

$$\left|z+(\lambda-1)\frac{\alpha_n}{|a_n|}\right| \leq \left[b+\sqrt{2}.\sqrt{a^2+b^2}\right] \div |a_n|, \tag{9}$$

where

$$a=\lambda|\alpha_n|+|\beta_n| \text{ and } b=|\alpha_{n-1}|+|\beta_{n-1}|. \tag{10}$$

Proof : Consider the polynomial

$$F(z)=(1-z)P(z)=-a_n z^{n+1}+(a_n-a_{n-1})z^n+\sum_{j=0}^{n-1}(a_j-a_{j-1})z^j, \qquad (a_{-1}=0)$$

$$= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + i(\beta_n - \beta_{n-1})z^n + \sum_{j=0}^{n-1}\{(\alpha_j - \alpha_{j-1}) + i(\beta_j - \beta_{j-1})\}z^j.$$

Now let $|z| > 1$, then

$$|F(z)| = \Big| -a_n z^{n+1} - (\lambda\alpha_n - \alpha_n)z^n + (\lambda\alpha_n - \alpha_{n-1})z^n +$$

$$i(\beta_n - \beta_{n-1})z^n + \sum_{j=0}^{n-1}(\alpha_j - \alpha_{j-1})z^j + \sum_{j=0}^{n-1}(\beta_j - \beta_{j-1})z^j \Big|$$

$$= |z|^n \Big\| (a_n z + (\lambda - 1)\alpha_n) - \{\{(\lambda\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})\} +$$

$$\sum_{j=0}^{n-1}(\alpha_j - \alpha_{j-1})z^{j-n} + i\sum_{j=0}^{n-1}(\beta_j - \beta_{j-1})z^{j-n} \Big]$$

$$= |z|^n \left| \left[F_1(\lambda,\alpha,z) - \{F_2(\lambda,\alpha,\beta) + F_3(\alpha,z) + F_4(\beta,z) \right] \right|. \tag{11}$$

Where,

$$F_1(\lambda,\alpha,z) = a_n z + (\lambda - 1)\alpha_n, F_2(\lambda,\alpha,\beta) = \{(\lambda\alpha_n - \alpha_{n-1}) + i(\beta_n - \beta_{n-1})\};$$

$$F_3(\alpha,z) = \sum_{j=0}^{n-1}(\alpha_j - \alpha_{j-1})z^{j-n}, F_4(\beta,z) = i\sum_{j=0}^{n-1}(\beta_j - \beta_{j-1})z^{j-n}. \tag{12}$$

To simplify Eq (11), we use the Lemma due to Govil & Rehman [5] given as :

Lemma: If $|\arg a_j - \beta| \le \alpha \le \pi/2$ *for some* $t > 0$, $|ta_j| \ge |a_{j-1}|$, then

$$|ta_j - a_{j-1}| \le \{(|ta_j| - |a_{j-1}|)\cos\alpha + (|ta_j| + |ta_{j-1}|)\sin\alpha\} \tag{13}$$

Now from Eq (11) we have :

$$|F(z)| \ge |z|^n \left[|F(z,\lambda,\alpha| - |F_5(\lambda,\alpha,\beta,z| \right], \text{(by T. inequality)} \tag{14}$$

where,

$$F_5(\lambda,\alpha,\beta,z) = F_2(\lambda,\alpha,\beta) + F_3(\alpha,z) + F_4(\beta,z) \tag{15}$$

Using Triangular inequality we have:

$$|F_5(\lambda,\alpha,\beta,z)| \le |F_2(\lambda,\alpha,\beta)| + |F_3(\alpha,z)| + |F_4(\beta,z)| \tag{16}$$

Using (12), we have

$|F_2(\lambda,\alpha,\beta)| \le |\lambda\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|$ (by T. Inequality)

$\le (|\lambda\alpha_n| - |\alpha_{n-1}| + |\beta_n| - |\beta_{n-1}|)\cos\alpha + (|\lambda\alpha_n| + |\alpha_{n-1}| + |\beta_n| + |\beta_{n-1}|)\sin\alpha$ (using lemma) (17)

Also it is easy to find $|F_3(\alpha,z)| \le \sum_{j=o}^{n-1} |(\alpha_j - \alpha_{j-1})| \le |\alpha_{n-1}|$, [Using T. inequality & eq.(8)] (18)

& $|z|^{j-n} < 1, |\alpha_{-1}| = 0$

and $|F_4(\beta,z)| \le |i| \sum_{j=0}^{n-1} |(\beta_j - \beta_{j-1})| \le |\beta_{n-1}|$ (19)

(taking $\beta_{-1} = 0$)

Now from Eq. (16), taking in to account the result of eqs. (17), (18) & (19), and noting Eq. (10) we write equation (14) as:-

$$|F(z)| \ge |z|^n \left[|a_n z + (\lambda-1)\alpha_n| - \{(b + (a-b)\cos\alpha + (a+b)\sin\alpha)\} \right] \quad (20)$$

Thus for $|z| > 1, |F(z)| > 0,\ if$

$$|a_n z + (\lambda-1)\alpha_n| > (b + (a-b)\cos\alpha + (a+b)\sin\alpha),$$

which gives :

$$\left| z + (\lambda-1)\frac{\alpha_n}{|a_n|} \right| > [b + (a-b)\cos\alpha + (a+b)\sin\alpha] \div |a_n|. \quad (21)$$

Above equation shows that the zeros of F(z) having moduli greater than 1 lie in the circle:

$$\left| z + (\lambda-1)\frac{\alpha_n}{|a_n|} \right| \le \frac{1}{|a_n|}\{b + (a-b)\cos\alpha + (a+b)\sin\alpha\} \quad (22)$$

In view of max $(p\cos\alpha + q\sin\alpha) = \sqrt{p^2+q^2}$, from above we observe that the zeros of F(z) whose modulus is greater than one lie in the disc defined by Eq. (9) of Theorem (5). It can be easily verified that the zeros of F(z) whose moduli is less than or equal to one, also lie in the circle defined by Eq. (9), in view of all the zeros of P(z) lying in the disc given by Eq. (9). Hence the theorem is proved.

Following the above and in the passing we note the following theorems:

Theorem 6 : Let $P(z) = \sum_{j=0}^{n} a_j.z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j) = \alpha_j$ and Im $(a_j) = \beta_j, j = 0,1,\ldots\ldots\ldots\ldots\ldots, n\, and\, If\ for\ \lambda \ge 1\ such that$

$$\lambda\alpha_n \ge \alpha_{n-1} \ge \ldots\ldots\ldots\ldots\ldots\ldots\alpha_1 \ge \alpha > 0, \quad (23)$$

then all the zeros of P(z) lie in

$$\left|z+(\lambda-1)\frac{\alpha_n}{|a_n|}\right| \le \left[(\alpha_{n-1}+\beta_{n-1})+\{2\{(\lambda\alpha_n+\beta_n)^2+(\alpha_{n-1}+\beta_{n-1})^2\}\}^{1/2}\right]\div|a_n| \qquad (24)$$

Cor. If we choose $\lambda = \dfrac{\alpha_{n-1}}{\alpha_n}$, *then*

$$\left|z+\frac{\alpha_{n-1}-\alpha_n}{|a_n|}\right| \le \left[(\alpha_{n-1}+\beta_{n-1})+\{2\{(\alpha_{n-1}+\beta_n)^2+(\alpha_{n-1}+\beta_{n-1})^2\}\}^{1/2}\right]\div|a_n| \qquad (25)$$

Theorem 7: Let $p(z)=\sum_{j=0}^{n} a_j.z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ *and* $\operatorname{Im}(a_j)=\beta_j$ for $j=0,1,\ldots\ldots\ldots\ldots,n.$ If for some $\lambda \ge 1$ *and* $t>0$,

$\lambda t^n\alpha_n \ge t^{n-1}\alpha_{n-1}\ldots\ldots\ldots\ldots\ldots\ldots \ge t.\alpha_1 \ge \alpha_0$, then all the zero of P(z) lie in the disc.

$$\left|z+\frac{(\lambda-1)t\alpha_n}{|a_n|}\right| \le \left[(t^{n-1}\alpha_{n-1}+\beta_{n-1})+\{2(\lambda t^n\alpha_n+\beta_n)^2\right.$$

$$\left. \cdot\,(t^{n-1}\alpha_{n-1}+\beta_{n-1})\}^{1/2}\right]\div|a_n|t^{n-1}. \qquad (26)$$

Proof_:- Using above theorem (6) to the polynomial P(tz), in view of the symmetry we arrive at the result given by (26).

Applying theorem 6 to the polynomial $\{-iP(tz)\}$, we easily have the following:

Theorem 8:- Let $P(z)=\sum_{j=0}^{n} a_j.z^j$ be a polynomial of degree n with $\operatorname{Re}(a_j)=\alpha_j$ and Im $(a_j)=\beta_j$, $j=0,1,\ldots\ldots\ldots\ldots\ldots\ldots\ldots n.$ If for some $\mu>1$ *and* $t>0$

$\mu t^n\beta_n \ge t^{n-1}\beta_{n-1} \ge \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ge t\beta_1 \ge \beta_{0,}$

Then all the zeros lie in

$$\left|z+(\mu-1)\frac{t\beta_n}{|a_n|}\right| \le \left[(\alpha_{n-1}+t^{n-1}\beta_{n-1})+\{2\{(\alpha_n+\mu t^n\beta_n)^2+\right.$$

$$\left.(\alpha_{n-1}+t^{n-1}\beta_{n-1})^2\}^{1/2}\right]\div|a_n|t^{n-1}$$

$$\left|z+(\mu-1)\frac{t\beta_n}{|a_n|}\right| \le \left[(\alpha_{n-1}+t^{n-1}\beta_{n-1})+\{2\{(\alpha_n+\mu t^n\beta_n)^2\right.$$

$$\left.+(\alpha_{n-1}+t^{n-1}\beta_{n-1})^2\}^{1/2}\right]\div|a_n|t^{n-1} \qquad (27)$$

Theorem 9 :- If t = 1, then subject to $\mu\beta_n \geq \beta_{n-1} \geq \dots\dots\dots\dots \geq \beta_1 \geq \beta_0$, from above theorem we have all the zeros lying in the disc.

$$\left|z+(\mu-1)\frac{\beta_n}{|a_n|}\right| \leq \Big[(\alpha_{n-1}+\beta_{n-1})+\{2\{(\alpha_n+\mu\beta_n)^2$$

$$+(\alpha_{n-1}+\beta_{n-1})^2\}\}^{1/2}\Big] \div |a_n| \tag{28}$$

3. Illustration

Here we present some examples in the tabulated form as given below to show that the main theorems 4 & 5, give better results. We therefore consider the polynomial:

$$P_n(z) = \sum_{j=0}^{n} a_j.z^j$$

We consider some polynomials $P_n(z) = \sum_{j=0}^{n} a_j.z^j$ corresponding to n=2, 3 & 4 and compare the bounds obtained by others with our present bounds and we also give the location of the zeros of polynomials corresponding to these values of n.

n	$a_j = \alpha_j + i\beta_j$	Approximate zeros of polynomials $P_n(z)$	Bound for the zeros of polynomial with the values of λ & reference of equation
2	$a_2 = 2+3i,$ $a_1 = -2-2i,$ $a_0 = -5-5i,$ With constraint $\lambda\alpha_2 \geq \alpha_1 \geq \alpha_0$	$z_1 = 3.17 - 0.905i,$ $z_2 = 2.5 + 0.75i.$	$\lvert z+1.1\rvert \leq 9.4$ for $\lambda = 2.1$ eq. (2)
			$\lvert z+1.1\rvert \leq 9.9$ for $\lambda = 3,$ eq.(3) due to Rather & Shakeel [9].
			$\lvert z+1.1\rvert \leq 3.6$ for $\lambda = 3,$ in eq. (9)
3	$a_3 = 2+3i,$ $a_2 = 2+4i,$ $a_1 = -9-9i,$ $a_0 = -10-10i,$ With constraint $\lambda\alpha_3 \geq \alpha_2 \geq \alpha$,Corresponding to $\lambda = 1$	$z_1 = -2+0i,$ $z_2 = 3.17 - 0.905i,$ $z_3 = 2.5 + 0.75i$	$\lvert z\rvert \leq 20.6$ for eq. (2) due to Govil & Mc-Tune [6].
			$\lvert z\rvert \leq 20.6$ for eq.(3), due to Rather & Shakeel [9]
			$\lvert z\rvert \leq 4.7$ eq. (9) of present paper.
4	$a_4 = 6, \alpha_3 = 4$ $a_2 = 3, \alpha_1 = 2,$ $a_0 = -100,$ With constraint $\lambda\alpha_4 \geq \alpha_3 \geq \alpha_2$ where , $\lambda = 1$	$z_1 = 1.81,$ $z_2 = -2.16,$ $z_3 = -0.15 + 2.06i$ $z_4 = -0.15 - 2.06i$	$\lvert z\rvert \leq 34.33$ eq. (2) in [6] and eq. (3) in [9] as stated above.
			$0.26 \leq \lvert z\rvert \leq 6.023$ due to Dewan & Govil [3].
			$\lvert z\rvert \leq 2.3$ eq. (9), present paper.

From the above table we note that our bounds for the zeros of $P_n(z)$ corresponding to n=2, 3 & 4 are substantially shaper than those obtained by others. Moreover, corresponding to n=4 and on comparing bound due to Dewan & Govil we obtain the annulus $0.2 \leq \lvert z\rvert \leq 2.3,$ which may in this case represent the smaller region containing the zeros of $P_4(z)$.

Remark:

We also note that choosing $\lambda = 3$ & using above eq. (4) due to Rather & Shakeel [9], all the zeros of $P_2(z)$ lie in the approximate disc: $|z+2| \le 8.8.$ However if we choose $\mu = 3$ such that $\lambda = 3$ & $m = 2$ or $\lambda = 2$ & $m = 5/3$, then we obtain the same bound by using of Eq. (6). However the generalized theorem 4 has the advantage that corresponding to the fixed value of λ we can vary m & μ to obtain a number of discs containing all the zeros of given polynomial and select the possible optimal bound given by (7a) for some particular values of $\lambda, m,$ & μ. It is easy to find that all the above bounds for $P_n(z)$ from Theorem 6 to Theorem 9 give sharper bounds for a given $n \in N$, the set of natural numbers.

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