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A new class of meromorphically multivalent functions with applications to generalized hypergeometric functions

R.K. Raina[a], H.M. Srivastava[b,*]

[a] *Department of Mathematics, College of Technology and Engineering, Maharana Pratap University of Agriculture and Technology, Udaipur 313001, Rajasthan, India*

[b] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

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Abstract

In the present paper we introduce a new class of meromorphically multivalent functions, which is defined by means of a Hadamard product (or convolution) involving some suitably normalized meromorphically p-valent functions. A characterization property giving the coefficient bounds is obtained for this class of functions. The other related properties, which are investigated in this paper, include distortion and the radii of starlikeness and convexity. We also consider several applications of our main results to generalized hypergeometric functions, whose special cases yield some known results given recently by J.-L. Liu, and H. M. Srivastava [Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, Math. Comput. Modelling 39 (2004) 21–34].

Keywords: Meromorphic functions; p-Valent functions; Hadamard product (or convolution); Starlike functions; Convex functions; Generalized hypergeometric function; Linear operator; Dziok–Srivastava operator; Fox–Wright generalized hypergeometric function; Coefficient bounds; Distortion properties; Radii of starlikeness and convexity

1. Introduction and definitions

Let Σ_p denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = z^{-p} + \sum_{n=p}^{\infty} a_n z^n \qquad (a_n \geqq 0;\ p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disk $\mathbb{U}^* := \mathbb{U}^*(1)$, where

$$\mathbb{U}^*(r) = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < r \ (0 < r \leqq 1)\} = \mathbb{U}(r) \setminus \{0\} \quad (\mathbb{U}(1) \equiv \mathbb{U}).$$

* Corresponding author. Tel.: +1 250 721 7455; fax: +1 250 721 8962.
E-mail addresses: rainark_7@hotmail.com (R.K. Raina), harimsri@math.uvic.ca (H.M. Srivastava).

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A function $f \in \Sigma_p$ is said to be meromorphically p-valent starlike of order ρ $(0 \leqq \rho < p)$ in $\mathbb{U}(r)$ if

$$\Re\left(\frac{z f'(z)}{f(z)}\right) < -\rho \quad (z \in \mathbb{U}(r); 0 < r \leqq 1; 0 \leqq \rho < p). \tag{2}$$

On the other hand, a function $f \in \Sigma_p$ is said to be meromorphically p-valent convex of order ρ $(0 \leqq \rho < p)$ in $\mathbb{U}(r)$ if

$$\Re\left(\frac{z f''(z)}{f'(z)}\right) < -\rho - 1 \quad (z \in \mathbb{U}(r); 0 < r \leqq 1; 0 \leqq \rho < p). \tag{3}$$

The Hadamard product (or convolution) of the function f defined by (1) with the functions g and h given, respectively, by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p} \quad (b_n \geqq 0;\ p \in \mathbb{N}) \tag{4}$$

and

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} c_n z^{n-p} \quad (c_n \geqq 0;\ p \in \mathbb{N}) \tag{5}$$

can be expressed as follows:

$$(f * g)(z) = z^{-p} + \sum_{n=p}^{\infty} a_n b_{n+p} z^n = (g * f)(z) \tag{6}$$

and

$$(f * h)(z) = z^{-p} + \sum_{n=p}^{\infty} a_n c_{n+p} z^n = (h * f)(z), \tag{7}$$

where we have assumed that

$$b_j = c_j = 0 \quad (j = 1, \ldots, 2p - 1;\ p \in \mathbb{N}).$$

Suppose that $f(z)$ and $g(z)$ are two analytic functions in the unit disk $\mathbb{U}$. Then we say that the function $g(z)$ is subordinate to the function $f(z)$ and we write

$$g(z) \prec f(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$g(z) = f\big(w(z)\big) \quad (z \in \mathbb{U}).$$

By applying the above subordination definition, we introduce here a new class $\mathfrak{S}_m(g, h; A, B, \gamma)$ of meromorphically multivalent functions, which is defined as follows.

Definition 1. A function $f \in \Sigma_p$ is said to be in the class $\mathfrak{S}_m(g, h; A, B, \gamma)$ if it satisfies the following subordination property:

$$\gamma \frac{\big((f * g)(z)\big)^{(m)}}{\big((f * h)(z)\big)^{(m)}} \prec \gamma - \frac{p(A - B)z}{1 + Bz} \quad (z \in \mathbb{U}^*) \tag{8}$$

$$(\gamma > 0; 0 \leqq B < A \leqq 1; b_n \geqq c_n \geqq 0\ (n \geqq p); p \geqq m; m \in \mathbb{N}_0^* := \{2j - 1 : j \in \mathbb{N}\} \cup \{0\}), \tag{9}$$

provided that

$$\left((f*h)(z)\right)^{(m)} \neq 0 \quad (z \in \mathbb{U}^*).$$

The purpose of this paper is to investigate the coefficient estimates, distortion properties and the radii of convexity and starlikeness for the above-defined class of meromorphically p-valent functions. Several applications of the main results involving generalized hypergeometric functions are considered. The relevance to some known results is also pointed out. In the concluding section (Section 4), we briefly indicate the possibilities of deducing additional results from our main theorems.

2. Characterization and other related properties

In this section we begin by proving a characterization property which provides a necessary and sufficient condition for a function $f \in \Sigma_p$ of the form (1) to belong to the class $\mathfrak{S}_m(g, h; A, B, \gamma)$ of meromorphically p-valent functions with positive coefficients.

Theorem 1. *A function $f \in \Sigma_p$ of the form (1) belongs to the class $\mathfrak{S}_m(g, h; A, B, \gamma)$ if and only if*

$$\sum_{n=p}^{\infty} a_n[\gamma(1+B)b_{n+p} + \{p(A-B) - \gamma(1+B)\}c_{n+p}]\binom{n}{m} \leqq p(A-B)\binom{p+m-1}{m}. \tag{10}$$

The sharpness in (10) is achieved for the functions $f_n(z)$ given by

$$f_n(z) = z^{-p} + \frac{p(A-B)\binom{p+m-1}{m}}{[\gamma(1+B)b_{n+p} + \{p(A-B) - \gamma(1+B)\}c_{n+p}]\binom{n}{m}} \quad (n = p, p+1, p+2, \ldots; p \in \mathbb{N}). \tag{11}$$

Proof. Let f of the form (1) belong to the class $\mathfrak{S}_m(g, h; A, B, \gamma)$. Then, in view of (6) to (8), we find for $m \in \mathbb{N}_0^*$ that

$$\left|\frac{\gamma \sum_{n=p}^{\infty} a_n(b_{n+p} - c_{n+p})\binom{n}{m} z^{n+p}}{p(A-B)\binom{p+m-1}{m} - \sum_{n=p}^{\infty} a_n[\gamma B b_{n+p} + \{p(A-B) - \gamma B\}c_{n+p}]\binom{n}{m} z^{n+p}}\right| < 1 \quad (z \in \mathbb{U}). \tag{12}$$

Putting $z = r$ $(0 \leqq r < 1)$, and noting the fact that the denominator in the inequality (12) remains positive by virtue of the constraints stated in (9) for all $r \in [0, 1)$, we easily arrive at the desired inequality (10) by letting $z \to 1-$ in (12).

Conversely, if we assume that the inequality (10) holds true in the simplified form (12), it can readily be shown that

$$\left|\frac{\gamma\left\{\left((f*g)(z)\right)^{(m)} - \left((f*h)(z)\right)^{(m)}\right\}}{\gamma B\left((f*g)(z)\right)^{(m)} + \{p(A-B)\gamma B - \gamma B\}\left((f*h)(z)\right)^{(m)}}\right| < 1 \quad (z \in \mathbb{U}),$$

which implies that $f \in \mathfrak{S}_m(g, h; A, B, \gamma)$.

Finally, it is observed that the result is sharp and that the extremal functions $f_n(z)$ are given by (11). □

Theorem 1 immediately yields the following result.

Corollary 1. *If a function $f \in \Sigma_p$ of the form (1) belongs to the class $\mathfrak{S}_m(g, h; A, B, \gamma)$, then*

$$a_n \leqq \frac{p(A-B)\binom{p+m-1}{m}}{[\gamma(1+B)b_{n+p} + \{p(A-B) - \gamma(1+B)\}c_{n+p}]\binom{n}{m}} \quad (n = p, p+1, p+2, \ldots; p \in \mathbb{N}), \tag{13}$$

where the equality holds true for the functions $f_n(z)$ given by (11).

We now state the following growth and distortion properties for the class $\mathfrak{S}_m(g, h; A, B, \gamma)$, which can be proved analogously to similar results proved by Liu and Srivastava [1, p. 26, Theorem 4] (see also [2,3] and [4]).

Theorem 2. *Let a function $f \in \Sigma_p$ of the form (1) belong to the class $\mathfrak{S}_m(g, h; A, B, \gamma)$. If the sequence $\{\eta_n\}$ is nondecreasing, then*

$$r^{-p} - \frac{p(A-B)\binom{p+m-1}{m}}{\eta_n} r^n \leqq |f(z)| \leqq r^{-p} + \frac{p(A-B)\binom{p+m-1}{m}}{\eta_n} r^n \quad (0 < |z| = r < 1), \tag{14}$$

where

$$\eta_n = [\gamma(1+B)b_{n+p} + \{p(A-B) - \gamma(1+B)\}c_{n+p}]\binom{n}{m} \quad (n = p, p+1, p+2, \ldots; p \in \mathbb{N}). \tag{15}$$

If the sequence $\{\frac{\eta_n}{n}\}$ is nondecreasing, then

$$pr^{-p} - \frac{p^2(A-B)\binom{p+m-1}{m}}{\eta_n} r^{n-1} \leqq |f'(z)| \leqq pr^{-p} + \frac{p^2(A-B)\binom{p+m-1}{m}}{\eta_n} r^{n-1} \quad (0 < |z| = r < 1). \tag{16}$$

The results (14) and (16) are sharp with the extremal functions $f_n(z)$ given by (11).

We next determine the radii of meromorphically p-valent starlikeness and meromorphically p-valent convexity of the class $\mathfrak{S}_m(g, h; A, B, \gamma)$, which are given by Theorem 3 below.

Theorem 3. *Let a function $f \in \Sigma_p$ of the form (1) belong to the class $\mathfrak{S}_m(g, h; A, B, \gamma)$. Then*
(i) *f is meromorphically p-valent starlike of order ρ $(0 \leqq \rho < p)$ in the disk $|z| < r_1$, where*

$$r_1 = \inf_{n \geqq p} \left(\frac{(p-\rho)\eta_n}{p(A-B)(n+\rho)\binom{p+m-1}{m}} \right)^{1/(n+p)}; \tag{17}$$

(ii) *f is meromorphically p-valent convex of order ρ $(0 \leqq \rho < p)$ in the disk $|z| < r_2$, where*

$$r_2 = \inf_{n \geqq p} \left(\frac{(p-\rho)\eta_n}{n(n+\rho)(A-B)\binom{p+m-1}{m}} \right)^{1/(n+p)}. \tag{18}$$

The sequence $\{\eta_n\}$ occurring in (17) and (18) is given by (15).

Proof. To prove (i), we observe from the inequality (2) that the function f of the form (1) is meromorphically p-valent starlike of order ρ $(0 \leqq \rho < p)$ in the disk $\mathbb{U}(r)$ if

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} - p + 2\rho} \right| \leqq 1 \quad (z \in \mathbb{U}(r); 0 < r \leqq 1; 0 \leqq \rho < p; p \in \mathbb{N}). \tag{19}$$

For $|z| = r$, the inequality in (19) is true if

$$\sum_{n=p}^{\infty} \left(\frac{n+\rho}{p-\rho} \right) a_n r^{n+p} \leqq 1. \tag{20}$$

Comparing (20) with the coefficient inequality (10), we conclude that the function f is meromorphically p-valent starlike of order ρ $(0 \leqq \rho < p)$ in the disk $|z| < r_1$ with r_1 given precisely by (17).

The proof of (ii) is similar to that of (i) detailed above; it is, therefore, being omitted here. □

3. Applications involving generalized hypergeometric functions

In order to consider some applications of Theorems 1–3 to the generalized hypergeometric functions, we first set the sequences $\{b_n\}$ and $\{c_n\}$, which are involved in (4) and (5), as follows:

$$b_n = \left(\frac{\alpha_1 + n}{\alpha_1}\right) c_n = \frac{(\alpha_1 + 1)_n (\alpha_2)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n n!}, \tag{21}$$

where

$$c_n = \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n n!}. \tag{22}$$

Here we also assume that

$$\alpha_j > 0 \quad (j = 1, \ldots, q) \quad \text{and} \quad \beta_j > 0 \quad (j = 1, \ldots, s).$$

The corresponding functions $g(z)$ and $h(z)$ then become

$$g(z) = z^{-p}\ {}_qF_s(\alpha_1 + 1, \alpha_2, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) \tag{23}$$

and

$$h(z) = z^{-p}\ {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z), \tag{24}$$

where

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s;\ z)$$

is the familiar generalized hypergeometric function defined by (see, for example, [5, p. 19])

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s;\ z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!} \quad (q \leqq s + 1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U}). \tag{25}$$

Making use of (24), the convolution defined by (7) can be used to represent the Dziok–Srivastava linear operator

$$\mathcal{H}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) : \Sigma_p \to \Sigma_p$$

by means of the following relation (cf. [4, p. 3]):

$$\begin{aligned}(\mathcal{H}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) f)(z) &= z^{-p}\ {}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s;\ z) * f(z) \\ &= z^{-p} + \sum_{n=p}^{\infty} a_n\, c_{n+p}\, z^n = (f * h)(z),\end{aligned} \tag{26}$$

where c_n is given by (22). Also, in view of (21), the Hadamard product (or convolution) defined by (6) represents a relation similar to (26), involving the Dziok–Srivastava operator

$$\mathcal{H}_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s),$$

which is given by

$$\begin{aligned}(\mathcal{H}_p(\alpha_1 + 1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s) f)(z) &= z^{-p}\ {}_qF_s(\alpha_1 + 1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s;\ z) * f(z) \\ &= z^{-p} + \sum_{n=p}^{\infty} a_n \left(\frac{\alpha_1 + n + p}{\alpha_1}\right) c_{n+p}\, z^n = (f * g)(z),\end{aligned} \tag{27}$$

where c_n is given by (22).

The subordination relation (8) in conjunction with (26) and (27) takes the following form:

$$\gamma \frac{\left(\left(\mathcal{H}_p(\alpha_1+1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f\right)(z)\right)^{(m)}}{\left((\mathcal{H}_p(\alpha_1,\ldots,\alpha_q;\ \beta_1,\ldots,\beta_s)f)(z)\right)^{(m)}} \prec \gamma - \frac{p(A-B)z}{1+Bz}$$
$$(\gamma > 0;\, 0 \leqq B < A \leqq 1;\, p \geqq m;\ p \in \mathbb{N};\, m \in \mathbb{N}_0^*). \tag{28}$$

Definition 2. A function $f \in \Sigma_p$ of the form (1) is said to be in the class $\mathfrak{S}_{p,q,s,m}(\alpha_1; A, B, \gamma)$ if it satisfies the subordination relation (28) above.

The following consequences of Theorem 1 to 3 can be deduced by applying (21) and (22) along with Definition 2.

Corollary 2. *A function $f \in \Sigma_p$ of the form (1) belongs to the class $\mathfrak{S}_{p,q,s,m}(\alpha_1; A, B, \gamma)$ if and only if*

$$\sum_{n=p}^{\infty} a_n \left[\frac{\gamma(1+B)(n+p)}{\alpha_1} + p(A-B)\right]\binom{n}{m} c_{n+p} \leqq p(A-B)\binom{p+m-1}{m}, \tag{29}$$

where c_n is given by (22).

Corollary 3. *Let a function $f \in \Sigma_p$ of the form (1) belong to the class $\mathfrak{S}_{p,q,s,m}(\alpha_1; A, B, \gamma)$. If the sequence $\{\eta_n^*\}$ is nondecreasing, then*

$$r^{-p} - \frac{p(A-B)\binom{p+m-1}{m}}{\eta_n^*} r^n \leqq |f(z)| \leqq r^{-p} + \frac{p(A-B)\binom{p+m-1}{m}}{\eta_n^*} r^n \quad (0 < |z| = r < 1), \tag{30}$$

where

$$\eta_n^* = \left[\frac{\gamma(1+B)(n+p)}{\alpha_1} + p(A-B)\right]\binom{n}{m} c_{n+p} \quad (n = p, p+1, p+2, \ldots;\, p \in \mathbb{N}). \tag{31}$$

If the sequence $\{\frac{\eta_n^*}{n}\}$ is nondecreasing, then

$$pr^{-p} - \frac{p^2(A-B)\binom{p+m-1}{m}}{\eta_n^*} r^{n-1} \leqq |f'(z)| \leqq pr^{-p} + \frac{p^2(A-B)\binom{p+m-1}{m}}{\eta_n^*} r^{n-1} \quad (0 < |z| = r < 1). \tag{32}$$

Corollary 4. *Let a function $f \in \Sigma_p$ of the form (1) belong to the class $\mathfrak{S}_{p,q,s,m}(\alpha_1; A, B, \gamma)$. Then*
(i) *f is meromorphically p-valent starlike of order ρ $(0 \leqq \rho < p)$ in the disk $|z| < r_1^*$, where*

$$r_1^* = \inf_{n \geqq p} \left(\frac{(p-\rho)\eta_n^*}{p(A-B)(n+\rho)\binom{p+m-1}{m}}\right)^{1/(n+p)}; \tag{33}$$

(ii) *f is meromorphically p-valent convex of order ρ $(0 \leqq \rho < p)$ in the disk $|z| < r_2^*$, where*

$$r_2^* = \inf_{n \geqq p} \left(\frac{(p-\rho)\eta_n^*}{n(n+\rho)(A-B)\binom{p+m-1}{m}}\right)^{1/(n+p)}. \tag{34}$$

The sequence $\{\eta_n^*\}$ occurring in (33) and (34) is given by (31).

Remark. For $\gamma = \alpha_1$ and $m = 1$ in (28), the class $\mathfrak{S}_{p,q,s,m}(\alpha_1; A, B, \gamma)$ reduces to the known class $\Omega_{p,q,s}^{+}(\alpha_1; A, B)$, which was studied recently by Liu and Srivastava [1]. In these special cases, Corollaries 2–4 are seen to correspond, respectively, to the results of Liu and Srivastava [4, pp. 25–27, Theorems 3 to 5].

4. Concluding remarks and observations

We conclude this paper by observing that, in view of the subordination relation (8) which is expressed in terms of the convolution of the functions in (4) and (5) with the function defined by (1), involving arbitrary sequences, our main results can lead to several additional new results. In fact, by appropriately selecting these arbitrary sequences, the results presented in this paper would find further applications for the class of meromorphic functions which would incorporate a generalized form of the Dziok–Srivastava linear operator involving the Hadamard product (or convolution) of the function in (1) with the Fox–Wright generalization ${}_q\Psi_s$ of the hypergeometric function ${}_qF_s$ (see [5, p. 21]). Theorem 1 to 3 would thus eventually lead us further to new results for the class of functions (defined analogously to the class $\mathfrak{S}_m(\alpha_1; A, B, \gamma)$ by associating instead the Fox–Wright generalized hypergeometric function ${}_q\Psi_s$. These considerations can fruitfully be worked out by closely following the recent investigations by Dziok and Raina [3] and Dziok et al. [6]. We choose to skip further details in this regard.

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