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ENTROPY FUNCTIONAL FOR CONTINUOUS SYSTEMS OF FINITE ENTROPY*

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Abstract In this article, we introduce the concept of entropy functional for continuous systems on compact metric spaces, and prove some of its properties. We also extract the Kolmogorov entropy from the entropy functional.

Key words Entropy; entropy functional; invariant
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1 Introduction

The importance of the concept of entropy in mathematics and physics is referred to the importance of the variational principle; since indeed, entropy is a quantity that determines the direction of a thermodynamical process. We may have different versions of the definition of entropy, depending on the conditions of the discussed problem. But its intrinsic concept, given by the effect of a suitable distribution on the configuration space, is preserved in these different versions. Because of its importance in mathematics, physics, information theory, and even social sciences, entropy was studied extensively by many people. In information theory, entropy is a measure of the uncertainty associated with a random variable. The term by itself, in information theory, usually refers to the Shannon entropy, which quantifies, in the sense of an expected value, the information contained in a message, usually in units such as bits. Equivalently, the Shannon entropy is a measure of the average information content one is missing when one does not know the value of the random variable. The concept was introduced by Claude E. Shannon [17] in 1948. Kolmogorov [10] introduced the concept of entropy into ergodic theory. The definition of entropy was improved by Sinai in [18]. It measures the rate of increase in dynamical complexity as the system evolves with time. Adler, Konheim, and McAndrew [1] introduced the topological entropy as an invariant of topological conjugacy and also as an analogue of measure theoretic entropy. Later, Dinaburg [7] and Bowen [3] gave a new, but equivalent, definition of topological entropy that led to the variational principle connected the topological entropy and measure theoretic entropy. Shannon [17], McMillan [13], Breiman [4], and Brin and Katok [5] gave local approaches to entropy. Ruelle [16] and Pesin

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[14] extracted entropy from the Lyapunov exponents in smooth case. R. Mañé [12] related the exponential growth rate of the number of geodesics joining two points, with the topological entropy of the geodesic flows. Later on, different versions of the entropy of a dynamical system were defined. This article is an attempt to present an approach to the entropy of a continuous system $T : X \to X$ on a compact metric space X as a special value of a linear functional on $C(X)$. The idea is based on the feeling that any 'intelligent' point in the space is informed about the space, as it increases its experience by meeting different areas of the space under the dynamic of T. As disjoint points may have different 'intelligence', we may assign a weight factor to the local loss of information caused by the lack of experience of any point. To do this, given any invariant measure μ on the Borel sets of X, we introduce the linear entropy functional $\mathcal{L}_T(\cdot;\mu) : C(X) \to \mathbb{R}$, which is continuous when $h_\mu(T) < \infty$. The entropy functional is expected to have the fundamental properties of the entropy and also coincides with the entropy when there is no weight factor in the middle. As the main results are given for any invariant, but not necessarily ergodic, measures, the modeling could be considered even for non-physical systems.

In this article, $M(X)$ denotes the set of all probability measures on Borel sets of X. The set of all probability measures on X preserving T is denoted by $M(X,T)$. We also write $E(X,T)$ for the set of all ergodic measures of T. Finally, for $\mu \in M(X,T)$, $h_\mu(T)$ denotes the Kolmogorov entropy of T.

2 Entropy Functional

In this section, we introduce the concept of entropy functional for a continuous map $T : X \to X$ on a compact metric space X with finite entropy, and state some of its properties. We will finally extract the entropy of a system as a special case.

Definition 2.1 Suppose that $T : X \to X$ is a continuous map on the topological space X, $x \in X$ and A a Borel subset of X. Define

$$\omega_T(x,A) := \limsup_{n\to\infty} \frac{1}{n}\mathrm{card}(\{k \in \{0,1,\cdots,n-1\} : \ T^k(x) \in A\}).$$

Now, let $x \in X$ and $\xi = \{A_1, A_2, \cdots, A_n\}$ be a finite Borel partition of X. Define

$$\Omega_T(x,\xi) := -\sum_{j=1}^{n} \omega_T(x,A_j)\log\omega_T(x,A_j).$$

(We assume that $\log 0 = -\infty$ and $0 \times \infty = 0$.)

Finally, let $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\mathrm{diam}(\xi_n) \to 0$ as $n \to \infty$. The map $h^*_T(.;\mathfrak{U}) : X \to [0,\infty]$ is defined as

$$h^*_T(x;\mathfrak{U}) = \lim_{n\to\infty}\limsup_{l\to\infty}\frac{1}{l}\Omega_T(x,\bigvee_{i=0}^{l-1}T^{-i}\xi_n).$$

Note that, without loss of generality, we may assume that $\xi_n < \xi_{n+1}$, as otherwise we may replace ξ_n by $\eta_n = \bigvee_{k=1}^{n}\xi_k$. Hence, the sequence

$$a_n(x) = \limsup_{l\to\infty}\frac{1}{l}\Omega_T(x,\bigvee_{i=0}^{l-1}T^{-i}\xi_n)$$

is increasing in n and so $\lim_{n\to\infty} a_n(x)$ exists as a nonnegative extended real number. We also write h_T^* for $h_T^*(.;\mathfrak{U})$ when there is no confusion.

Definition 2.2 Suppose that $T : X \to X$ is a continuous map on the compact metric space X, and $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X such that $\text{diam}(\xi_n) \to 0$. Let $\mu \in M(X,T)$ be such that $h_\mu(T) < \infty$. The entropy functional of T (with respect to μ), $\mathcal{L}_T(.;\mu;\mathfrak{U}) : C(X) \to \mathbb{R}$, is defined as

$$\mathcal{L}_T(f;\mu;\mathfrak{U}) := \int_X f(x)h_T^*(x;\mathfrak{U})\text{d}\mu(x)$$

for all $f \in C(X)$ (again $0 \times \infty := 0$). In other words, the entropy functional of T is an integral functional with the kernel h_T^*. We will show that the entropy functional is bounded when T has finite entropy.

In the following, we will prove the independence of entropy functional from the selection of the sequence of finite Borel partitions $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ in the previous definition. Before stating the theorem, we recall some classical results that we need in the sequel.

Theorem 2.3 ([19, Theorem 8.3]) Let $T : X \to X$ be a continuous map on the compact space X. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\text{diam}(\xi_n) \to 0$ as $n \to \infty$. For every $\mu \in M(X,T)$, we have $h_\mu(T) = \lim_{n\to\infty} h_\mu(T,\xi_n)$.

Theorem 2.4 (Choquet) Suppose that Y is a compact convex metrisable subset of a locally convex space E, and that $x_0 \in Y$. Then, there exists a probability measure τ on Y which represents x_0 and is supported by the extreme points of Y, that is, $\Phi(x_0) = \int_Y \Phi \text{d}\tau$ for every continuous linear functional Φ on E, and $\tau(\text{ext}(Y)) = 1$.

See Phelps [15] for a proof of Choquet's Theorem.

Let $\mu \in M(X,T)$ and $f : X \to \mathbb{R}$ be a bounded measurable function. As we know that $E(X,T)$ equals the extreme points of $M(X,T)$, applying the Choquet's Theorem for $E = \mathcal{M}(X)$, the space of finite regular Borel measures on X, and $Y = M(X,T)$, and using the linear functional $\Phi : \mathcal{M}(X) \to \mathbb{R}$ given by $\Phi(\mu) = \int_X f\text{d}\mu$, we have the following corollary.

Corollary 2.5 Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Then, for each $\mu \in M(X,T)$, there is a unique measure τ on the Borel subsets of the compact metrisable space $M(X,T)$, such that $\tau(E(X,T)) = 1$ and

$$\int_X f(x)\text{d}\mu(x) = \int_{E(X,T)} \left(\int_X f(x)\text{d}m(x)\right) \text{d}\tau(m)$$

for every bounded measurable function $f : X \to \mathbb{R}$.

Under the assumptions of Corollary 2.5, we write $\mu = \int_{E(X,T)} m\text{d}\tau(m)$, called the ergodic decomposition of μ.

Theorem 2.6 (Jacobs) Let $T : X \to X$ be a continuous map on a compact metrisable space. If $\mu \in M(X,T)$ and $\mu = \int_{E(X,T)} m\text{d}\tau(m)$ is the ergodic decomposition of μ, then we have:

(i) If ξ is a finite Borel partition of X, then, $h_\mu(T,\xi) = \int_{E(X,T)} h_m(T,\xi)\text{d}\tau(m)$.

(ii) $h_\mu(T) = \int_{E(X,T)} h_m(T)\text{d}\tau(m)$ (both sides could be ∞).

See [19, Theorem 8.4] for a proof of this Theorem.

Now, we are in a position to prove the independence of the entropy functional from the selection of the sequence of finite Borel partitions in Definition 2.2.

Theorem 2.7 Suppose that $T : X \to X$ is a continuous map on the compact metric space X. Let $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ and $\mathcal{V} = \{\eta_n\}_{n\in\mathbb{N}}$ be two sequences of finite Borel partitions of X, such that both $\mathrm{diam}(\xi_n)$ and $\mathrm{diam}(\eta_n) \to 0$. Then, for $\mu \in M(X,T)$,

$$\mathcal{L}_T(f;\mu;\mathfrak{U}) = \mathcal{L}_T(f;\mu;\mathcal{V})$$

for all $f \in C(X)$.

Proof First, let $m \in E(X,T)$. For any Borel set $A \subset X$ and $x \in X$, applying Birkhoff ergodic Theorem, we have

$$\omega_T(x,A) = m(A)$$

for almost all $x \in X$. Hence, if ξ is a finite Borel partition of X, then,

$$\Omega_T(x,\xi) = H_m(\xi)$$

for almost all $x \in X$. Thus, for each $n \in \mathbb{N}$, one can find Borel sets Y_n and Z_n with $m(Y_n) = m(Z_n) = 1$, such that

$$\limsup_{l\to\infty} \frac{1}{l}\Omega_T\left(x, \bigvee_{i=0}^{l-1} T^{-i}\xi_n\right) = h_m(T,\xi_n) \tag{2.1}$$

for all $x \in Y_n$, and

$$\limsup_{l\to\infty} \frac{1}{l}\Omega_T\left(x, \bigvee_{i=0}^{l-1} T^{-i}\eta_n\right) = h_m(T,\eta_n) \tag{2.2}$$

for all $x \in Z_n$.

Put $X_1 := \bigcap_{n=1}^{\infty}(Y_n \cap Z_n)$, then $m(X_1) = 1$, and for any $n \in \mathbb{N}$, the equalities in (2.1) and (2.2) hold for all $x \in X_1$. Therefore, if $x \in X_1$, then applying (2.1), (2.2), and Theorem 2.3, we have

$$\begin{aligned} h_T^*(x;\mathfrak{U}) &= \lim_{n\to\infty} \limsup_{l\to\infty} \frac{1}{l}\Omega_T(x, \bigvee_{i=0}^{l-1} T^{-i}\xi_n) \\ &= \lim_{n\to\infty} h_m(T,\xi_n) = h_m(T) = \lim_{n\to\infty} h_m(T,\eta_n) \\ &= \lim_{n\to\infty} \limsup_{l\to\infty} \frac{1}{l}\Omega_T(x, \bigvee_{i=0}^{l-1} T^{-i}\eta_n) = h_T^*(x,\mathcal{V}). \end{aligned}$$

So, if $f \in C(X)$, then,

$$f(x)h_T^*(x;\mathfrak{U}) = f(x)h_T^*(x;\mathcal{V})$$

for all $x \in X_1$. Therefore,

$$\begin{aligned} \int_X f(x)h_T^*(x;\mathfrak{U})dm(x) &= \int_{X_1} f(x)h_T^*(x;\mathfrak{U})\mathrm{d}m(x) \\ &= \int_{X_1} f(x)h_T^*(x;\mathcal{V})\mathrm{d}m(x) \\ &= \int_X f(x)h_T^*(x;\mathcal{V})\mathrm{d}m(x). \end{aligned} \tag{2.3}$$

Now, let $\mu \in M(X,T)$, and let $\mu = \int_X m\mathrm{d}\tau(m)$ be the ergodic decomposition of μ. For $n \in \mathbb{N}$, put $h_n^*(.;\mathfrak{U}) := \min\{h_T^*(.;\mathfrak{U}), n\}$ and $h_n^*(.;\mathcal{V}) := \min\{h_T^*(.;\mathcal{V}), n\}$. Then, $\{h_n^*(.;\mathfrak{U})\}_{n\in\mathbb{N}}$ and

$\{h_n^*(.;\mathcal{V})\}_{n\in\mathbb{N}}$ are increasing sequences of bounded measurable maps on X, such that $h_n^*(.;\mathfrak{U}) \uparrow h_T^*(.;\mathfrak{U})$ and $h_n^*(.;\mathcal{V}) \uparrow h_T^*(.;\mathcal{V})$.

First, let $f \in C^+(X)$. Then, $fh_n^*(.;\mathfrak{U}) \uparrow fh_T^*(.;\mathfrak{U})$ and $fh_n^*(.;\mathcal{V}) \uparrow fh_T^*(.;\mathcal{V})$. Applying (2.3), Corollary 2.5, and Monotone Convergence Theorem, we have

$$\begin{aligned}
\mathcal{L}_T(f;\mu;\mathfrak{U}) &= \int_X f(x)h_T^*(x;\mathfrak{U})\mathrm{d}\mu(x) \\
&= \lim_{n\to\infty}\int_X f(x)h_n^*(x;\mathfrak{U})\mathrm{d}\mu(x) \\
&= \lim_{n\to\infty}\int_{E(X,T)}\left(\int_X f(x)h_n^*(x;\mathfrak{U})\mathrm{d}m(x)\right)\mathrm{d}\tau(m) \\
&= \int_{E(X,T)}\left(\int_X f(x)h_T^*(x;\mathfrak{U})\mathrm{d}m(x)\right)\mathrm{d}\tau(m) \\
&= \int_{E(X,T)}\left(\int_X f(x)h_T^*(x;\mathcal{V})\mathrm{d}m(x)\right)\mathrm{d}\tau(m) \\
&= \lim_{n\to\infty}\int_{E(X,T)}\left(\int_X f(x)h_n^*(x;\mathcal{V})\mathrm{d}m(x)\right)\mathrm{d}\tau(m) \\
&= \lim_{n\to\infty}\int_X f(x)h_n^*(x;\mathcal{V})\mathrm{d}\mu(x) \\
&= \int_X f(x)h_T^*(x;\mathcal{V})\mathrm{d}\mu(x) \\
&= \mathcal{L}_T(f;\mu;\mathcal{V}).
\end{aligned}$$

Finally, for $f \in C(X)$, let $f^+ = \max\{f,0\}$ and $f^- = \max\{-f,0\}$. Then, $f^+, f^- \in C^+(X)$ and $f = f^+ - f^-$. The result follows from the linearity of $f \to \mathcal{L}_T(f;\mu;\mathfrak{U})$.

Remark By Theorem 2.7, we conclude that the definition of entropy functional is independent of the selection of the sequence of finite Borel partitions. Therefore, given any invariant measure μ and any sequence of finite Borel partitions $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ with $\mathrm{diam}(\xi_n) \to 0$, we have the (unique) entropy functional $\mathcal{L}_T(.;\mu;\mathfrak{U})$. So, we can write $\mathcal{L}_T(.;\mu)$ for $\mathcal{L}_T(.;\mu;\mathfrak{U})$ without confusion.

Theorem 2.8 Suppose that $T : X \to X$ is a continuous map on the compact metric space X such that $h_{\mathrm{top}}(T) < \infty$. Then,

(i) Given any $\mu \in M(X,T)$, the entropy functional $f \to \mathcal{L}_T(f;\mu)$ is linear.

(ii) Given any $f \in C(X)$, the map $\mu \to \mathcal{L}_T(f;\mu)$ is affine.

(iii) If $\mu \in M(X,T)$ and $\mu = \int_{E(X,T)} m\mathrm{d}\tau(m)$ is the ergodic decomposition of μ, then,

$$\mathcal{L}_T(f;\mu) = \int_{E(X,T)} \mathcal{L}_T(f;m)\mathrm{d}\tau(m)$$

for all $f \in C(X)$.

(iv) If $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ are topologically conjugate continuous maps via the homeomorphism $\phi : X_1 \to X_2$, and $\mu \in M(X_1,T_1)$, then,

$$\mathcal{L}_{T_1}(f;\mu) = \mathcal{L}_{T_2}(f\phi^{-1};\mu\phi^{-1})$$

for all $f \in C(X_1)$.

Proof Note that the condition $h_{\text{top}}(T) < \infty$ guarantees that $\mathcal{L}_T(f;\mu)$ is well-defined for all $\mu \in M(X,T)$.

(i) and (ii) are trivial.

(iii) Let $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\text{diam}(\xi_n) \to 0$. First, let $f \in C^+(X)$. For $n \geq 1$, let $h_n^* := \min\{h_T^*(\cdot;\mathfrak{U}), n\}$. Then, $\{fh_n^*\}_{n\in\mathbb{N}}$ is an increasing sequence of bounded measurable functions and $fh_n^* \uparrow fh_T^*$ on X. Applying Monotone Convergence Theorem and Corollary 2.5, we have

$$\begin{aligned}
\mathcal{L}_T(f;\mu) &= \int_X f(x)h_T^*(x;\mathfrak{U})\mathrm{d}\mu(x)\\
&= \lim_{n\to\infty}\int_X f(x)h_n^*(x)\mathrm{d}\mu(x)\\
&= \lim_{n\to\infty}\int_{E(X,T)}\left(\int_X f(x)h_n^*(x)dm(x)\right)\mathrm{d}\tau(m)\\
&= \int_{E(X,T)}\left(\int_X f(x)h_T^*(x;\mathfrak{U})dm(x)\right)\mathrm{d}\tau(m)\\
&= \int_{E(X,T)}\mathcal{L}_T(f;m)\mathrm{d}\tau(m).
\end{aligned}$$

For $f \in C(X)$, write $f = f^+ - f^-$, where $f^+, f^- \in C^+(X)$.

(iv) For $x \in X$ and the Borel set $A \subseteq X_1$, we have $\omega_{T_1}(x,A) = \omega_{T_2}(\phi(x),\phi(A))$. Therefore, $\Omega_{T_1}(x,\xi) = \Omega_{T_2}(\phi(x),\phi(\xi))$ for any finite Borel partition ξ. Now, for any sequence $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ of finite Borel partitions of X with $\text{diam}(\xi_n) \to 0$, by the definition of $h_T^*(\cdot;\mathfrak{U})$, we have $h_{T_1}^*(\cdot;\mathfrak{U}) = h_{T_2}^*(\cdot;\phi(\mathfrak{U}))o\phi$. Note that $\phi(\mathfrak{U}) = \{\phi(\xi_n)\}_{n\in\mathbb{N}}$ and $\text{diam}(\phi(\xi_n)) \to 0$. Let $\mu \in M(X_1,T_1)$, and $f \in C(X_1)$. Then,

$$\begin{aligned}
\mathcal{L}_{T_1}(f;\mu) &= \int_{X_1} f(x)h_{T_1}^*(x;\mathfrak{U})\mathrm{d}\mu(x)\\
&= \int_{X_1} f(x)h_{T_2}^*(\phi(x);\phi(\mathfrak{U}))\mathrm{d}\mu(x)\\
&= \int_{X_2} f(\phi^{-1}(x))h_{T_2}^*(x;\phi(\mathfrak{U}))d(\mu\phi^{-1})(x)\\
&= \mathcal{L}_{T_2}(f\phi^{-1};\mu\phi^{-1}).
\end{aligned}$$

In the following theorem, we extract Kolmogorov entropy from the entropy functional as a special case.

Theorem 2.9 Suppose that $T : X \to X$ is a continuous map on the compact metric space X, and let $\mu \in M(X,T)$. Then,

(i) $\mathcal{L}_T(1;\mu) = h_\mu(T)$.

(ii) The entropy functional $f \to \mathcal{L}_T(f;\mu)$ is a continuous linear functional on $C(X)$, and $\|\mathcal{L}_T(\cdot;\mu)\| = h_\mu(T)$.

Proof (i) Let $\mathfrak{U} = \{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\text{diam}(\xi_n) \to 0$. First, let $m \in E(X,T)$. As in the proof of Theorem 2.7, we have

$$h_T^*(x;\mathfrak{U}) = h_m(T)$$

for almost all $x \in X$. Therefore,

$$\mathcal{L}_T(1; m) = \int_X h_T^*(x; \mathfrak{U}) \mathrm{d}m(x) = h_m(T).$$

Now, let $\mu \in M(X,T)$, and $\mu = \int_{E(X,T)} m \mathrm{d}\tau(m)$ be the ergodic decomposition of μ. Applying Theorems 2.6 (ii) and 2.8 (iii), we have

$$\mathcal{L}_T(1; \mu) = \int_{E(X,T)} \mathcal{L}_T(1; m) \mathrm{d}\tau(m) = \int_{E(X,T)} h_m(T) \mathrm{d}\tau(m) = h_\mu(T).$$

(ii) Let $\mathfrak{U} = \{\xi_n\}_{n \in \mathbb{N}}$ be a sequence of finite Borel partitions of X, such that $\mathrm{diam}(\xi_n) \to 0$. Let $f \in C(X)$, then,

$$\begin{aligned} |\mathcal{L}_T(f; \mu)| &= \left| \int_X f(x) h_T^*(x; \mathfrak{U}) \mathrm{d}\mu(x) \right| \leq \int_X |f(x)| h_T^*(x; \mathfrak{U}) \mathrm{d}\mu(x) \\ &\leq \|f\|_\infty \int_X h_T^*(x; \mathfrak{U}) \mathrm{d}\mu(x) = \|f\|_\infty \mathcal{L}_T(1; \mu) = \|f\|_\infty h_\mu(T). \end{aligned}$$

Therefore, the entropy functional is continuous and $\|\mathcal{L}_T(\cdot; \mu)\| \leq h_\mu(T)$. The equality holds by (i).

Discussions and concluding remarks

(i) In this article, we introduce the entropy functional for continuous maps, with finite entropy, on compact metric spaces . It is a continuous linear functional on $C(X)$ such that its norm equals the entropy of T. It generates the entropy of system as a special case. It is indeed an integral functional with the kernel h_T^*. The map h_T^* is indeed a local entropy map and so the value $\mathcal{L}_T(f; \mu)$ may be interpreted as the average entropy of T weightened by f. Theorem 2.8 generalizes some classical properties of the entropy to the entropy functional. More precisely, Theorem 2.8(ii) is the generalized form of the property that, the entropy map $\mu \longmapsto h_\mu(T)$ is affine. Theorem 2.8(iii) is the generalized Jacobs Theorem, and Theorem 2.8(iv) generalizes the invariance of the entropy of a system, under topological conjugacy, to the entropy functional. Also, Theorem 2.9 (ii) is a type of variational principle. Finally, if the dynamical system T is ergodic (with respect to μ), then we may easily deduce that $\mathcal{L}_T(f; \mu) = h_\mu(T)$ for any distribution f. This equality is justified, because the ergodic dynamical systems are, some how, 'memoryless' in nature.

(ii) As mentioned in Section 1, the entropy functional $\mathcal{L}_T(\cdot; \mu)$ is defined, in general, for a non-ergodic measure μ, that is, for non-physical systems. In case of ergodicity, where the system is memoryless, one may see that

$$\mathcal{L}_T(f; \mu) = \left(\int_X f \mathrm{d}\mu \right) h_\mu(T).$$

So, the properties given in Theorem 2.8 are reduced to the classical properties of $h_\mu(T)$. But when μ is not ergodic, $\mathcal{L}_T(f; \mu)$ is a generalized form of the classical entropy, satisfying the known properties of Kolmogorov entropy.

(iii) The map $h_T^* : X \to [0, \infty]$ is indeed a local entropy map, since, by Theorem 2.9(i),

$$\int_X h_T^* \mathrm{d}\mu = h_\mu(T)$$

for all $\mu \in M(X,T)$.

One should note that, unlike the Brin-Katok local entropy, the definition of h_T^* does not depend on any measure μ and is, some how, universal in the sense that it generates all entropies $\{h_\mu(T)\}_{\mu\in M(X,T)}$.

(iv) The idea of the definition of h_T^* is based on the relationship between "experience" and "information", in the sense that, the more one experiences an event the more he is informed about it. To be more precise, $\omega_T(x,A)$ in Definition 2.1 is the average time in which x spends in the Borel set A under the dynamic T. Consequently, $-\log\omega_T(x,A)$ may be regarded as the measure of how to encounter the orbit of x in A surprises us. Also, $\Omega_T(x,\xi)$ is the average of information attained by observing the orbit of x in each fragment of the partition ξ. Finally, $a_n(x)$ in Definition 2.1 is the average information attained by successive observations of the orbit of x in each fragment of ξ. The last limit in the definition of h_T^* is for the increasement of the resolution of the partitioning of X.

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