

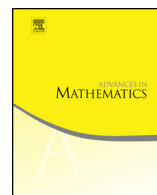


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Gaps between zeros of $\zeta(s)$ and the distribution of zeros of $\zeta'(s)$ Maksym Radziwiłł^{*,1}

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ABSTRACT

We assume the Riemann Hypothesis in this paper. We settle a conjecture of Farmer and Ki in a stronger form. Roughly speaking we show that there is a positive proportion of small gaps between consecutive zeros of the zeta-function $\zeta(s)$ if and only if there is a positive proportion of zeros of $\zeta'(s)$ lying very closely to the half-line. Our work has applications to the Siegel zero problem. We provide a criterion for the non-existence of the Siegel zero, solely in terms of the distribution of the zeros of $\zeta'(s)$. Finally on the Riemann Hypothesis and the Pair Correlation Conjecture we obtain near optimal bounds for the number of zeros of $\zeta'(s)$ lying very closely to the half-line. Such bounds are relevant to a deeper understanding of Levinson's method, allowing us to place one-third of the zeros of the Riemann zeta-function on the half-line.

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1. Introduction

The inter-relation between the *horizontal* distribution of zeros of $\zeta(s)$ (denoted $\rho = \beta + i\gamma$) and the *horizontal* distribution of the zeros of $\zeta'(s)$ (denoted $\rho' = \beta' + i\gamma'$) is

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the basis of Levinson’s method [15] which allowed Levinson to place one third of the zeros of $\zeta(s)$ on the critical line. Further improvements of the method, due to Conrey [3], put slightly more than forty percent of the zeros on the line (see also [2,8] for recent progress).

Recently it has been understood that the *horizontal* distribution of the zeros of $\zeta'(s)$ is also related to the *vertical* distribution of zeros of $\zeta(s)$. As a first attempt at capturing such a relationship we have the following conjecture of Soundararajan [19].

Note: Throughout we assume the Riemann Hypothesis. We recall that $\beta' \geq \frac{1}{2}$ for all non-trivial zeros of $\zeta'(s)$ (see [21]) and that this is equivalent to the Riemann Hypothesis.

Conjecture 1. (See Soundararajan [19].) *We have*

$$\liminf_{\gamma \rightarrow \infty} (\gamma^+ - \gamma) \log \gamma = 0 \tag{A}$$

with γ^+ the ordinate succeeding γ , if and only if

$$\liminf_{\gamma' \rightarrow \infty} \left(\beta' - \frac{1}{2} \right) \log \gamma' = 0. \tag{B}$$

Zhang [22] shows that (A) \implies (B) (see also [11] for a partial converse). Ki [14] obtained a necessary and sufficient condition for the negation of (B). Ki’s result shows that zeros ρ' with $(\beta' - \frac{1}{2}) \log \gamma' = o(1)$ arise not only from small gaps between zeros of $\zeta(s)$ but also, for example, from clusters of regularly spaced zeros of $\zeta(s)$. Therefore given our current knowledge about the zeros of $\zeta(s)$ it is possible for (B) and the negation of (A) to co-exist. The assertion (A) is arithmetically very interesting, since, following an idea of Montgomery (made explicit by Conrey and Iwaniec in [4]) if there are many small gaps between consecutive zeros of $\zeta(s)$ then the class number of $\mathbb{Q}(\sqrt{-d})$ is large and there are no Siegel zeros.

A more recent attempt at capturing the relation between the distribution of zeros of $\zeta(s)$ and $\zeta'(s)$ is due to Farmer and Ki [6]. Let $w(x)$ be the indicator function of the unit interval. Following Farmer and Ki we introduce two distribution functions,

$$m'(\varepsilon) := \liminf_{T \rightarrow \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma' \leq 2T} w\left(\frac{(\beta' - \frac{1}{2}) \log T}{\varepsilon}\right),$$

$$m(\varepsilon) := \liminf_{T \rightarrow \infty} \frac{2\pi}{T \log T} \sum_{T \leq \gamma \leq 2T} w\left(\frac{(\gamma^+ - \gamma) \log T}{\varepsilon}\right).$$

These are indeed distribution functions, since in a vertical rectangle of height T , both $\zeta(s)$ and $\zeta'(s)$ have asymptotically $N(T) \sim (T/2\pi) \log T$ zeros (see [1]), and it is conjectured that $m'(v) \rightarrow 1$ as $v \rightarrow \infty$, whereas it is known that $m(v) \rightarrow 1$ as $v \rightarrow \infty$ (see [19,10]).

Zhang shows in [22] that if $m(\varepsilon) > 0$ for all $\varepsilon > 0$, then $m'(\varepsilon) > 0$. An analogue of Soundararajan’s conjecture would assert that $m(\varepsilon) > 0$ for all $\varepsilon > 0$ if and only if

$m'(\varepsilon) > 0$ for all $\varepsilon > 0$. As explained by Farmer and Ki in [6] if for example the zeros are well-spaced with sporadic large gaps, something we cannot rule out at present, then in principle $m'(\varepsilon) > 0$ is not enough to imply $m(\varepsilon) > 0$. Farmer and Ki propose the following alternative conjecture.

Conjecture 2. (See Farmer and Ki [6].) *If $m'(\varepsilon) \gg \varepsilon^v$ with a $v < 2$ as $\varepsilon \rightarrow 0$ then $m(\varepsilon) > 0$ for all $\varepsilon > 0$.*

This is a realistic conjecture since we expect that $m'(\varepsilon) \sim (8/9\pi)\varepsilon^{3/2}$ as $\varepsilon \rightarrow 0$ (see [5]). Farmer and Ki comment “we intend this as a general conjecture, applying to the Riemann zeta-function but also to other cases such as a sequence of polynomials with all zeros on the unit circle” and that “stronger statements should be true for the zeta function”. Our main result is a proof of **Conjecture 2** in a stronger and quantitative form for the Riemann zeta-function.

Main Theorem. *Let $A, \delta > 0$ be given.*

- *If $m'(\varepsilon) \gg \varepsilon^A$ as $\varepsilon \rightarrow 0$ then $m(\varepsilon^{1/2}) \gg \varepsilon^{A+\delta}$ for all $\varepsilon \leq 1$.*
- *If $m(\varepsilon^{1/2}) \gg \varepsilon^A$ as $\varepsilon \rightarrow 0$ then $m'(\varepsilon) \gg \varepsilon^{A+\delta}$ for all $\varepsilon \leq 1$.*

We conjecture that $m'(\varepsilon) \asymp m(\varepsilon^{1/2})$ provided that one of $m(\varepsilon)$ or $m'(\varepsilon)$ is $\gg \varepsilon^A$ for some $A > 0$. This is consistent with the expectation that $m(\varepsilon) \sim (\pi/6)\varepsilon^3$ and $m'(\varepsilon) \sim (8/9\pi)\varepsilon^{3/2}$ as $\varepsilon \rightarrow 0$ (see [5]). Our Main Theorem could be restated as saying that

$$\log m(\varepsilon) \sim \log m'(\varepsilon^{1/2})$$

as $\varepsilon \rightarrow 0$ provided that one of $m(\varepsilon)$ or $m'(\varepsilon)$ is greater than ε^A . As a consequence of the Main Theorem we obtain estimates for $m'(\varepsilon)$ assuming the Pair Correlation Conjecture.

Corollary 1. *Assume the Pair Correlation Conjecture. Let $\delta > 0$. Then*

$$\varepsilon^{3/2+\delta} \ll m'(\varepsilon) \ll \varepsilon^{3/2-\delta}$$

as $\varepsilon \rightarrow 0$.

An assumption on the zero distribution in **Corollary 1** is inevitable, since $m'(\varepsilon) \rightarrow 0$ implies that almost all the zeros of $\zeta(s)$ are simple. **Corollary 1** allows one to quantify the loss in Levinson’s method coming from the zeros of $\zeta'(s)$ lying closely to the half-line. Unfortunately **Corollary 1** is a conditional result, and as such it cannot be used to put a greater proportion of the zeros of $\zeta(s)$ on the half-line (see [7] for related work).

A final consequence of our work is a criterion for the non-existence of the Siegel zero in terms of the zeros of $\zeta'(s)$. We state it only for completeness since a stronger result has been obtained by Farmer and Ki [6].

Corollary 2. *Let $A > 0$. If $m'(\varepsilon) \gg \varepsilon^A$, for all $\varepsilon > 0$, then for primitive characters χ modulo q ,*

$$L(1; \chi) > (\log q)^{-18}$$

for all q sufficiently large.

Proof. If $m'(\varepsilon) \gg \varepsilon^A$ for every $\varepsilon > 0$ then $m(1/4) > 0$ by our Main Theorem, hence $L(1; \chi) > (\log q)^{-18}$ for all q sufficiently large by Theorem 1.1 of Conrey–Iwaniec, [4]. \square

With some care it is possible to turn the above corollary into an effective result. By Dirichlet’s formula [Corollary 2](#) also implies that the class number of $\mathbb{Q}(\sqrt{-d})$ is at least as large as $c\sqrt{d}(\log d)^{-18}$ with c constant.

Farmer and Ki show that if $m'(\varepsilon) \gg \exp(-\varepsilon^{-1/2+\delta})$ as $\varepsilon \rightarrow 0$, for some $\delta > 0$, then there are $N(T)/\log \log T$ ordinates of zeros of $\zeta(s)$ lying in $[T; 2T]$ and such that $(\gamma^+ - \gamma) \log \gamma = o(1)$ as $T \rightarrow \infty$. Using the result of Conrey and Iwaniec [4] this is enough to rule out the existence of Siegel zeros. It is an interesting question to determine whether, given the current technology, one can increase the exponent $\frac{1}{2}$ in Farmer and Ki’s assumption $m'(\varepsilon) \gg \exp(-\varepsilon^{-1/2+\delta})$ and still guarantee the non-existence of Siegel zeros.

2. Main ideas

The first part of our Main Theorem follows from the stronger [Theorem 1](#) below.

Theorem 1. *Let $A, \delta > 0$. There is a constant $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ and $m'(\varepsilon) \geq c\varepsilon^A$ then $m(\varepsilon^{1/2-\delta}) \geq (c/8)\varepsilon^A$.*

The approximate value of $C(\delta, A)$ is $(B\delta/A)^{32A/\delta}$ with B an absolute constant. [Theorem 1](#) follows from two technical propositions which we now describe. Given a zero $\rho' = \beta' + i\gamma'$ of $\zeta'(s)$ we denote by $\rho_c = \frac{1}{2} + i\gamma_c$ the zero of $\zeta(s)$ lying closest to ρ' . If there are two choices of ρ_c then we pick the one lying closer to the origin. For any ordinate γ of a zero of $\zeta(s)$ we denote by γ^+ the ordinate succeeding γ and by γ^- the ordinate preceding γ . We denote by γ^\pm the ordinate closest to γ . [Theorem 1](#) follows quickly from the following proposition.

Proposition 1. *Let $0 < \delta, \varepsilon < 1$. Let $S_{\varepsilon, \delta}(T)$ be a set of zeros $\rho' = \beta' + i\gamma'$ of $\zeta'(s)$ such that $T \leq \gamma' \leq 2T$, $\beta' - \frac{1}{2} \leq \varepsilon/\log T$ and*

$$|\gamma_c - \gamma_c^\pm| > \varepsilon^{1/2-\delta}/\log T.$$

There is a $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ then $|S_{\varepsilon, \delta}(T)| \leq \varepsilon^A \cdot T \log T$.

The proof of [Proposition 1](#) rests on a proposition describing the structure the roots of $\zeta'(s)$ lying close to the half-line. The proposition which we are about to state complements with a corresponding upper bound the pointwise lower bound,

$$|\rho' - \rho_c| \geq \sqrt{\frac{2(\beta' - \frac{1}{2})}{\log T}}$$

valid for all $\rho' = \beta' + i\gamma'$ (see [\[19\]](#)). It might be of independent interest.

Proposition 2. *Let $0 < \delta < 1$, $0 < \varepsilon < c$ with $c > 0$ an absolute constant. Let T be large and $\mathcal{Z} := \mathcal{Z}_{\varepsilon, \delta}(T)$ be a set of $\delta/\log T$ well-spaced ordinates of zeros ρ' such that $\rho' \neq \rho$, $\beta' - \frac{1}{2} \leq \varepsilon/\log T$ and $T \leq \gamma' \leq 2T$. If $|\mathcal{Z}_{\varepsilon, \delta}(T)| \gg \varepsilon^A \cdot T \log T$ then, for any given $\kappa > 0$, all but $\kappa|\mathcal{Z}|$ elements $\rho' \in \mathcal{Z}$ satisfy the inequality,*

$$\sqrt{\frac{\beta' - \frac{1}{2}}{\log T}} \ll |\rho' - \rho_c| \ll \sqrt{A \log(\varepsilon \kappa \delta)^{-1}} \cdot \sqrt{\frac{\beta' - \frac{1}{2}}{\log T}}.$$

The proof of the converse part of our Main Theorem builds on ideas of Zhang, and follows from the following more precise statement valid for any fixed $\varepsilon > 0$.

Theorem 2. *Let $A, \delta > 0$. There is a $C = C(\delta, A)$ such that if $0 < \varepsilon < C$ and $m(\varepsilon^{1/2}) \geq c\varepsilon^A$ then $m'(\varepsilon) \geq (c/4)\varepsilon^{A+\delta}$.*

The paper is organized as follows. Most of the paper, all the way until [Section 7](#), is devoted to the proof of the propositions above and the deduction of [Theorem 1](#) from them. Following [Section 7](#) we prove [Theorem 2](#) and [Corollary 1](#).

3. Lemma on Dirichlet polynomials

Define

$$A_N(s) := \sum_{n \leq N} \frac{\Lambda(n)W_N(n)}{n^s}$$

with

$$W_N(n) = \begin{cases} 1 & \text{for } 1 \leq n \leq N^{1/2} \\ \log(N/n)/\log N & \text{for } N^{1/2} < n \leq N \end{cases}.$$

The lemma below is due to Selberg.

Lemma 1. *Let $\sigma = \frac{1}{2} + 2/\log N$, with $N \leq T$. Then for $T \leq t \leq 2T$,*

$$\sum_{\rho} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} \ll |A_N(s)| + \log T.$$

Proof. This is Eq. (2.2) in [17]. \square

Using the explicit formula we obtain an upper bound for the number of zeros in a small window $[t - 2\pi K/\log t; t + 2\pi K/\log t]$, in terms of the Dirichlet polynomial

$$B_N(s) := \sum_{n \leq N} \frac{\Lambda(n)}{n^s} \cdot \left(1 - \frac{\log n}{\log N}\right).$$

We have the following lemma.

Lemma 2. For $T \leq t \leq 2T$ and $N \leq T$,

$$N\left(t + \frac{\pi}{\log N}\right) - N\left(t - \frac{\pi}{\log N}\right) \ll \frac{\log T}{\log N} + \frac{|B_N(\frac{1}{2} + it)|}{\log N}.$$

Proof. Let

$$F_\Delta(v) = \left(\frac{\sin \pi \Delta v}{\pi \Delta v}\right)^2$$

be the Fejer kernel. The Fourier transform of $F_\Delta(v)$ is for $|x| < \Delta$,

$$\widehat{F}_\Delta(x) := \int_{-\infty}^{\infty} F_\Delta(t) e^{-2\pi i t x} dx = \frac{1}{\Delta} \left(1 - \frac{x}{\Delta}\right)$$

and $\widehat{F}_\Delta(x) = 0$ for $|x| > \Delta$. By the explicit formula (see Lemma 1 in [12]),

$$\begin{aligned} \sum_{\gamma} F_\Delta(\gamma - t) &= O(e^{\pi \Delta/2} \cdot t^{-2} + 1/\Delta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} F_\Delta(u - t) \cdot \Re \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{i u}{2}\right) du \\ &\quad - \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \left(\widehat{F}_\Delta\left(\frac{\log n}{2\pi}\right) + \widehat{F}_\Delta\left(\frac{-\log n}{2\pi}\right)\right). \end{aligned} \tag{1}$$

The integral over u is bounded by $\ll (\log t)/\Delta$. On the other hand the prime sum is bounded by,

$$\left| \frac{1}{2\pi \Delta} \sum_{n \leq e^{2\pi \Delta}} \frac{\Lambda(n)}{n^{1/2+it}} \cdot \left(1 - \left|\frac{\log n}{2\pi \Delta}\right|\right) \right|.$$

Finally $(\pi/2)^2 \sum_{\gamma} F_\Delta(\gamma - t)$ is an upper bound for the number of zeros in the interval going from $t - 1/(2\Delta)$ to $t + 1/(2\Delta)$. If $T \leq t \leq 2T$ we choose $2\pi \Delta = \log N$ and we are done. \square

In order to understand the average behavior of the Dirichlet polynomials $A_N(s)$ and $B_N(s)$ we use a version of the large sieve.

Lemma 3. *Let $A(s)$ be a Dirichlet polynomial with positive coefficients and of length N . Let $s_r = \sigma_r + it_r$ be points with $T \leq t_r \leq 2T$ and $0 \leq \sigma_r - \frac{1}{2} \leq \varepsilon/\log T$ for some small $\varepsilon > 0$. Suppose that $|t_i - t_j| \geq \delta/\log T$ for $i \neq j$, with $100\varepsilon < \delta < 1$. Then, for $N^k \leq T$,*

$$\sum_{s_r} |A(s_r)|^{2k} \leq \frac{60 \log T}{\delta} \int_{-2T}^{2T} \left| A\left(\frac{1}{2} + it\right) \right|^{2k} dt.$$

Proof. Let $D(s) = A(s)^k$. For any s we have, with \mathcal{C} a circle of radius $\delta/(2\log T)$ around s ,

$$|D(s)|^2 \leq \frac{4(\log T)^2}{\pi\delta^2} \iint_{\mathcal{C}} |D(x + iy)|^2 dx dy.$$

Summing over all $s = s_r$, since the circles are disjoint we obtain,

$$\sum_{s_r} |D(s_r)|^2 \leq \frac{4(\log T)^2}{\pi\delta^2} \int_{\frac{1}{2}-\delta/\log T}^{\frac{1}{2}+\delta/\log T} \int_{T^{-1}}^{2T+1} |D(\sigma + it)|^2 dt d\sigma.$$

Since the coefficients of $D(s)$ are positive, and D is of length at most T , by a majorant principle (see Chapter 7, Theorem 3 in [16]), the inner integral is bounded by

$$\leq 3e^{2\delta} \int_{-2T}^{2T} \left| D\left(\frac{1}{2} + it\right) \right|^2 dt.$$

Since in addition $\delta < 1$, the claim follows (we obtain a constant of $24e^2/\pi < 60$). \square

Combining the above lemma with Chebyshev’s inequality allows us to understand the average size of the Dirichlet polynomials $A_N(s)$ and $B_N(s)$.

Lemma 4. *Let $s_r = \sigma_r + it_r$ be a set of well-spaced points as appearing in the statement of Lemma 3. Suppose that $N^k \leq T/\log T$. The number of points s_r for which we have*

$$|A_N(s_r)| > ke \log N \quad \text{or} \quad |B_N(s_r)| > ke \log N$$

is bounded above by $\ll (e^{-k}/\delta)T \log T$.

Proof. Let $L_N(s)$ be either $A_N(s)$ or $B_N(s)$. Let

$$D_N(s) = \sum_{n \leq N} \frac{A(n)}{n^s}.$$

By a majorant principle (see Chapter 7, Theorem 3 in [16]) we have,

$$\int_{-2T}^{2T} \left| L_N \left(\frac{1}{2} + it \right) \right|^{2k} dt \leq 3 \int_{-2T}^{2T} \left| D_N \left(\frac{1}{2} + it \right) \right|^{2k} dt.$$

By Soundararajan’s lemma 3 in [20], for $N^k \leq T/\log T$ we have,

$$\int_{-2T}^{2T} \left| D_N \left(\frac{1}{2} + it \right) \right|^{2k} dt \ll k! T (\log N)^{2k}.$$

Therefore, for $N^k \leq T/\log T$, by the previous lemma,

$$\sum |L_N(s_r)|^{2k} \ll \frac{k!}{\delta} \cdot T \log T (\log N)^{2k}.$$

It follows that for $N^k \leq T/\log T$, the number of points s_r for which $|L_N(s_r)| > B \log N$ is less than,

$$\ll \left(\frac{k}{B} \right)^k \cdot (T/\delta) \log T.$$

Choosing $B = ke$ we conclude that the number of points for which $|L_N(s_r)| > ke \log N$ is bounded by $(e^{-k}/\delta)T \log T$ as desired. \square

Lemma 5. Let $0 < c < 1$. Uniformly in $T \leq t \leq 2T$, $N \leq T$ and $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \frac{1}{\log N}$,

$$\frac{\zeta'}{\zeta}(s) = \sum_{|s-\rho| < c/\log T} \frac{1}{s-\rho} + O\left(\frac{\log T}{c} \cdot \mathcal{E}_{T,N}(s)\right)$$

where

$$\mathcal{E}_{T,N}(s) := \frac{1}{\log N} \cdot \left(\left| A_N \left(\frac{1}{2} + \frac{2}{\log N} + it \right) \right| + \left| B_N \left(\frac{1}{2} + it \right) \right| \right) + \frac{\log T}{\log N}.$$

Furthermore, if s_r is a set of well-spaced points as in Lemma 3, and $N^k \leq T/\log T$, then

$$\sum_{s_r} |\mathcal{E}_{T,N}(s_r)|^{2k} \ll ((Ck)^{2k}/\delta) T \log T$$

with $C > 0$ an absolute constant.

Proof. In [18] Selberg establishes this result for $s = \frac{1}{2} + it$ and with $c = 1$. We will follow closely Selberg’s work [18] and outline the few modifications here. Let $s_N = \frac{1}{2} + \frac{2}{\log N} + it$.

Following Selberg (see Eq. (12) in [18]), for $T \leq t \leq 2T$,

$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s_N) + (s - s_N) \left(\frac{\zeta'}{\zeta} \right)'(s_N) + \sum_{\rho} \frac{(s - s_N)^2}{(s_N - \rho)^2 (s - \rho)} + O(T^{-2}).$$

According to [17] we have $(\zeta'/\zeta)(s_N) \ll |A_N(s_N)| + \log T$. Using Lemma 1

$$\left(\frac{\zeta'}{\zeta} \right)'(s_N) = \sum_{\rho} \frac{1}{(s_N - \rho)^2} + O(T^{-1}) \ll \log N \cdot (|A_N(s_N)| + \log T).$$

Combining the above equations we get

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{(s - s_N)^2}{(s_N - \rho)^2 \cdot (s - \rho)} + O(|A_N(s_N)| + \log T).$$

We now subtract $\sum_{|s-\rho|<c/\log T} (s - \rho)^{-1}$ from the main term. We find that

$$\begin{aligned} \sum_{|s-\rho|>c/\log T} \frac{(s - s_N)^2}{(s_N - \rho)^2 \cdot (s - \rho)} &\ll \frac{\log T}{c} \cdot \frac{1}{(\log N)^2} \sum_{\rho} \frac{1}{|s_N - \rho|^2} \\ &\ll \frac{\log T}{c \log N} \cdot (|A_N(s_N)| + \log T). \end{aligned}$$

We also find that

$$\begin{aligned} &\sum_{|s-\rho|<c/\log T} \left(\frac{(s - s_N)^2}{(s_N - \rho)^2 \cdot (s - \rho)} - \frac{1}{s - \rho} \right) \\ &= \sum_{|s-\rho|<c/\log T} \frac{s - s_N}{(s_N - \rho)^2} - \sum_{|s-\rho|<c/\log T} \frac{1}{s_N - \rho}. \end{aligned}$$

Using Lemma 1 and Lemma 2 the above expression is bounded by

$$\begin{aligned} &\ll (|A_N(s_N)| + \log T) + \log N \cdot \left(N \left(t + \frac{\pi}{\log N} \right) - N \left(t - \frac{\pi}{\log N} \right) \right) \\ &\ll (|A_N(s_N)| + \log T) + \log N \cdot \left(\frac{\log T}{\log N} + \frac{|B_N(\frac{1}{2} + it)|}{\log N} \right) \\ &\ll |A_N(s_N)| + \log T + \left| B_N \left(\frac{1}{2} + it \right) \right|. \end{aligned}$$

Combining the above inequalities we conclude that

$$\frac{\zeta'}{\zeta}(s) = \sum_{|s-\rho|<c/\log T} \frac{1}{s - \rho} + O \left(\frac{\log T}{c \log N} \cdot |A_N(s_N)| + \frac{\log^2 T}{c \log N} + \left| B_N \left(\frac{1}{2} + it \right) \right| \right).$$

Now it remains to estimate the moments of $\mathcal{E}_{T,N}$. Write $s_r = \sigma_r + it_r$. We have,

$$\sum_{s_r} |\mathcal{E}_{T,N}(s_r)|^{2k} \ll \left(\frac{C}{\log N}\right)^{2k} \cdot \left(\sum_{s_r} \left|A_N\left(\frac{1}{2} + \frac{2}{\log N} + it_r\right)\right|^{2k} + \sum_{s_r} \left|B_N\left(\frac{1}{2} + it_r\right)\right|^{2k}\right) + ((Ck)^{2k}/\delta)T \log T$$

with $C > 0$ an absolute constant. Using [Lemma 3](#) and proceeding as in [Lemma 4](#) we find that the $2k$ -th moments of the Dirichlet polynomials A_N and B_N is bounded above by $(k!/\delta)T \log T(\log N)^{2k}$. Hence we conclude that the $2k$ -th moment of $\mathcal{E}_{N,T}$ is bounded above by $((Ck)^{2k}/\delta)T \log T$. \square

4. Proof of [Proposition 2](#)

The proof of [Proposition 2](#) rests on the following lemma.

Lemma 6. *If $\rho' \neq \rho$ then,*

$$\frac{1}{2} \cdot \log \gamma' = \sum_{\rho} \frac{\beta' - \frac{1}{2}}{(\beta' - \frac{1}{2})^2 + (\gamma' - \gamma)^2} + O(1). \tag{2}$$

Proof. According to [\[20\]](#), Eq. (4),

$$-\Re \frac{\zeta'}{\zeta}(s) = \frac{1}{2} \log t - \sum_{\rho} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2} + O(1).$$

This is obtained in [\[20\]](#) by writing down Hadamard’s factorization formula and performing some standard simplifications. Taking $s = \rho'$ in the above formula gives the claim. \square

We will show that on average the zero $\rho = \rho_c$ dominates, the claim then follows shortly.

Proof of [Proposition 2](#). By the comment preceding [Proposition 2](#) only the upper bound needs to be proven. Let $N \leq T$ to be fixed later. In the formula (2) the contribution of the ρ ’s for which $|\gamma - \gamma'| < \pi(\log N)^{-1}$ is bounded above by

$$\begin{aligned} &\ll \left(N\left(\gamma' + \frac{\pi}{\log N}\right) - N\left(\gamma' - \frac{\pi}{\log N}\right)\right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} \\ &\ll \left(\frac{\log T}{\log N} + \frac{1}{\log N} \cdot \left|B_N\left(\frac{1}{2} + i\gamma'\right)\right|\right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} \end{aligned} \tag{3}$$

by [Lemma 2](#). On the other hand, to bound the contribution of the ρ ’s for which $|\gamma - \gamma'| > \pi(\log N)^{-1}$ we notice that if $|\gamma' - \gamma| > \pi(\log N)^{-1}$ then

$$\left(\beta' - \frac{1}{2}\right)^2 + (\gamma - \gamma')^2 \gg (2/\log N)^2 + (\gamma - \gamma')^2.$$

Therefore in (2) the contribution of the ρ 's with $|\gamma - \gamma'| > \pi(\log N)^{-1}$ is bounded above by

$$\begin{aligned} &\ll \left(\beta' - \frac{1}{2}\right) \log N \cdot \left(\sum_{\rho} \frac{2/\log N}{(2/\log N)^2 + (\gamma - \gamma')^2}\right) \\ &\ll \left(\beta' - \frac{1}{2}\right) \log N \cdot \left(\log T + \left|A_N\left(\frac{1}{2} + \frac{1}{\log N} + i\gamma'\right)\right|\right) \end{aligned} \tag{4}$$

by Lemma 1. Combining (2), (3) and (4) we conclude that

$$\begin{aligned} \log T &\ll \left(\frac{\log T}{\log N} + \frac{1}{\log N} \cdot \left|B_N\left(\frac{1}{2} + i\gamma'\right)\right|\right) \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} \\ &\quad + \left(\beta' - \frac{1}{2}\right) \log N \cdot \left(\log T + \left|A_N\left(\frac{1}{2} + \frac{1}{\log N} + i\gamma'\right)\right|\right). \end{aligned} \tag{5}$$

By Lemma 4 the number of $\rho' \in \mathcal{Z}_{\varepsilon, \delta}$ for which $|B_N(\frac{1}{2} + i\gamma')| > k\varepsilon \log N$ is bounded above by $c(e^{-k}/\delta)T \log T$ with c a constant. Similarly the number of $\rho' \in \mathcal{Z}_{\varepsilon, \delta}$ for which $|A_N(\frac{1}{2} + 1/\log N + i\gamma')| > k\varepsilon \log N$ is also bounded above by $c(e^{-k}/\delta)T \log T$. Choose k so that $c(e^{-k}/\delta)T \log T \leq (\kappa/2)|\mathcal{Z}_{\varepsilon, \delta}|$. Since $|\mathcal{Z}_{\varepsilon, \delta}| \geq c_1 \varepsilon^A T \log T$ we can take k to be the closest integer to $c_2 A \log(\kappa \varepsilon \delta)^{-1}$ with c_2 an absolute constant. Choose N to be the largest integer such that $N^k \leq T/\log T$. With this choice of k and N it follows that for at most $\kappa|\mathcal{Z}_{\varepsilon, \delta}|$ elements $\rho' \in \mathcal{Z}_{\varepsilon, \delta}$ we have $|B_N(\frac{1}{2} + i\gamma')| \geq k\varepsilon \log N$ or $|A_N(\frac{1}{2} + i\gamma')| \geq k\varepsilon \log N$. It follows that for all but at most $\kappa|\mathcal{Z}_{\varepsilon, \delta}|$ of the $\rho' \in \mathcal{Z}_{\varepsilon, \delta}$ we have,

$$c \log T \leq k \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2} + \frac{1}{k} \cdot \left(\beta' - \frac{1}{2}\right) \cdot (\log T)^2$$

with $c > 0$ an absolute constant. If ε is chosen so that $\varepsilon < (c/2)$ then (since $\beta' - \frac{1}{2} < \varepsilon/\log T$) we obtain

$$(c/2) \log T \leq k \cdot \frac{\beta' - \frac{1}{2}}{|\rho_c - \rho'|^2}$$

hence $|\rho_c - \rho'|^2 \leq (2k(\beta' - \frac{1}{2})/c \log T)^{1/2}$ which gives the desired bound for all but at most $\kappa|\mathcal{Z}_{\varepsilon, \delta}|$ elements $\rho' \in \mathcal{Z}_{\varepsilon, \delta}$. (Recall that $k \ll A \log(\varepsilon \delta \kappa)^{-1}$.) \square

5. Proof of Proposition 1

The lemma below is critical, in that it allows us to produce a sufficiently dense well-spaced sequence of zeros of $\zeta'(s)$.

Lemma 7. (See Soundararajan [19].) Suppose that $\rho_1 = \frac{1}{2} + i\gamma_1$ and $\rho_2 = \frac{1}{2} + i\gamma_2$ are two consecutive zeros of $\zeta(s)$ with $T \leq \gamma_1 < \gamma_2 \leq 2T$ for large T . Then the box,

$$\left\{ s = \sigma + it: \frac{1}{2} \leq \sigma < \frac{1}{2} + 1/\log T, \gamma_1 < t < \gamma_2 \right\}$$

contains at most one zero (counted with multiplicity) of $\zeta'(s)$.

Proof. The only way that ρ' can lie on the critical line is if $\rho' = \rho$. Since $\gamma_1 < t < \gamma_2$ this possibility is excluded. As for the box $\frac{1}{2} < \sigma < \frac{1}{2} + 1/\log T$ we know by Soundararajan’s work [19] (see Proposition 1.6) that the box $\frac{1}{2} < \sigma < \frac{1}{2} + 1/\log T$ with t in $[\gamma_1, \gamma_2]$ can contain at most one zero of $\zeta'(s)$, counted with multiplicity. \square

We are now ready to prove Proposition 1.

Proof of Proposition 1. Suppose that $S = S_{\varepsilon, \delta}(T) > \varepsilon^A \cdot T \log T$. We will show that this leads to a contradiction when $0 < \varepsilon < C(\delta, A)$ with $C(\delta, A)$ some explicit constant depending only on δ and A (for example we could take $C(\delta, A) = (c\delta/A)^{32A/\delta}$ with $c > 0$ an absolute constant). Since each $\rho' \in S$ satisfies $\gamma_c^- \leq \gamma' \leq \gamma_c^+$ and $|\gamma_c - \gamma_c^\pm| > \varepsilon^{1/2-\delta}/\log T$, by the above lemma for each $\rho' \in S$ there is at most one zero of $\zeta'(s)$ in $[\gamma_c^-, \gamma_c]$ and at most one zero of $\zeta'(s)$ in $[\gamma_c, \gamma_c^+]$.

We construct a subset S' of S by skipping every second element in S . This produces a subset of at least $(1/2)|S|$ elements, with the property that the ordinates of elements of S' are $\varepsilon^{1/2-\delta}/\log T$ well-spaced, because $|\gamma_c - \gamma_c^\pm| \geq \varepsilon^{1/2-\delta}/\log T$ for each $\rho' \in S$.

By Proposition 2, we have for at least half of the $\rho' \in S'$,

$$|\gamma' - \gamma_c| \leq |\rho' - \rho_c| \leq \frac{C\sqrt{A\varepsilon \log(\varepsilon)^{-1}}}{\log T} \tag{6}$$

with $C > 0$ an absolute constant. We call S'' the subset of S' satisfying the above inequality. Since $|\gamma_c^\pm - \gamma_c| > \varepsilon^{1/2-\delta}/\log T$ for each $\rho' \in S''$ the interval $|\gamma' - t| \leq \varepsilon^{1/2-\delta}/\log T$ contains exactly one ordinate of a zero of $\zeta(s)$ (namely γ_c) once ε is chosen so small so as to make the right-hand side of (6) less than $\varepsilon^{1/2-\delta}/\log T$ (for example $\varepsilon < (\delta/CA)^{2/\delta}$ would suffice).

Using Lemma 5, we have at $s = \rho' \in S$,

$$\sum_{|s-\rho| < c/\log T} \frac{1}{s-\rho} \ll \frac{\log T}{c} \cdot |\mathcal{E}_{T,N}(s)|.$$

Choose $s = \rho' \in S''$, $c = \varepsilon^{1/2-\delta}$ and N the largest integer such that $N^k \leq T/\log T$ with a k to be fixed later (ultimately $k = \lceil (A+1)/\delta \rceil$). By our previous remark the left-hand side of the above expression consists of only one term $(\rho' - \rho_c)^{-1}$. Raising the above expression to the $2k$ -th power and then summing over all $\rho' \in S''$ we obtain

$$\begin{aligned} \sum_{\rho' \in S''} \frac{1}{|\rho' - \rho_c|^{2k}} &\ll \varepsilon^{-k+2k\delta} \cdot (C \log T)^{2k} \sum_{\rho' \in S''} |\mathcal{E}_{T,N}(\rho')|^{2k} \\ &\ll \varepsilon^{-k+2k\delta} \cdot ((Ck)^{2k} / \varepsilon^{1/2-\delta}) \cdot T(\log T)^{2k+1} \end{aligned} \tag{7}$$

by Lemma 5, with $C > 0$ an absolute constant (not necessarily the same in each occurrence). Since for each $\rho' \in S''$ we have,

$$|\rho' - \rho_c| \ll \frac{\sqrt{A\varepsilon \log(\varepsilon)^{-1}}}{\log T}$$

the left-hand side of (7) is at least

$$\begin{aligned} \sum_{\rho' \in S''} \frac{1}{|\rho' - \rho_c|^{2k}} &\gg |S''| \cdot (C/A)^k \cdot \varepsilon^{-k} (\log(\varepsilon)^{-1})^{-k} (\log T)^{2k} \\ &\gg (C/A)^k \cdot \varepsilon^{A-k} \cdot (\log(\varepsilon)^{-1})^{-k} \cdot T(\log T)^{2k+1} \end{aligned} \tag{8}$$

since $|S''| \gg \varepsilon^A T \log T$. Combining the upper bound (7) and the lower bound (8) we get

$$\varepsilon^{A-k} (\log(\varepsilon)^{-1})^{-k} \leq \varepsilon^{-k-1/2+(2k+1)\delta} \cdot (CAk)^{2k}$$

with $C > 0$ an absolute constant. The above inequality simplifies to

$$\varepsilon^{A+1/2} \leq (CAk)^{2k} \cdot \varepsilon^{(2k+1)\delta} \cdot (\log(\varepsilon)^{-1})^k.$$

Using the inequality $(\log x) \leq x^\delta / \delta$ we obtain

$$\varepsilon^{A+1/2} \leq (CAk/\delta)^{2k} \cdot \varepsilon^{k\delta}.$$

Choosing k to be the smallest integers with $k\delta > A + 1$ we obtain a contradiction once $\varepsilon < (2C\delta^2/A^2)^{16A/\delta}$ with C an absolute constant. (Note: We have certainly not tried to optimize the constant $C(\delta, A)$.) \square

6. Proof of Theorem 1

Let T be large. By assumption each interval $[T; 2T]$ contains at least $c\varepsilon^A N(T)$ ordinates $T \leq \gamma' \leq 2T$ with $\beta' - \frac{1}{2} < \varepsilon / \log T$. If $\rho' = \rho$ for more than half of these zeros of $\zeta'(s)$, then we have $\geq (c/2)\varepsilon^A N(T)$ zeros ρ with $\gamma^+ = \gamma$ and so we are done.

Thus we can assume that there are $\geq (c/2)\varepsilon^A N(T)$ zeros ρ' with $T \leq \gamma' \leq 2T$, $\rho' \neq \rho$ and $\beta' - \frac{1}{2} < \varepsilon / \log T$. We denote the set of such ρ' by S . By Lemma 7 between any two consecutive zeros of $\zeta(s)$ there is at most one $\rho' \in S$. For each $\rho' \in S$ consider two possibilities

- (1) $|\gamma_c^\pm - \gamma_c| \leq \varepsilon^{1/2-\delta} / \log T$
- (2) $|\gamma_c^\pm - \gamma_c| > \varepsilon^{1/2-\delta} / \log T$.

Call S_2 the subset of S for which the second possibility holds. If the second possibility holds for at least one half of the elements in S then $|S_2| \geq (c/2)\varepsilon^A T \log T$. But this is impossible by Proposition 1 once ε is less than $(c/4)C(\delta, A + 1)$, with $C(\delta, A)$ as in the statement of Proposition 1. Therefore the second possibility can hold for *at most* one half of the elements in S . Hence the first possibility holds for *at least* a half of the elements in S . Call S_1 the subset of S for which the first possibility holds.

By Lemma 7, there are no two $\rho' \in S_1$ lying between the same tuple of consecutive zeros of $\zeta(s)$. Every $\rho' \in S_1$ lies either between $[\gamma_c^-, \gamma_c]$ or $[\gamma_c; \gamma_c^+]$ and moreover one of these intervals is of length $\leq \varepsilon^{1/2-\delta} / \log \gamma_c$. Skipping every second $\rho' \in S_1$ we make sure that no two $\rho_1 \in S_1$ and $\rho_2 \in S_1$ lie between the same set of consecutive zeros. Therefore every second $\rho' \in S_1$ gives rise to one (new) zero γ (namely γ_c or γ_c^-) with $(\gamma^+ - \gamma) \log \gamma \leq \varepsilon^{1/2-\delta}$. Thus we have at least $(1/2)|S_1| \geq (c/8)\varepsilon^A \cdot T \log T$ zeros $T \leq \gamma \leq 2T$ such that $(\gamma^+ - \gamma) \log \gamma \leq \varepsilon^{1/2-\delta}$.

7. Lemma: Zeros of the Riemann zeta-function

In this section we collect a few facts concerning the zeros of the Riemann zeta-function. They will be used in the proof of Theorem 2 and Corollary 1. We first need Gonek’s lemma.

Lemma 8. (See Gonek [13].) *For $x = a/b$ not an integer and $a, b \leq N$, then,*

$$\sum_{T \leq \gamma \leq 2T} x^{i\gamma} \ll N \log^2 T.$$

Proof. As noted by Ford and Zaharescu (Lemma 1, [9]), it follows from Gonek’s work that,

$$\sum_{T \leq \gamma \leq 2T} x^{1/2+i\gamma} = -\frac{\Lambda(n_x)}{2\pi} \frac{e^{iT \log(x/n_x)} - 1}{i \log(x/n_x)} + O\left(x \log^2(2xT) + \frac{\log 2T}{\log x}\right)$$

where n_x is the integer closest to x . Since x is not an integer we have $x \neq n_x$. Therefore the closest that $|x/n_x| = |a/(bn_x)|$ can be to 1 is when bn_x is equal to $a \pm 1$. This shows that $|\log(x/n_x)| \gg a^{-1} \gg N^{-1}$. Therefore the main term in the above equation is bounded by $N \log T$. This gives a bound of $\sum_{T \leq \gamma \leq 2T} x^{i\gamma} \ll (N/\sqrt{x}) \log T + \sqrt{x} \log^2 T$ for $x > 1$. For $x < 1$ this bound is reversed to $\sqrt{x} N \log T + \log^2 T / \sqrt{x}$. In either case the final bound is $\ll N \log^2 T$ because $N^{-1} \leq |x| \leq N$. □

A quick consequence of the above lemma is a bound for Dirichlet polynomials.

Lemma 9. *Let $B_N(s)$ be as in Lemma 2. If $N^k \leq \sqrt{T}$ then,*

$$\sum_{T \leq \gamma \leq 2T} \left| B_N\left(\frac{1}{2} + i\gamma\right) \right|^{2k} \ll (Ck)^k \cdot T \log T \cdot (\log N)^{2k}$$

for some absolute constant $C > 0$.

Proof. The trivial bound gives,

$$\sum_{\substack{p^k \leq N \\ k > 1}} \frac{\log p}{p^{k/2+kit}} \cdot \left(1 - \frac{\log p^k}{\log N}\right) \ll \sum_{\substack{p^k \leq N \\ k > 1}} \frac{\log p}{p^{k/2}} \ll \log N.$$

Therefore,

$$\begin{aligned} \sum_{T \leq \gamma \leq 2T} \left| B_N \left(\frac{1}{2} + i\gamma \right) \right|^{2k} &\ll C^k \sum_{T \leq \gamma \leq 2T} \left| \sum_{p \leq N} \frac{\log p}{p^{1/2+i\gamma}} \cdot \left(1 - \frac{\log p}{\log N}\right) \right|^{2k} \\ &\quad + T \log T \cdot (C \log N)^{2k} \end{aligned}$$

with $C > 0$ some absolute constant. We denote the coefficients of the Dirichlet polynomial over primes by $a(p)$. We have,

$$\sum_{T \leq \gamma \leq 2T} \left| \sum_{p \leq N} a(p)p^{-i\gamma} \right|^{2k} = \sum_{\substack{p_1, \dots, p_k \leq N \\ q_1, \dots, q_k \leq N}} a(p_1) \dots a(p_k) a(q_1) \dots a(q_k) \sum_{T \leq \gamma \leq 2T} \left(\frac{p_1 \dots p_k}{q_1 \dots q_k} \right)^{i\gamma}$$

The diagonal terms $p_1 \dots p_k = q_1 \dots q_k$ contribute at most

$$\ll T \log T \cdot k! \cdot \left(2 \sum_{p \leq N} |a(p)|^2 \right)^k \ll (Ck)^k T \log T \cdot (\log N)^{2k}$$

because given q_1, \dots, q_k all the solutions to the equation $p_1 \dots p_k = q_1 \dots q_k$ are obtained by pairing together each prime p_i with some other prime q_j , and there is at most $k!$ such pairings. To bound the off-diagonal terms $p_1 \dots p_k \neq q_1 \dots q_k$ we notice that $p_1 \dots p_k \leq N^k \leq \sqrt{T}$ and similarly that $q_1 \dots q_k \leq N^k \leq \sqrt{T}$. Therefore by Gonek’s lemma

$$\sum_{T \leq \gamma \leq 2T} \left(\frac{p_1 \dots p_k}{q_1 \dots q_k} \right)^{i\gamma} \leq \sqrt{T} \log^2 T.$$

Since $\sum_{p \leq N} a(p) \ll \sqrt{N}$ it follows that the off-diagonal terms contribute at most $C^k N^k \cdot \sqrt{T} \log^2 T \ll C^k T \log^2 T$, which is less than the main term as soon as $k > 0$. \square

An immediate consequence of the above lemma is the following.

Lemma 10. *Let $T \leq t \leq 2T$. Then,*

$$\sum_{T \leq \gamma \leq 2T} \left| N \left(\gamma + \frac{2\pi}{\log T} \right) - N \left(\gamma - \frac{2\pi}{\log T} \right) \right|^{2k} \ll (Ck)^{2k} \cdot T \log T \tag{9}$$

with $C > 0$ an absolute constant.

Proof. Let N be the largest integer such that $N^k \leq \sqrt{T}$. We have

$$\begin{aligned} N\left(\gamma + \frac{2\pi}{\log T}\right) - N\left(\gamma - \frac{2\pi}{\log T}\right) &\leq N\left(\gamma + \frac{\pi}{\log N}\right) - N\left(\gamma - \frac{\pi}{\log N}\right) \\ &\ll \frac{\log T}{\log N} + \frac{|B_N(\frac{1}{2} + i\gamma)|}{\log N} \end{aligned}$$

by Lemma 2. Raising the above expression to the $2k$ -th power and then summing over all $T \leq \gamma \leq 2T$ we obtain

$$\begin{aligned} \sum_{T \leq \gamma \leq 2T} \left| N\left(\gamma + \frac{2\pi}{\log T}\right) - N\left(\gamma - \frac{2\pi}{\log T}\right) \right|^{2k} &\ll (Ck)^{2k} \cdot T \log T \\ &\quad + \frac{C^{2k}}{(\log N)^{2k}} \sum_{T \leq \gamma \leq 2T} \left| B_N\left(\frac{1}{2} + i\gamma\right) \right|^{2k} \end{aligned}$$

with $C > 0$ an absolute constant. By the previous lemma the sum over $T \leq \gamma \leq 2T$ is $\ll (Ck)^k \cdot T \log T \cdot (\log N)^{2k}$ and so the claim follows. \square

Corollary 3. Let $A > 0$ and $\delta > 0$ be given. If $0 < \varepsilon < C(\delta, A)$, with $C(\delta, A)$ depending only on δ and A , then,

$$\#\left\{ T \leq \gamma \leq 2T: N\left(\gamma + \frac{2\pi}{\log T}\right) - N\left(\gamma - \frac{2\pi}{\log T}\right) > \varepsilon^{-\delta} \right\} \leq \varepsilon^A \cdot T \log T.$$

Proof. According to (9) the number of $T \leq \gamma \leq 2T$ for which the interval $[\gamma - 2\pi/\log T; \gamma + 2\pi/\log T]$ contains more than $\varepsilon^{-\delta}$ ordinates of zeros is bounded above by $\varepsilon^{2k\delta}(Ck)^{2k} \cdot T \log T$. Choose $k = \lceil A/\delta \rceil$. Then $\varepsilon^{2k\delta}(Ck)^{2k} \leq \varepsilon^A$ provided that $\varepsilon \leq (CA/\delta)^{-4/\delta}$ with $C > 0$ an absolute constant. \square

8. Proof of Theorem 2

We will require the following two lemmas.

Lemma 11. (See Zhang [22].) Let $\varepsilon < 1$. If $\rho = \frac{1}{2} + i\gamma$ is a zero of $\zeta(s)$ such that γ is sufficiently large and $(\gamma^+ - \gamma) \log \gamma < \varepsilon$ then there exists a zero ρ' of $\zeta'(s)$ such that

$$|\rho' - \rho| \leq \frac{2\varepsilon}{\log \gamma}.$$

Lemma 12. (See Soundararajan [19].) We have,

$$|\rho' - \rho_c|^2 \geq \frac{2(\beta' - \frac{1}{2})}{\log \gamma'}.$$

We are now ready to prove [Theorem 2](#).

Proof of Theorem 2. Suppose that there are at least $c\varepsilon^A \cdot T \log T$ zeros $T \leq \gamma \leq 2T$ such that $(\gamma^+ - \gamma) \log \gamma \leq \varepsilon^{1/2}$. Denote this set by S . If $\gamma^+ = \gamma$ for at least a half of the elements in S then $\rho' = \rho$ and hence $\beta' = \frac{1}{2}$ for at least $(c/2)\varepsilon^A \cdot T \log T$ zeros.

Hence suppose that $\gamma^+ > \gamma$ for at least half of the elements in S and denote the subset of such elements by S_1 . By [Corollary 3](#), the number of $T \leq \gamma \leq 2T$ such that the interval $[\gamma - 2\pi/\log T; \gamma + 2\pi/\log T]$ contains more than $\varepsilon^{-\delta}$ zeros is $\leq (c/4)\varepsilon^A \cdot T \log T$, provided that ε is small enough with respect to δ and A . Therefore there is a subset S_2 of S_1 of cardinality $\geq (c/4)\varepsilon^A \cdot T \log T$ with the properties that $0 < (\gamma^+ - \gamma) \log \gamma < \varepsilon^{1/2}$ and the number of zeros in the interval $[\gamma - 2\pi/\log T, \gamma + 2\pi/\log T]$ is less than $\varepsilon^{-\delta}$.

By [Lemma 11](#) each $\rho \in S_2$ gives rise to a zero ρ' such that $|\rho' - \rho| \leq 2\sqrt{\varepsilon}/\log T$. By [Lemma 12](#) the zero ρ' satisfies $(\beta' - \frac{1}{2}) \log \gamma \leq \varepsilon$. Furthermore the interval $|t - \gamma| < 2\sqrt{\varepsilon}/\log T$ contains at most $\varepsilon^{-\delta}$ zeros. Therefore striking out at most $\varepsilon^{-\delta}$ zeros from S_2 we obtain each time a new and distinct zero ρ' of $\zeta'(s)$. It follows that $\varepsilon^\delta |S_2|$ is a lower bound for the number of zeros ρ' with $(\beta' - \frac{1}{2}) \log \gamma \leq \varepsilon$. Hence $m'(\varepsilon) \geq (c/4)\varepsilon^{A+\delta}$, as desired. \square

9. Proof of [Corollary 1](#)

The Pair Correlation Conjecture asserts that the number of zeros $T \leq \gamma_1, \gamma_2 \leq 2T$ for which $2\pi\alpha/\log T < \gamma_1 - \gamma_2 \leq 2\pi\beta/\log T$ is asymptotically

$$N(T) \cdot \int_{\alpha}^{\beta} \left(1 - \left(\frac{\sin(\pi u)}{\pi u} \right)^2 + \delta(u) \right) du$$

with δ denoting Dirac’s delta function. Here we derive a simple consequence of the Pair Correlation Conjecture for small gaps between *consecutive* zeros of the Riemann zeta-function. The lower bound is not optimal but sufficient for our needs.

Lemma 13. *Assume the Pair Correlation Conjecture. Let $\delta > 0$ be given. Then $\varepsilon^{3+\delta} \ll m(\varepsilon) \ll \varepsilon^3$ provided that $0 < \varepsilon < C(\delta)$ with $C(\delta)$ a constant depending only on δ .*

Proof. The Pair Correlation Conjecture asserts that the number of distinct zeros $T \leq \gamma_1, \gamma_2 \leq 2T$ for which $0 \leq \gamma_1 - \gamma_2 \leq 2\pi\alpha/\log T$ is asymptotically $N(T) \cdot f(\alpha)$ with $f(\alpha)$ such that $f(\alpha) \sim c \cdot \alpha^3$ as $\alpha \rightarrow 0$. The number of $T \leq \gamma \leq 2T$ such that $(\gamma^+ - \gamma) \log \gamma \leq \varepsilon$ is less than the number of distinct $T \leq \gamma_1, \gamma_2 \leq 2T$ for which $0 \leq \gamma_1 - \gamma_2 \leq 2\pi\varepsilon/\log T$ therefore $m(\varepsilon) \leq f(\varepsilon) \ll \varepsilon^3$.

Now consider the set of $T \leq \gamma_1, \gamma_2 \leq 2T$ for which $\frac{\varepsilon}{2} \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon$. Denote by S the set of $T \leq \gamma_1 \leq 2T$ for which the interval $[\gamma_1 - 2\pi/\log T; \gamma_1 + 2\pi/\log T]$ contains at most $\varepsilon^{-\delta}$ zeros. By [Corollary 3](#), the zeros $T \leq \gamma_1 \leq 2T$ with $\gamma_1 \notin S$ have cardinality

$\leq \varepsilon^A \cdot T \log T$ provided that $0 < \varepsilon < C(\delta, A)$ (we choose $A = 100$ for example). We have

$$\sum_{\substack{T \leq \gamma_1, \gamma_2 \leq 2T \\ \frac{\varepsilon}{2} \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon}} 1 = \sum_{\gamma_1 \in S} \sum_{\substack{T \leq \gamma_2 \leq 2T \\ \frac{\varepsilon}{2} \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon}} 1 + \sum_{\gamma_1 \notin S} \sum_{\substack{T \leq \gamma_2 \leq 2T \\ \frac{\varepsilon}{2} \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon}} 1. \tag{10}$$

Since $\gamma_1 \in S$ there can be at most $\varepsilon^{-\delta}$ zeros γ_2 satisfying $\varepsilon/2 \leq |\gamma_1 - \gamma_2| \log \gamma_1 \leq \varepsilon$. Therefore the first sum is bounded by

$$\sum_{\substack{\gamma_1 \in S \\ (\gamma_1^+ - \gamma_1) \log \gamma_1 \leq \varepsilon}} \varepsilon^{-\delta} \ll \varepsilon^{-\delta} \cdot m(\varepsilon) \cdot T \log T$$

because for each $\gamma_1 \in S$ the inner sum over γ_2 is $\leq \varepsilon^{-\delta}$ if $(\gamma_1^+ - \gamma_1) \log \gamma_1 \leq \varepsilon$ and is 0 otherwise. On the other hand the second sum is by Cauchy–Schwarz less than,

$$\begin{aligned} & |S|^{1/2} \cdot \left(\sum_{T \leq \gamma_1 \leq 2T} \left(\sum_{\substack{T \leq \gamma_2 \leq 2T \\ \frac{\varepsilon}{2} \leq (\gamma_1 - \gamma_2) \log \gamma_1 \leq \varepsilon}} 1 \right)^2 \right)^{1/2} \\ & \leq |S|^{1/2} \cdot \left(\sum_{T \leq \gamma_1 \leq 2T} \left(N\left(\gamma_1 + \frac{2\pi}{\log T}\right) - N\left(\gamma_1 - \frac{2\pi}{\log T}\right) \right)^2 \right)^{1/2} \ll \varepsilon^{A/2} \cdot T \log T \end{aligned}$$

by Lemma 10. By the Pair Correlation Conjecture the left-hand side of (10) is asymptotically $C \cdot N(T) \cdot \varepsilon^3$ for some absolute constant $C > 0$. Combining the above three equations we get $C\varepsilon^3 \leq m(\varepsilon)\varepsilon^{-\delta} + C_1\varepsilon^{A/2}$ for some absolute constant $C, C_1 > 0$. Therefore if ε is small enough then $\varepsilon^{3+\delta} \ll m(\varepsilon)$. \square

We are now ready to prove Corollary 1.

Proof of Corollary 1. By the previous lemma, on the Pair Correlation, we have $m(\varepsilon^{1/2}) \gg \varepsilon^{3/2+\delta}$ for all $C(\delta) > \varepsilon > 0$. Therefore by the second part of our Main Theorem we get $m'(\varepsilon) \gg \varepsilon^{3/2+\delta}$ for all $C(\delta) > \varepsilon > 0$. Now suppose to the contrary that there is an $\eta > 0$ and a sequence of $\varepsilon \rightarrow 0$ such that $m'(\varepsilon) \gg \varepsilon^{3/2-\eta}$. Then, by Theorem 1 on the same sub-sequence of $\varepsilon \rightarrow 0$ we have $m(\varepsilon^{1/2-\delta}) \gg \varepsilon^{3/2-\eta}$. However by the Pair Correlation Conjecture we have $\varepsilon^{3/2-3\delta} \gg m(\varepsilon^{1/2-\delta}) \gg \varepsilon^{3/2-\eta}$. Choosing $0 < \delta < (1/3)\eta$ and letting $\varepsilon \rightarrow 0$ along the subsequence, we obtain a contradiction. \square

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References

- [1] B.C. Berndt, The number of zeros for $\zeta^{(k)}(s)$, *J. Lond. Math. Soc.* (2) 2 (1970) 577–580.
- [2] H.M. Bui, B. Conrey, M.P. Young, More than 41% of the zeros of the zeta function are on the critical line, *Acta Arith.* 150 (1) (2011) 35–64.
- [3] J.B. Conrey, More than two fifths of the zeros of the Riemann zeta function are on the critical line, *J. Reine Angew. Math.* 399 (1989) 1–26.
- [4] B. Conrey, H. Iwaniec, Spacing of zeros of Hecke L -functions and the class number problem, *Acta Arith.* 103 (3) (2002) 259–312.
- [5] E. Dueñez, D.W. Farmer, S. Froehlich, C.P. Hughes, F. Mezzadri, T. Phan, Roots of the derivative of the Riemann-zeta function and of characteristic polynomials, *Nonlinearity* 23 (10) (2010) 2599–2621.
- [6] D.W. Farmer, H. Ki, Landau–Siegel zeros and zeros of the derivative of the Riemann zeta function, *Adv. Math.* 230 (4–6) (2012) 2048–2064.
- [7] S. Feng, A note on the zeros of the derivative of the Riemann zeta function near the critical line, *Acta Arith.* 120 (1) (2005) 59–68.
- [8] S. Feng, Zeros of the Riemann zeta function on the critical line, *J. Number Theory* 132 (4) (2012) 511–542.
- [9] K. Ford, A. Zaharescu, On the distribution of imaginary parts of zeros of the Riemann zeta function, *J. Reine Angew. Math.* 579 (2005) 145–158.
- [10] A. Fujii, On the zeros of Dirichlet L -functions. II, *Trans. Amer. Math. Soc.* 267 (1) (1981) 33–40, with corrections to: “On the zeros of Dirichlet L -functions. I” [*Trans. Amer. Math. Soc.* 196 (1974) 225–235, MR 50 #2096] and subsequent papers.
- [11] M.Z. Garaev, C.Y. Yıldırım, On small distances between ordinates of zeros of $\zeta(s)$ and $\zeta'(s)$, *Int. Math. Res. Not. IMRN* 2007 (21) (2007), Art. ID rnm091, 14 pp.
- [12] D.A. Goldston, S.M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, *Bull. Lond. Math. Soc.* 39 (3) (2007) 482–486.
- [13] S.M. Gonek, An explicit formula of Landau and its applications to the theory of the zeta-function, in: *A Tribute to Emil Grosswald: Number Theory and Related Analysis*, in: *Contemp. Math.*, vol. 143, Amer. Math. Soc., Providence, RI, 1993, pp. 395–413.
- [14] H. Ki, The zeros of the derivative of the Riemann zeta function near the critical line, *Int. Math. Res. Not. IMRN* 2008 (16) (2008), Art. ID rnn064, 23 pp.
- [15] N. Levinson, More than one third of zeros of Riemann’s zeta-function are on $\sigma = 1/2$, *Adv. Math.* 13 (1974) 383–436.
- [16] H.L. Montgomery, *Ten Lectures on the Interface between Analytic Number Theory and Harmonic Analysis*, CBMS Regional Conference Series in Mathematics, vol. 84, 1994, published for the Conference Board of the Mathematical Sciences, Washington, DC.
- [17] A. Selberg, On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$, *Avh. Nor. Vidensk. Akad. Oslo I* 1944 (1) (1944) 27.
- [18] A. Selberg, On the value distribution of the derivative of the Riemann zeta-function, unpublished manuscript available at <http://publications.ias.edu/selberg/section/2483>.
- [19] K. Soundararajan, The horizontal distribution of zeros of $\zeta'(s)$, *Duke Math. J.* 91 (1) (1998) 33–59.
- [20] K. Soundararajan, Moments of the Riemann zeta function, *Ann. of Math.* (2) 170 (2) (2009) 981–993.
- [21] A. Speiser, Geometrisches zur Riemannschen Zetafunktion, *Math. Ann.* 110 (1) (1935) 514–521.
- [22] Y. Zhang, On the zeros of $\zeta'(s)$ near the critical line, *Duke Math. J.* 110 (3) (2001) 555–572.