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BEST APPROXIMATION FOR WEIERSTRASS TRANSFORM CONNECTED WITH SPHERICAL MEAN OPERATOR*

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Abstract Using reproducing kernels for Hilbert spaces, we give best approximation for Weierstrass transform associated with spherical mean operator. Also, estimates of extremal functions are checked.

Key words Weierstrass transform; spherical mean operator best approximation; reproducing kernel; extremal function

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1 Introduction

The spherical mean operator is defined on $\mathcal{C}_*\big(\mathbb{R}\times\mathbb{R}^n\big)$ (the space of continuous functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable) by

$$\mathcal{R}(f)(r,x)=\int_{S^n} f\big(r\eta,x+r\xi\big)\mathrm{d}\sigma_n(\eta,\xi),$$

where S^n is the unit sphere $\big\{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n;\ \eta^2+|\xi|^2=1\big\}$ in $\mathbb{R}\times\mathbb{R}^n$ and σ_n the surface measure on S^n normalized with total measure one.

For all non negative measurable functions f and g defined on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, we have the relation of duality

$$\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\int_0^\infty\int_{\mathbb{R}^n}\mathcal{R}(f)(r,x)g(r,x)r^n\mathrm{d}r\mathrm{d}x=\sqrt{\frac{2}{\pi}}\int_0^\infty\int_{\mathbb{R}^n} f(r,x)\,{}^t\mathcal{R}(g)(r,x)\mathrm{d}r\mathrm{d}x$$

where ${}^t\mathcal{R}$ is the operator defined by

$${}^t\mathcal{R}(g)(r,x)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} g\big(\sqrt{r^2+|x-y|^2},y\big)\mathrm{d}y.$$

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The spherical mean operator $\mathcal{R}$ and its dual ${}^t\mathcal{R}$ play an important role and have many applications, for example; in image processing of the so-called synthetic aperture radar (SAR) data [7, 8], or in the linearized inverse scattering problem in acoustics [5]. These operators were studied by many authors from many points of view [1, 5, 11, 12].

In [11, 12], the second author with the others associated to the spherical mean operator $\mathcal{R}$, the Fourier transform $\mathcal{F}$ defined by

$$\mathcal{F}(f)(\mu,\lambda)=\int_0^\infty\int_{\mathbb{R}^n}f(r,x)\varphi_{\mu,\lambda}(r,x)\mathrm{d}\nu_n(r,x),$$

where $\varphi_{\mu,\lambda}(r,x)=\mathcal{R}\big(\cos(\mu.)e^{-\mathrm{i}\langle\lambda|.\rangle}\big)(r,x)$, $\mathrm{d}\nu_n$ is the measure defined on $[0,+\infty[\times\mathbb{R}^n$ by

$$\mathrm{d}\nu_n(r,x)=\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}r^n\mathrm{d}r\otimes\frac{\mathrm{d}x}{(2\pi)^{\frac{n}{2}}}.$$

They constructed the harmonic analysis related to the transform $\mathcal{F}$ (inversion formula, Plancherel formula, Paley-Wiener theorem, Plancherel theorem, etc.).

Our investigation in this article consists of defining and studying the Weierstrass transform $\mathcal{W}_t$ associated with the spherical mean operator $\mathcal{R}$. This transform is defined by

$$\mathcal{W}_t(f)(r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}\mathcal{E}_t((r,x),(s,y))f(s,-y)\mathrm{d}\nu_n(s,y),$$

where $\mathcal{E}_t((r,x),(s,y))$, $t>0$; is the heat kernel associated with the spherical mean operator which will be defined later. This integral transform which generalizes the usual Weierstrass transform [10, 13, 14] solves the heat equation

$$\Xi u\big((r,x);t\big)=\frac{\partial}{\partial t}u\big((r,x);t\big),$$

where

$$\Xi=\Big(\frac{\partial}{\partial r^2}\Big)^2+\frac{n}{r}\frac{\partial}{\partial r}+\sum_{j=1}^n\Big(\frac{\partial}{\partial x_j}\Big)^2.$$

Building on the ideas of Saitoh, Matsuura, Fujiwara, and Yamada [6, 13–15, 17], and using the theory of reproducing kernels [2], we give best approximation of this transform and nice estimates of the associated extremal function.

Let $L^2(\mathrm{d}\nu_n)$ be the Hilbert space of square integrable functions on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $\mathrm{d}\nu_n$ and $\langle.|.\rangle_{\nu_n}$ its inner product.

For $\alpha\in\mathbb{R}$, we consider the Sobolev type space's $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ consisting of functions $f\in L^2(\mathrm{d}\nu_n)$, such that the function

$$(\mu,\lambda)\longmapsto(1+\mu^2+2|\lambda|^2)^{\alpha/2}\mathcal{F}(f)(\mu,\lambda)$$

is square integrable on the set

$$\Gamma_+=[0,+\infty[\times\mathbb{R}^n\cup\big\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ 0\le\mu\le|\lambda|\big\}$$

with respect to the measure $\mathrm{d}\gamma_n$ defined on the set Γ_+ by

$$\iint_{\Gamma_+}g(\mu,\lambda)\mathrm{d}\gamma_{(\frac{n-1}{2},n)}(\mu,\lambda)$$

$$= \frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\Big(\int_0^{\infty}\int_{\mathbb{R}^n} g(\mu,\lambda)(\mu^2+|\lambda|^2)^{\frac{n-1}{2}}\mu \mathrm{d}\mu \mathrm{d}\lambda$$
$$+\int_{\mathbb{R}^n}\int_0^{|\lambda|} g(i\mu,\lambda)(|\lambda|^2-\mu^2)^{\frac{n-1}{2}}\mu \mathrm{d}\mu \mathrm{d}\lambda\Big).$$

Then, for $\alpha > \frac{2n+1}{2}$, $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ is a Hilbert space when endowed with the inner product

$$\langle f|g\rangle_\alpha = \iint_{\Gamma^+}(1+\mu^2+2|\lambda|^2)^\alpha \mathcal{F}(f)(\mu,\lambda)\overline{\mathcal{F}(g)(\mu,\lambda)}\mathrm{d}\gamma_n(\mu,\lambda).$$

Moreover, the function defined on $\big([0,+\infty[\times\mathbb{R}^n\big)^2$ by

$$K_\alpha\big((r,x),(s,y)\big) = \iint_{\Gamma^+}\frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{(1+\mu^2+2|\lambda|^2)^\alpha}\mathrm{d}\gamma_n(\mu,\lambda)$$

is a reproducing kernel of the space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$, where $\varphi_{\mu,\lambda}$ is the eigenfunction that will be defined in the first section. Using the properties of the Fourier transform $\mathcal{F}$ and its connection with the convolution product, we show that the Weierstrass transform $\mathcal{W}_t$ is a bounded linear operator from $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ into $L^2(\mathrm{d}\nu_n)$ and that, for all $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$,

$$\|\mathcal{W}_t(f)\|_{2,\nu_n} \le \|f\|_\alpha.$$

Next, for $\rho>0$, we define on the space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ the new inner product by setting

$$\langle f|g\rangle_{\alpha,\rho} = \rho\langle f|g\rangle_\alpha + \langle \mathcal{W}_t(f)|\mathcal{W}_t(g)\rangle_{\nu_n}.$$

We show that $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ is a Hilbert space when equipped with the inner product $\langle f|g\rangle_{\alpha,\rho}$, and we exhibit a reproducing kernel, that is,

$$\mathcal{K}_{\alpha,\rho}\big((r,x),(s,y)\big) = \iint_{\Gamma^+}\frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda).$$

The last section of this article is devoted to study the extremal function. More precisely, for all $\alpha>\frac{2n+1}{2}$, $\rho>0$, and $g\in L^2(\mathrm{d}\nu_n)$, the infimum of

$$\Big\{\rho\|f\|_\alpha^2 + \|g-\mathcal{W}_t(f)\|_{2,\nu_n}^2;\quad f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)\Big\}$$

is attained at function $f^*_{\rho,g}$, called the extremal function. We establish also the following estimates

- For all $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ and $g=\mathcal{W}_t(f)$,

$$\lim_{\rho\to o^+}\big\|f^*_{\rho,g}-f\big\|_\alpha = 0.$$

- For all $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ and $g=\mathcal{W}_t(f)$,

$$\lim_{\rho\to o^+} f^*_{\rho,g}(r,x) = f(r,x), \quad \text{uniformly}.$$

2 The Spherical Mean Operator

In this section, we recall some harmonic analysis results related to the convolution product and the Fourier transform associated with the spherical mean operator. For more details, we refer to [11, 12].

For all $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$; we denote by $\varphi_{\mu,\lambda}$ the function defined by

$$\varphi_{\mu,\lambda}(r,x)=\mathcal{R}\big(\cos(\mu.)e^{-i\langle\lambda|.\rangle}\big)(r,x),$$

then we have

$$\varphi_{\mu,\lambda}(r,x)=j_{\frac{n-1}{2}}\big(r\sqrt{\mu^2+\lambda^2}\big)e^{-i\langle\lambda|x\rangle}, \tag{2.1}$$

where $\lambda^2=\lambda_1^2+\cdots+\lambda_n^2$; $\lambda=\big(\lambda_1,\cdots,\lambda_n\big)\in\mathbb{C}^n$, $\langle\lambda\mid x\rangle=\lambda_1x_1+\cdots+\lambda_nx_n$; $x=\big(x_1,\cdots,x_n\big)\in\mathbb{R}^n$, $j_{\frac{n-1}{2}}$ is the modified Bessel function given by

$$j_{\frac{n-1}{2}}(s)=2^{\frac{n-1}{2}}\Gamma\Big(\frac{n+1}{2}\Big)\frac{J_{\frac{n-1}{2}}(s)}{s^{\frac{n-1}{2}}}=\Gamma\Big(\frac{n+1}{2}\Big)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\frac{n+1}{2})}\big(\frac{s}{2}\big)^{2k}$$

and $J_{\frac{n-1}{2}}$ is the usual Bessel function of first kind of order $\frac{n-1}{2}$ [3, 4, 9, 16].

Also, using the relation (2.1) and the properties of the modified Bessel function $j_{\frac{n-1}{2}}$, we deduce that the function $\varphi_{\mu,\lambda}$ satisfies the following properties [11, 12]:

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}\big|\varphi_{\mu,\lambda}(r,x)\big|=1, \tag{2.2}$$

if and only if (μ,λ) belongs to the set Γ defined by

$$\Gamma=\mathbb{R}\times\mathbb{R}^n\ \cup\ \big\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ |\mu|\le\ |\lambda|\big\}, \tag{2.3}$$

For all $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, the function $\varphi_{\mu,\lambda}$ is the unique solution of the system

$$\begin{cases}\dfrac{\partial u}{\partial x_j}(r,x)=-i\ \lambda_j\ u(r,x);\ 1\le j\le n\\ Lu(r,x)=-\mu^2\ u(r,x)\\ u(0,0)=1;\ \dfrac{\partial u}{\partial r}\big(0,x_1,\cdots,x_n\big)=0;\ \forall\ \big(x_1,\cdots,x_n\big)\in\mathbb{R}^n\end{cases}$$

where

$$L=\frac{\partial^2}{\partial r^2}+\frac{n}{r}\frac{\partial}{\partial r}-\sum_{j=1}^{n}\Big(\frac{\partial}{\partial x_j}\Big)^2.$$

In the following, we denote by

- $\mathrm{d}\nu_n$ the measure defined on $[0,+\infty[\times\mathbb{R}^n$, by

$$\mathrm{d}\nu_n(r,x)=\frac{r^n\ dr}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}\otimes\frac{\mathrm{d}x}{(2\pi)^{\frac{n}{2}}},$$

where $\mathrm{d}x$ is the Lebesgue measure on $\mathbb{R}^n$.

- $L^p\big(\mathrm{d}\nu_n\big);\ p\in\ [1,+\infty]$, the space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$ satisfying

$$\|f\|_{p,\nu_n}=\begin{cases}\Big(\displaystyle\int_0^\infty\int_{\mathbb{R}^n}\big|f(r,x)\big|^p\mathrm{d}\nu_n(r,x)\Big)^{\frac{1}{p}}<+\infty, & \text{if } 1\le p<+\infty\\ \displaystyle\operatorname*{ess\ sup}_{(r,x)\in[0,+\infty[\times\mathbb{R}^n}\big|f(r,x)\big|<+\infty, & \text{if } p=+\infty.\end{cases}$$

- $\langle\ \mid\ \rangle_{\nu_n}$ the inner product of $L^2\big(\mathrm{d}\nu_n\big)$ defined by

$$\langle f\mid g\rangle_{\nu_n}=\int_0^\infty\int_{\mathbb{R}^n}f(r,x)\overline{g(r,x)}\mathrm{d}\nu_n(r,x).$$

- Γ_+ the subset of Γ, given by

$$\Gamma_+=[0,+\infty[\times\mathbb{R}^n\cup\big\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n;\ 0\le\mu\le|\lambda|\big\}.$$

- $\mathcal{B}_{\Gamma_+}$ the $\sigma-$algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+}=\big\{\theta^{-1}(B);\ B\in\mathcal{B}\text{ or }\big([0,+\infty[\times\mathbb{R}^n\big)\big\},$$

where θ is the bijective function defined on Γ_+ by $\theta(\mu,\lambda)=\big(\sqrt{\mu^2+|\lambda|^2},\ \lambda\big)$.

- $\mathrm{d}\gamma_n$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by

$$\forall A\in\mathcal{B}_{\Gamma_+};\gamma_n(A)=\nu_n\big(\theta(A)\big).$$

- $L^p\big(\mathrm{d}\gamma_n\big),p\in\ [1,+\infty]$, the space of measurable functions g on Γ_+ satisfying

$$\|g\|_{p,\gamma_n}=\begin{cases}\Big(\displaystyle\iint_{\Gamma_+}\big|g(\mu,\lambda)\big|^p\mathrm{d}\gamma_n(\mu,\lambda)\Big)^{\frac{1}{p}}<+\infty, & \text{if } 1\le p<+\infty\\ \displaystyle\operatorname*{ess\ sup}_{(\mu,\lambda)\in\Gamma_+}\big|g(\mu,\lambda)\big|<+\infty, & \text{if } p=+\infty.\end{cases}$$

- $\langle\mid\rangle_{\gamma_n}$ the inner product in $L^2\big(\mathrm{d}\gamma_n\big)$ given by

$$\langle f\mid g\rangle_{\gamma_n}=\iint_{\Gamma_+}f(\mu,\lambda)\overline{g(\mu,\lambda)}\mathrm{d}\gamma_n(\mu,\lambda).$$

Then, we have the following result.

Proposition 2.1 i) For all non-negative measurable function g on Γ_+(respectively integrable on Γ_+ with respect to the measure $\mathrm{d}\gamma_n$), we have

$$\iint_{\Gamma_+}g(\mu,\lambda)\mathrm{d}\gamma_n(\mu,\lambda)=\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n}{2})(2\pi)^{\frac{n}{2}}}\Big(\int_0^\infty\int_{\mathbb{R}^n}g(\mu,\lambda)(\mu^2+|\lambda|^2)^{\frac{n-1}{2}}\mu\mathrm{d}\mu\mathrm{d}\lambda+\int_{\mathbb{R}^n}\int_0^{|\lambda|}g(i\mu,\lambda)(|\lambda|^2-\mu^2)^{\frac{n-1}{2}}\mu\mathrm{d}\mu\mathrm{d}\lambda\Big).$$

ii) For all non-negative measurable function f on $[0,+\infty[\times\mathbb{R}^n$ (respectively integrable on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $\mathrm{d}\nu_n$), the function $f\circ\theta$ is measurable on Γ_+ (respectively integrable on Γ_+ with respect to the measure $\mathrm{d}\gamma_n$) and we have

$$\iint_{\Gamma_+}f\circ\theta(\mu,\lambda)\mathrm{d}\gamma_n(\mu,\lambda)=\int_0^\infty\int_{\mathbb{R}^n}f(r,x)\mathrm{d}\nu_n(r,x).\tag{2.4}$$

In the following, we shall define the translation operators and the convolution product associated with the spherical mean operator. For this purpose, we use the product formula for the function $\varphi_{\mu,\lambda}$, for all $(r,x),(s,y)\in\mathbb{R}\times\mathbb{R}^n$;

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}\int_0^{\pi}\varphi_{\mu,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)(\sin\theta)^{n-1}d\theta. \quad (2.5)$$

Definition 2.1 i) The translation operator associated with the spherical mean operator is defined on $L^1(\mathrm{d}\nu_n)$ by $\forall\ (r,x),\ (s,y)\in[0,+\infty[\times\mathbb{R}^n$;

$$\tau_{(r,x)}(f)(s,y)=\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}\int_0^{\pi}f\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)(\sin\theta)^{n-1}d\theta.$$

ii) The convolution product of $f,g\in L^1(\mathrm{d}\nu_n)$ is defined by

$$f*g(r,x)=\int_{\mathbb{R}^n}\int_0^{+\infty}\tau_{(r,-x)}\check{f}(s,y)g(s,y)\mathrm{d}\nu_n(s,y),\quad \text{for any}\quad (r,x)\in[0,+\infty[\times\mathbb{R}^n,$$

where $\check{f}(s,y)=f(s,-y)$.

We have the following properties.

• The relation (2.5) can be written as

$$\tau_{(r,x)}(\varphi_{\mu,\lambda})(s,y)=\varphi_{\mu,\lambda}(r,x)\ \varphi_{\mu,\lambda}(s,y).$$

• If $f\in L^p(\mathrm{d}\nu_n),\ 1\le p\le+\infty$, then for all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function $\tau_{(s,y)}(f)$ belongs to $L^p(\mathrm{d}\nu_n)$, and we have

$$\left\|\tau_{(s,y)}(f)\right\|_{p,\nu_n}\le\|f\|_{p,\nu_n}. \quad (2.6)$$

In particular, for all $f\in L^1(\mathrm{d}\nu_n)$ and $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function $\tau_{(s,y)}(f)$ belongs to $L^1(\mathrm{d}\nu_n)$ and we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\tau_{(s,y)}(f)(r,x)\mathrm{d}\nu_n(r,x)=\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\mathrm{d}\nu_n(r,x). \quad (2.7)$$

• For $f\in L^1(\mathrm{d}\nu_n),\ g\in L^p(\mathrm{d}\nu_n),\ 1\le p<+\infty$, the function $f*g$ belongs to $L^p(\mathrm{d}\nu_n)$, and we have

$$\|f*g\|_{p,\nu_n}\le\|f\|_{1,\nu_n}\|g\|_{p,\nu_n}. \quad (2.8)$$

In the sequel, we shall define also the Fourier transform $\mathcal{F}$ associated with the spherical mean operator and give some properties that we use later.

Definition 2.2 The Fourier transform $\mathcal{F}$ associated with the spherical mean operator is defined on $L^1\big(\mathrm{d}\nu_n\big)$ by

$$\forall(\mu,\lambda)\in\Gamma;\ \mathcal{F}(f)(\mu,\lambda)=\int_0^{\infty}\int_{\mathbb{R}^n}f(r,x)\varphi_{\mu,\lambda}(r,x)\mathrm{d}\nu_n(r,x),$$

where $\varphi_{\mu,\lambda}$ is the function given by (2.1) and Γ is the set defined by (2.3).

The Fourier transform $\mathcal{F}$ satisfies the following properties.

• For every $f\in L^1(\mathrm{d}\nu_n)$ and $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, we have

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathcal{F}\left(\tau_{(r,x)}(f)\right)(\mu,\lambda)=\overline{\varphi_{\mu,\lambda}(r,x)}\mathcal{F}(f)(\mu,\lambda). \quad (2.9)$$

• For $f, g \in L^1(\mathrm{d}\nu_n)$, we have

$$\forall(\mu,\lambda) \in \Gamma, \quad \mathcal{F}\left(f * g\right)(\mu,\lambda) = \mathcal{F}(f)(\mu,\lambda)\mathcal{F}(g)(\mu,\lambda).$$

Theorem 2.1 (Inversion formula for $\mathcal{F}$) Let $f \in L^1(\mathrm{d}\nu_n)$ such that $\mathcal{F}(f) \in L^1(\mathrm{d}\gamma_n)$, then for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have

$$f(r,x) = \int\int_{\Gamma^+} \mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}\mathrm{d}\gamma_n(\mu,\lambda).$$

Theorem 2.2 (Plancherel) The Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2\big(\mathrm{d}\nu_n\big)$ onto $L^2\big(\mathrm{d}\gamma_n\big)$. In particular, for all $f, g \in L^2\big(\mathrm{d}\nu_n\big)$, we have $\big($the Parseval's formula$\big)$

$$\langle f \mid g\rangle_{\nu_n} = \langle \mathcal{F}(f) \mid \mathcal{F}(g)\rangle_{\gamma_n}. \tag{2.10}$$

Remark 2.1

i) From the relation (2.2), it follows that the Fourier transform $\mathcal{F}$ is a bounded linear operator from $L^1\big(\mathrm{d}\nu_n\big)$ into $L^\infty\big(\mathrm{d}\gamma_n\big)$ and that, for all $f \in L^1\big(\mathrm{d}\nu_n\big)$,

$$\|\mathcal{F}(f)\|_{\infty,\gamma_n} \leq \|f\|_{1,\nu_n}.$$

ii) Let $f \in L^1(\mathrm{d}\nu_n)$ and $g \in L^2(\mathrm{d}\nu_n)$, then by relation (2.8), the function $f * g$ belongs to $L^2(\mathrm{d}\nu_n)$; moreover,

$$\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$$

iii) For all $f, g \in L^2(\mathrm{d}\nu_n)$, the function $f * g$ belongs to the space $\mathcal{C}_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ consisting of continuous functions h on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, and such that

$$\lim_{r^2+|x|^2\to+\infty} h(r,x) = 0.$$

Moreover,

$$f * g = \mathcal{F}^{-1}\big(\mathcal{F}(f)\mathcal{F}(g)\big). \tag{2.11}$$

3 Weierstrass Transform Associated with Spherical Mean Operator

In this section, we shall define and study the Weierstrass transform associated with the spherical mean operator $\mathcal{R}$. For this we define some Hilbert spaces and exhibit their reproducing kernels.

Let α be a real number such that $\alpha > \frac{2n+1}{2}$. We denote by

• $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ the subspace of $L^2(\mathrm{d}\nu_n)$ consisting of functions f such that the application

$$(\mu,\lambda) \longmapsto \big(1+\mu^2+2|\lambda|^2\big)^{\frac{\alpha}{2}}\mathcal{F}(f)(\mu,\lambda)$$

belongs to $L^2(\mathrm{d}\gamma_n)$.

• $\langle . \mid .\rangle_\alpha$ the inner product on $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ defined by

$$\langle f \mid g\rangle_\alpha = \iint_{\Gamma+} \big(1+\mu^2+2|\lambda|^2\big)^\alpha\mathcal{F}(f)(\mu,\lambda)\overline{\mathcal{F}(g)(\mu,\lambda)}\mathrm{d}\gamma_n(\mu,\lambda) \tag{3.1}$$

and $\| \ . \ \|_\alpha$ the norm defined on $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ by

$$\| f \|_\alpha^2 = \langle f \mid f\rangle_\alpha.$$

Remark 3.1

i) Let $f\in L^2(\mathrm{d}\nu_n)$ such that $\mathcal{F}(f)\in L^1(\mathrm{d}\gamma_n)$, then for almost every (r,x), we have

$$f(r,x)=\iint_{\Gamma+}\mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}\mathrm{d}\gamma_n(\mu,\lambda).$$

ii) For $\alpha>\frac{2n+1}{2}$, the function

$$(\mu,\lambda)\longmapsto\frac{1}{\big(1+\mu^2+2|\lambda|^2\big)^{\frac{\alpha}{2}}}$$

belongs to $L^2(\mathrm{d}\gamma_n)$. Consequently, for all $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$, $\mathcal{F}(f)$ belongs to $L^1(\mathrm{d}\gamma_n)\cap L^2(\mathrm{d}\gamma_n)$ and then,

$$f(r,x)=\iint_{\Gamma+}\mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}\mathrm{d}\gamma_n(\mu,\lambda)\quad\text{a.e.}\tag{3.2}$$

Proposition 3.1 For $\alpha>\frac{2n+1}{2}$, the function K_α defined on $\big([0,+\infty[\times\mathbb{R}^n\big)^2$ by

$$K_\alpha\big((r,x);(s,y)\big)=\iint_{\Gamma+}\frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{\big(1+\mu^2+2|\lambda|^2\big)^\alpha}\mathrm{d}\gamma_n(\mu,\lambda)$$

is a reproducing kernel for the Hilbert space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$, that is,

i) For all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function

$$(r,x)\longmapsto K_\alpha\big((r,x);(s,y)\big)$$

belongs to the space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$.

ii) (The reproducing property) For all $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ and $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, we have

$$\langle f\mid K_\alpha\big((\cdot,\cdot);(s,y)\big)\rangle_\alpha=f(s,y).$$

Proof i) By relation (2.2) and the hypothesis $\alpha>\frac{2n+1}{2}$, we deduce that, for all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function

$$(\mu,\lambda)\longmapsto\frac{\varphi_{\mu,\lambda}(s,y)}{\big(1+\mu^2+2|\lambda|^2\big)^\alpha}$$

belongs to $L^1(\mathrm{d}\gamma_n)\cap L^2(\mathrm{d}\gamma_n)$. Consequently, the kernel K_α is well defined and we have

$$K_\alpha\big((r,x);(s,y)\big)=\mathcal{F}^{-1}\left(\frac{\varphi_{\mu,\lambda}(s,y)}{\big(1+\mu^2+2|\lambda|^2\big)^\alpha}\right)(r,x).$$

This shows that, for all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function $K_\alpha\big((\cdot,\cdot);(s,y)\big)$ belongs to $L^2(\mathrm{d}\nu_n)$. Moreover,

$$\mathcal{F}\Big(K_\alpha\big((\cdot,\cdot);(s,y)\big)\Big)(\mu,\lambda)=\frac{\varphi_{\mu,\lambda}(s,y)}{\big(1+\mu^2+2|\lambda|^2\big)^\alpha}.\tag{3.3}$$

Again by relation (2.2), we get

$$\big|\big(1+\mu^2+2|\lambda|^2\big)^{\frac{\alpha}{2}}\mathcal{F}\big(K_\alpha\big((\cdot,\cdot);(s,y)\big)(\mu,\lambda)\big| \leq \frac{1}{\big(1+\mu^2+2|\lambda|^2\big)^{\frac{\alpha}{2}}}.$$

This implies that for all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function $K_\alpha\big((\cdot,\cdot);(s,y)\big)$ belongs to the space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$.

ii) Let $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$. For all $(s,y)\in\ [0,+\infty[\times\mathbb{R}^n$, we have

$$\begin{aligned}\langle f\mid K_\alpha\big((\cdot,\cdot);(s,y)\big)\rangle_\alpha = &\iint_{\Gamma+}\big(1+\mu^2+2|\lambda|^2\big)^{\alpha}\mathcal{F}(f)(\mu,\lambda)\\ &\times\overline{\mathcal{F}\big(K_\alpha\big((\cdot,\cdot);(s,y)\big)\big)}(\mu,\lambda)\mathrm{d}\gamma_n(\mu,\lambda)\end{aligned}$$

and by the relation (3.3), we obtain

$$\langle f\mid K_\alpha\big((\cdot,\cdot);(s,y)\big)\rangle_\alpha = \iint_{\Gamma+}\mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)}\mathrm{d}\gamma_n(\mu,\lambda).$$

We complete the proof by using the relation (3.2).

The heat equation associated with the spherical mean operator $\mathcal{R}$ is

$$\Xi u\big((r,x);t\big) = \frac{\partial}{\partial t}u\big((r,x);t\big), \tag{3.4}$$

where

$$\Xi = \left(\frac{\partial}{\partial r^2}\right)^2 + \frac{n}{r}\frac{\partial}{\partial r} + \sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}\right)^2.$$

In the following, we will define the heat kernel that will be a solution of the heat equation (3.4).

Let E be the kernel defined by

$$E\big((r,x);t\big) = \mathcal{F}^{-1}\big(e^{-t(\mu^2+2|\lambda|^2)}\big)(r,x) = \frac{1}{(2t)^{\frac{2n+1}{2}}}e^{-\frac{r^2+|x|^2}{4t}}, \tag{3.5}$$

then, the function E solves the heat equation (3.4) and we have

Definition 3.1 The heat kernel associated with the spherical mean operator $\mathcal{R}$ is defined on $\big([0,+\infty[\times\mathbb{R}^n\big)^2$ by

$$\mathcal{E}_t\big((r,x);(s,y)\big) = \tau_{(r,x)}\big(E((\cdot,\cdot);t)\big)(s,y) \tag{3.6}$$

$$= \tau_{(s,y)}\big(E((\cdot,\cdot);t)\big)(r,x) \tag{3.7}$$

$$= \frac{1}{(2t)^{\frac{2n+1}{2}}}e^{-\frac{r^2+s^2}{4t}}e^{-\frac{|x+y|^2}{4t}}j_{\frac{n-1}{2}}\left(\frac{irs}{2t}\right).$$

Then, we have the following properties.

i) For all $t>0, \mathcal{E}_t>0$.

ii) From the relations (2.6), (3.5), and (3.6), we deduce that for all $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{E}_t\big((r,x);(\cdot,\cdot)\big)$ belongs to $L^1(\mathrm{d}\nu_n)$, and by (2.9), we have

$$\mathcal{F}\big(\mathcal{E}_t\big((r,x);(\cdot,\cdot)\big)\big)(\mu,\lambda) = \overline{\varphi_{\mu,\lambda}(r,x)}e^{-t(\mu^2+2|\lambda|^2)}. \tag{3.8}$$

iii) From the relations (2.6), (2.7), (3.5), and (3.7), we deduce that, for all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{E}_t\big((\cdot,\cdot);(s,y)\big)$ belongs to $L^1(\mathrm{d}\nu_n)$ and we have

$$\int_0^\infty\int_{\mathbb{R}^n}\mathcal{E}_t\big((r,x);(s,y)\big)\mathrm{d}\nu_n(r,x)=\int_0^\infty\int_{\mathbb{R}^n}E\big((r,x);t\big)\mathrm{d}\nu_n(r,x)=1. \tag{3.9}$$

iv) Using the relation (3.7) and the fact that the function

$$((r,x);t)\longmapsto E((r,x);t)$$

solves the heat equation, we deduce that, for all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function

$$((r,x);t)\longmapsto \mathcal{E}_t\big((r,x);(s,y)\big)$$

solves the same equation.

In the following, we shall define the Weierstrass transform associated with the spherical mean operator and establish some properties that we would use in the next section.

Definition 3.2 The Weierstrass transform $\mathcal{W}_t$ associated with the spherical mean operator is defined on $L^2(\mathrm{d}\nu_n)$ by

$$\mathcal{W}_t(f)(r,x)=\big(E((\cdot,\cdot),t)*f\big)(r,x) \tag{3.10}$$

$$=\int_0^{+\infty}\int_{\mathbb{R}^n}\mathcal{E}_t((r,x),(s,y))f(s,-y)\mathrm{d}\nu_n(s,y). \tag{3.11}$$

Proposition 3.2 i) For all $f\in L^2(\mathrm{d}\nu_n)$, the function

$$((r,x);t)\longmapsto \mathcal{W}_t(f)(r,x)$$

solves heat equation (3.4), with the initial condition

$$\lim_{t\to0^+}\mathcal{W}_t(f)=f;\quad \text{in } L^2(\mathrm{d}\nu_n).$$

ii) For all $t>0$ and $\alpha>\frac{2n+1}{2}$, the transform $\mathcal{W}_t$ is a bounded linear operator from $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ into $L^2(\mathrm{d}\nu_n)$.

Moreover, for all $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$,

$$\|\mathcal{W}_t(f)\|_{2,\nu_n}\le\|f\|_\alpha.$$

Proof i) From relation (3.11), the derivative theorem's and the fact that, for all $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, the function

$$((r,x);t)\longmapsto\mathcal{E}_t((r,x),(s,y))$$

solves the heat equation; we deduce that, for all $f\in L^2(\mathrm{d}\nu_n)$, the function $((r,x);t)\longmapsto\mathcal{W}_t(f)(r,x)$ is a solution of (3.4).

From relation (3.9), we deduce that the family $\big(E((\cdot,\cdot);t)\big)_{t>0}$ is an approximate identity. In particular, for all $f\in L^2(\mathrm{d}\nu_n)$,

$$\lim_{t\to0^+}\|E((\cdot,\cdot);t)*f-f\|_{2,\nu_n}=0.$$

ii) From relations (2.8) and (3.10), we deduce that, for all $f \in L^2(\mathrm{d}\nu_n)$, we have

$$\|\mathcal{W}_t(f)\|_{2,\nu_n} = \|E((\cdot,\cdot),t) * f\|_{2,\nu_n} \leq \|f\|_{2,\nu_n}$$

and by Plancherel theorem, we get

$$\|\mathcal{W}_t(f)\|_{2,\nu_n} \leq \|\mathcal{F}(f)\|_{2,\gamma_n} \leq \|f\|_\alpha .$$

The second assertion of Proposition 3.2 allows us to define a new inner product on the space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$. More precisely, for all positive real numbers ρ, t and for $\alpha > \frac{2n+1}{2}$, we denote by

- $\langle .|. \rangle_{\alpha,\rho}$ the inner product defined on the space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ by

$$\langle f|g \rangle_{\alpha,\rho} = \rho\langle f|g\rangle_\alpha + \langle \mathcal{W}_t(f)|\mathcal{W}_t(g)\rangle_{\nu_n}.$$

- $\mathbf{H}_{\alpha,\rho}\big([0,+\infty[\times\mathbb{R}^n\big)$ the space $\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ equipped with the inner product $\langle .|. \rangle_{\alpha,\rho}$ and the norm

$$\|f\|^2_{\alpha,\rho} = \rho\|f\|^2_\alpha + \|\mathcal{W}_t(f)\|^2_{2,\nu_n}.$$

Then, we have the following main result [13, 14].

Theorem 3.1 For all $\rho, t > 0$ and $\alpha > \frac{2n+1}{2}$, the Hilbert space $\mathbf{H}_{\alpha,\rho}\big([0,+\infty[\times\mathbb{R}^n\big)$ possesses the following reproducing kernel

$$\mathcal{K}_{\alpha,\rho}\big((r,x),(s,y)\big) = \iint_{\Gamma^+} \frac{\varphi_{\mu,-\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda),$$

that is,

i) For all $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)$ lies in $\mathbf{H}_{\alpha,\rho}\big([0,+\infty[\times\mathbb{R}^n\big)$.

ii) (The reproducing property.) For all $f \in \mathbf{H}_{\alpha,\rho}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $(s,y) \in [0,+\infty[\times\mathbb{R}^n$,

$$\langle f|\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\rangle_{\alpha,\rho} = f(s,y).$$

Proof i) From relation (2.2), for every $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, we have

$$\frac{\big|\varphi_{\mu,\lambda}(s,y)\big|}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}} \leq \frac{1}{\rho(1+\mu^2+2|\lambda|^2)^\alpha}.$$

Thus, from the hypothesis $\alpha > \frac{2n+1}{2}$, we deduce that the function

$$(\mu,\lambda) \longmapsto \frac{\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}$$

belongs to $L^1(\mathrm{d}\gamma_n) \cap L^2(\mathrm{d}\gamma_n)$.

Consequently, the kernel $\mathcal{K}_{\alpha,\rho}$ is well defined and we have

$$\mathcal{K}_{\alpha,\rho}\big((r,x),(s,y)\big) = \mathcal{F}^{-1}\left(\frac{\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\right)(r,x) \tag{3.12}$$

By Plancherel theorem, for all $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)$ belongs to $L^2(\mathrm{d}\nu_n)$ and we have

$$(1+\mu^2+2|\lambda|^2)^{\alpha/2}\mathcal{F}\Big(\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\Big)(\mu,\lambda) = \frac{(1+\mu^2+2|\lambda|^2)^{\alpha/2}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}.$$

Using again relation (2.2) and the fact that $\alpha > \frac{2n+1}{2}$; we deduce that the function

$$(\mu,\lambda) \longmapsto \big(1+\mu^2+2|\lambda|^2\big)^{\frac{\alpha}{2}} \mathcal{F}\Big(\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\Big)(\mu,\lambda)$$

belongs to the space $L^2(\mathrm{d}\gamma_n)$. This proves that, for all $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, the function $\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)$ belongs to $\mathbf{H}_{\alpha,\rho}\big([0,+\infty[\times\mathbb{R}^n\big)$.

ii) Let $f \in \mathbf{H}_{\alpha,\rho}\big([0,+\infty[\times\mathbb{R}^n\big)$. From relations (3.1), (3.12), and for all $(s,y) \in [0,+\infty[\times\mathbb{R}^n$, we get

$$\begin{aligned}\langle f|\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\rangle_\alpha &= \iint_{\Gamma^+} \big(1+\mu^2+2|\lambda|^2\big)^\alpha \mathcal{F}(f)(\mu,\lambda) \\ &\quad \times \overline{\mathcal{F}\big(\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot);(s,y)\big)\big)}(\mu,\lambda)\mathrm{d}\gamma_n(\mu,\lambda) \\ &= \iint_{\Gamma^+} \big(1+\mu^2+2|\lambda|^2\big)^\alpha \mathcal{F}(f)(\mu,\lambda) \\ &\quad \times \frac{\overline{\varphi_{\mu,\lambda}(s,y)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda). \end{aligned} \tag{3.13}$$

In contrast, by relations (2.11), (3.5), and (3.12), we obtain

$$\begin{aligned}\mathcal{W}_t\Big(\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\Big)(r,x) &= \Big(E((\cdot,\cdot),t) * \mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\Big)(r,x) \\ &= \mathcal{F}^{-1}\bigg(\frac{\varphi_{\mu,\lambda}(s,y)e^{-t(\mu^2+2|\lambda|^2)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\bigg)(r,x) \end{aligned} \tag{3.14}$$

and

$$\mathcal{W}_t(f)(r,x) = \mathcal{F}^{-1}\Big(e^{-t(\mu^2+2|\lambda|^2)}\mathcal{F}(f)\Big)(r,x). \tag{3.15}$$

So, by relation (2.10), we get

$$\begin{aligned}&\langle \mathcal{W}_t(f)|\mathcal{W}_t\Big(\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\Big)\rangle_{\nu_n} \\ &= \iint_{\Gamma^+} e^{-2t(\mu^2+2|\lambda|^2)}\mathcal{F}(f)(\mu,\lambda)\frac{\overline{\varphi_{\mu,\lambda}(s,y)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda). \end{aligned} \tag{3.16}$$

By combining relations (3.13) and (3.16), it follows that

$$\langle f|\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\rangle_{\alpha,\rho} = \iint_{\Gamma^+} \mathcal{F}(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(s,y)}\mathrm{d}\gamma_n(\mu,\lambda),$$

and the desired result arises from relation (3.2).

4 The Extremal Function

This section contains the main result of this article, that is, the existence and uniqueness of the extremal function related to the generalized Weierstrass transform studied in the previous section.

Theorem 4.1 Let $\alpha > \frac{2n+1}{2}$, $\rho > 0$, and $g \in L^2(\mathrm{d}\nu_n)$. Then, there exists a unique function $f^*_{\rho,g} \in \mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$, where the infimum of the set

$$\Big\{\rho\|f\|^2_\alpha + \|g-\mathcal{W}_t(f)\|^2_{2,\nu_n}, \quad f \in \mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)\Big\}$$

is attained. Moreover, the extremal function $f^*_{\rho,g}$ is given by

$$f^*_{\rho,g}(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} g(r,x)\mathbf{Q}_\rho((r,x),(s,y))\mathrm{d}\nu_n(r,x), \tag{4.1}$$

where

$$\begin{aligned}\mathbf{Q}_\rho((r,x),(s,y)) &= \iint_{\Gamma^+} \frac{e^{-t(\mu^2+2|\lambda|^2)}\overline{\varphi_{\mu,\lambda}(r,x)}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda)\\ &= \overline{\mathcal{W}_t(\mathcal{K}_{\alpha,\rho}((\cdot,\cdot),(s,y)))(r,x)}\end{aligned} \tag{4.2}$$

Proof The existence and uniqueness of the extremal function $f^*_{\rho,g}$ is given by ([10, 13, 14]). On the other hand, we have

$$f^*_{\rho,g}(s,y) = \langle g|\mathcal{W}_t\Big(\mathcal{K}_{\alpha,\rho}\big((\cdot,\cdot),(s,y)\big)\Big)\rangle_{\nu_n},$$

where $\mathcal{K}_{\alpha,\rho}$ is the reproducing kernel given by relation (3.12). Thus, from relation (3.14); we obtain

$$\begin{aligned}f^*_{\rho,g}(s,y) &= \langle g|\mathcal{F}^{-1}\Big(\frac{\varphi_{\mu,\lambda}(s,y)e^{-t(\mu^2+2|\lambda|^2)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\Big)\rangle_{\nu_n}\\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} g(r,x)\Big(\iint_{\Gamma^+}\frac{e^{-t(\mu^2+2|\lambda|^2)}\varphi_{\mu,\lambda}(s,y)\overline{\varphi_{\mu,\lambda}(r,x)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda)\Big)\mathrm{d}\nu_n(r,x)\\ &= \int_0^{+\infty}\int_{\mathbb{R}^n} g(r,x)\mathbf{Q}_\rho((r,x),(s,y))\mathrm{d}\nu_n(r,x),\end{aligned}$$

where

$$\mathbf{Q}_\rho((r,x),(s,y)) = \iint_{\Gamma^+} \frac{e^{-t(\mu^2+2|\lambda|^2)}\overline{\varphi_{\mu,\lambda}(r,x)}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha + e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda).$$

Corollary 4.1 Let $\alpha > \frac{2n+1}{2}$, $\rho > 0$, and $g \in L^2(\mathrm{d}\nu_n)$. The extremal function $f^*_{\rho,g}$ satisfies the inequality

$$\big\|f^*_{\rho,g}\big\|^2_{2,\nu_n} \le \frac{1}{4\rho}\frac{\Gamma(\alpha-n-\frac{1}{2})}{2^{2n+1}\Gamma(\alpha)}\int_0^{+\infty}\int_{\mathbb{R}^n} e^{r^2+|x|^2}|g(r,x)|^2\mathrm{d}\nu_n(r,x).$$

Proof From relation (4.1), for $g \in L^2(\mathrm{d}\nu_n)$, we have

$$f^*_{\rho,g}(s,y) = \int_0^{+\infty}\int_{\mathbb{R}^n} e^{-\frac{r^2+|x|^2}{2}}e^{\frac{r^2+|x|^2}{2}}g(r,x)\mathbf{Q}_\rho((r,x),(s,y))\mathrm{d}\nu_n(r,x).$$

By Hölder inequality, we get

$$\begin{aligned}\big|f^*_{\rho,g}(s,y)\big|^2 \le &\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} e^{-(r^2+|x|^2)}\mathrm{d}\nu_n(r,x)\Big)\\ &\times\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} e^{r^2+|x|^2}\big|g(r,x)\big|^2\big|\mathbf{Q}_\rho((r,x),(s,y))\big|^2\mathrm{d}\nu_n(r,x)\Big).\end{aligned}$$

Integrating over $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $\mathrm{d}\nu_n(s,y)$, and using Fubini theorem, we obtain

$$\begin{aligned}\big\|f^*_{\rho,g}\big\|^2_{2,\nu_n} \le &\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} e^{-(r^2+|x|^2)}\mathrm{d}\nu_n(r,x)\Big)\\ &\times\Big(\int_0^{+\infty}\int_{\mathbb{R}^n} e^{r^2+|x|^2}\,|g(r,x)|^2\,\|\mathbf{Q}_\rho((r,x),(\cdot,\cdot))\|^2_{2,\nu_n}\,\mathrm{d}\nu_n(r,x)\Big).\end{aligned} \tag{4.3}$$

However, by (4.2), we have

$$\begin{aligned}\mathbf{Q}_\rho((r,x),(s,y)) &= \mathcal{F}^{-1}\left(\frac{e^{-t(\mu^2+2|\lambda|^2)}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\right)(r,x)\\ &= \mathcal{F}^{-1}\left(\frac{e^{-t(\mu^2+2|\lambda|^2)}\varphi_{\mu,\lambda}(r,x)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\right)(s,y),\end{aligned}$$

then, by Plancherel theorem and the relation (2.2),

$$\begin{aligned}\|\mathbf{Q}_\rho((r,x);(\cdot,\cdot))\|^2_{2,\nu_n} &= \iint_{\Gamma^+}\frac{e^{-2t(\mu^2+2|\lambda|^2)}\left|\varphi_{\mu,\lambda}(r,x)\right|^2}{\left|\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}\right|^2}\mathrm{d}\gamma_n(\mu,\lambda)\\ &\le \iint_{\Gamma^+}\frac{e^{-2t(\mu^2+2|\lambda|^2)}}{\left|\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}\right|^2}\mathrm{d}\gamma_n(\mu,\lambda).\end{aligned}$$

But, for any $a,b>0$, $\frac{1}{(a^2+b^2)^2}\le\frac{1}{4a^2b^2}$. Thus, by relation (2.4), we deduce that, for all $(r,x)\in[0,+\infty[\times\mathbb{R}^n$,

$$\|\mathbf{Q}_\rho((r,x);(\cdot,\cdot))\|^2_{2,\nu_n}\le\frac{1}{4\rho}\iint_{\Gamma^+}\frac{\mathrm{d}\gamma_n(\mu,\lambda)}{(1+\mu^2+2|\lambda|^2)^\alpha}\le\frac{1}{4\rho}\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{\mathrm{d}\nu_n(u,v)}{(1+u^2+|v|^2)^\alpha}.\quad(4.4)$$

We complete the proof by using relations (4.3), (4.4), and the fact that

$$\int_0^{+\infty}\int_{\mathbb{R}^n}e^{-(r^2+|x|^2)}\mathrm{d}\nu_n(r,x)=\frac{1}{2^{n+\frac12}}$$

and

$$\int_0^{+\infty}\int_{\mathbb{R}^n}\frac{\mathrm{d}\nu_n(r,x)}{(1+r^2+|x|^2)^\alpha}=\frac{\Gamma(\alpha-n-\frac12)}{2^{n+\frac12}\Gamma(\alpha)}.$$

Corollary 4.2 Let $\alpha>\frac{2n+1}{2}$. Then, for all $g_1,g_2\in L^2(\mathrm{d}\nu_n)$, we have

$$\left\|f^*_{\rho,g_1}-f^*_{\rho,g_2}\right\|_\alpha\le\frac{\|g_1-g_2\|_{2,\nu_n}}{2\sqrt{\rho}}.$$

Proof From relation (2.2), it follows that, for all $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the function

$$(\mu,\lambda)\longmapsto\frac{e^{-t(\mu^2+2|\lambda|^2)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\varphi_{\mu,\lambda}(r,x)$$

belongs to $L^1(\mathrm{d}\gamma_n)\cap L^2(\mathrm{d}\gamma_n)$.

From relation (4.1) and the fact that

$$\mathbf{Q}_\rho((r,x),(s,y))=\mathcal{F}^{-1}\left(\frac{e^{-t(\mu^2+2|\lambda|^2)}\varphi_{\mu,\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\right)(r,x),$$

we deduce that, for all $g\in L^2(\mathrm{d}\nu_n)$ and $(s,y)\in[0,+\infty[\times\mathbb{R}^n$, we have

$$f^*_{\rho,g}(s,y)=\int_0^{+\infty}\int_{\mathbb{R}^n}g(r,x)\overline{\mathcal{F}^{-1}\left(\frac{e^{-t(\mu^2+2|\lambda|^2)}\varphi_{\mu,-\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\right)(r,x)}\mathrm{d}\nu_n(r,x).$$

Applying Parseval equality, we obtain

$$\begin{aligned}f^*_{\rho,g}(s,y) &= \iint_{\Gamma^+}\mathcal{F}(g)(\mu,\lambda)\frac{e^{-t(\mu^2+2|\lambda|^2)}\varphi_{\mu,-\lambda}(s,y)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda)\\ &= \mathcal{F}^{-1}\left(\mathcal{F}(g)(\mu,\lambda)\frac{e^{-t(\mu^2+2|\lambda|^2)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\right)(s,y),\end{aligned}$$

which implies that

$$\mathcal{F}(f^*_{\rho,g})(\mu,\lambda)=\mathcal{F}(g)(\mu,\lambda)\frac{e^{-t(\mu^2+2|\lambda|^2)}}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}, \tag{4.5}$$

then, for any $g_1,\ g_2\in L^2(\mathrm{d}\nu_n)$,

$$\left\|f^*_{\rho,g_1}-f^*_{\rho,g_2}\right\|^2_\alpha=\iint_{\Gamma+}\frac{(1+\mu^2+2|\lambda|^2)^\alpha e^{-2t(\mu^2+2|\lambda|^2)}\left|\mathcal{F}(g_1-g_2)(\mu,\lambda)\right|^2}{\left(\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}\right)^2}\mathrm{d}\gamma_n(\mu,\lambda).$$

From the fact that

$$\frac{1}{\left(\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}\right)^2}\ \leq\frac{1}{4\rho(1+\mu^2+2|\lambda|^2)^\alpha e^{-2t(\mu^2+2|\lambda|^2)}}$$

and the Plancherel formula, we deduce that

$$\left\|f^*_{\rho,g_1}-f^*_{\rho,g_2}\right\|^2_\alpha\leq\frac{1}{4\rho}\iint_{\Gamma+}\left|\mathcal{F}(g_1-g_2)(\mu,\lambda)\right|^2\mathrm{d}\gamma_n(\mu,\lambda)=\frac{1}{4\rho}\left\|g_1-g_2\right\|^2_{2,\nu_n}.$$

Corollary 4.3 Let $\alpha>\frac{2n+1}{2}$. Then, for every $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ and $g=\mathcal{W}_t(f)$, we have

$$\lim_{\rho\longrightarrow0^+}\|f^*_{\rho,g}-f\|_\alpha=0.$$

Moreover, $\left(f^*_{\rho,g}\right)_{\rho>0}$ converges uniformely to f as $\rho\longrightarrow0$.

Proof Let $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$ and $g=\mathcal{W}_t(f)$. From proposition 3.2, we deduce that the function g belongs to $L^2(\mathrm{d}\nu_n)$, and by relations (3.15) and (4.5), we obtain

$$\mathcal{F}(f^*_{\rho,g}-f)(\mu,\lambda)=\frac{-\rho(1+\mu^2+2|\lambda|^2)^\alpha\mathcal{F}(f)(\mu,\lambda)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}. \tag{4.6}$$

Consequently,

$$\left\|f^*_{\rho,g}-f\right\|^2_\alpha=\iint_{\Gamma+}\frac{\rho^2(1+\mu^2+2|\lambda|^2)^{3\alpha}}{\left(\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}\right)^2}\left|\mathcal{F}(f)(\mu,\lambda)\right|^2\mathrm{d}\gamma_n(\mu,\lambda).$$

In contrast, for any $\rho>0$ and $(\mu,\lambda)\in\Gamma^+$, we have

$$\frac{\rho^2(1+\mu^2+2|\lambda|^2)^{3\alpha}}{\left(\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}\right)^2}\left|\mathcal{F}(f)(\mu,\lambda)\right|^2\leq(1+\mu^2+2|\lambda|^2)^\alpha\left|\mathcal{F}(f)(\mu,\lambda)\right|^2.$$

Then, using the dominate convergence theorem and the fact that $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$, we deduce that

$$\lim_{\rho\longrightarrow0^+}\left\|f^*_{\rho,g}-f\right\|_\alpha=0.$$

By relation (4.6), we deduce that for any $(\mu,\lambda)\in\Gamma^+$, we have

$$\left|\mathcal{F}\big(f^*_{\rho,g}-f\big)(\mu,\lambda)\right|\leq\left|\mathcal{F}(f)(\mu,\lambda)\right|.$$

Using Remark 2.1 and the fact that $f\in\mathbf{H}_\alpha\big([0,+\infty[\times\mathbb{R}^n\big)$, we deduce that the function

$$(\mu,\lambda)\longmapsto\mathcal{F}\big(f^*_{\rho,g}-f\big)(\mu,\lambda)$$

belongs to $L^1(\mathrm{d}\gamma_n)\cap L^2(\mathrm{d}\gamma_n)$.

Thus, by relation (4.6), it follows that

$$(f^*_{\rho,g}-f)(r,x)=\iint_{\Gamma^+}\frac{-\rho(1+\mu^2+2|\lambda|^2)^\alpha\mathcal{F}(f)(\mu,\lambda)\varphi_{\mu,-\lambda}(r,x)}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda).$$

Again by (2.2) and for any $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, we have

$$\big|(f^*_{\rho,g}-f)(r,x)\big|\le\iint_{\Gamma^+}\frac{\rho(1+\mu^2+2|\lambda|^2)^\alpha\big|\mathcal{F}(f)(\mu,\lambda)\big|}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\mathrm{d}\gamma_n(\mu,\lambda).$$

So, by the dominate convergence theorem and the fact that, for every $(\mu,\lambda)\in\Gamma^+$,

$$\frac{\rho(1+\mu^2+2|\lambda|^2)^\alpha\left|\mathcal{F}(f)(\mu,\lambda)\right|}{\rho(1+\mu^2+2|\lambda|^2)^\alpha+e^{-2t(\mu^2+2|\lambda|^2)}}\le\left|\mathcal{F}(f)(\mu,\lambda)\right|,$$

we deduce that

$$\sup_{(r,x)\in[0,+\infty[\times\mathbb{R}^n}\big|(f^*_{\rho,g}-f)(r,x)\big|\longrightarrow 0,\quad\text{as}\quad\rho\longrightarrow 0^+.$$

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