Real Analysis Qualifying Examination
Fall 2006
September 16, 2006, 10 am -12.30 pm

Instructions: Do all problems. Use only one side of each page. Write your name on each page. Do at most one problem on each page. Justify your answers. Where appropriate, state without proof, results that you use in your solutions.

1. Given a measure space $(X, \mathfrak{A}, \mu)$, let f be a nonnegative extended real-valued $\mathfrak{A}$-measurable function on a set $D \in \mathfrak{A}$ with $\mu(D) < \infty$. Let $D_n = \{x \in D : f(x) \geq n\}$ for $n \in \mathbb{N}$. Show that $\int_D f \, d\mu < \infty$ if and only if $\sum_{n \in \mathbb{N}} \mu(D_n) < \infty$.

2. Let $(X, \mathfrak{A}, \mu)$ be a σ-finite measure space and let f and g be extended real-valued $\mathfrak{A}$-measurable functions on X. Show that if $\int_E f \, d\mu = \int_E g \, d\mu$ for every $E \in \mathfrak{A}$ then $f = g$ a.e. on X.

3. Let f be an integrable function on a measure space $(X, \mathcal{B}, \mu)$. Show that:

   a) The set $\{f \neq 0\}$ is of σ-finite measure.

   b) If $f \geq 0$, then $f = \lim_n \phi_n$ pointwise for some increasing sequence of simple functions ϕ_n each of which vanishes outside a set of finite measure.

   c) For every $\varepsilon > 0$ there is a simple function ϕ such that

$$\int_X |f - \phi| \, d\mu < \varepsilon$$

4. Let $\{f_n\}$ and $\{g_n\}$ be sequences of measurable functions on the measurable set E, and let f and g be measurable functions on set E. Suppose $f_n \to f$ and $g_n \to g$ in measure. Is it true that $f_n^3 + g_n \to f^3 + g$ in measure if

   - $m(E) = 2$
   - $m(E) = \infty$

5. Suppose $f \in L^1[0,1]$. Let $F(x) = \int_0^x f(t)dt$. Let ϕ be a Lipshitz function (i.e. $|\phi(x) - \phi(y)| \leq M|x - y|$ for some $M < \infty$). Show that there exists $g \in L^1[0,1]$ such that $\phi(F(x)) = \int_0^x g(t)dt$.

6.
   - Suppose $f \in L^p[0,1]$, $p > 1$. Show that $\lim_{t \to 0} t^{-1+\frac{1}{p}} \int_0^t f(x)dx = 0$.
   - Suppose $\int_0^1 x^{-1}|f|^3 dx < \infty$. Show that $\lim_{t \to 0} t^{-1} \int_0^t f(x)dx = 0$.