Propositional fuzzy logics based on Frank t-norms: A comparison

Erich Peter Klement *
Department of Algebra, Stochastics and Knowledge-Based Mathematical Systems
Johannes Kepler University, A-4040 Linz, Austria

Mirko Navara †
Center for Machine Perception, Faculty of Electrical Engineering
Czech Technical University, CZ-166 27 Praha, Czech Republic

Abstract

Among various approaches to fuzzy logics, we have chosen two of them, which are built up in a similar way. Although starting from different basic logical connectives, they both use interpretations based on Frank t-norms. Different interpretations of the implication lead to different axiomatizations, but most logics studied here are complete. We compare the properties, advantages and disadvantages of the two approaches. We deal also with logics containing infinitary conjunctions, and we show that they are semantically "stronger" than all the other logics studied in this paper.

Key words: Fuzzy logic, many-valued logic, Frank t-norm

1 Frank t-norms

Triangular norms were introduced in the framework of probabilistic metric spaces [33, 32, 34], based on ideas first presented in [22], and they are applied in several fields, e.g., in fuzzy sets [35], fuzzy logics [2, 14, 28] and their applications, but also in the theory of generalized measures [1, 19] and nonlinear differential and difference equations [27].

A *triangular norm* (*t-norm* for short) is a commutative, associative, non-decreasing operation $T : [0,1]^2 \to [0,1]$ with neutral element 1. An immediate consequence of the commutativity, the monotonicity and the boundary

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condition is that, for each t-norm T and for each $x \in [0,1]$, the following boundary conditions are also fulfilled:

$$T(x,1) = T(1,x) = x, \tag{1}$$
$$T(x,0) = T(0,x) = 0, \tag{2}$$

which means that all t-norms coincide on the boundary of the unit square $[0,1]^2$. Algebraically speaking, $([0,1],T)$ is a commutative, linearly ordered semigroup with neutral element 1 and annihilator 0.

Three important t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ given, respectively, by $T_{\mathbf{M}}(x,y) = \min(x,y)$, $T_{\mathbf{P}}(x,y) = xy$ and $T_{\mathbf{L}}(x,y) = \max(0, x+y-1)$.

A *triangular conorm* (*t-conorm* for short) is a commutative, associative, non-decreasing operation $S : [0,1]^2 \to [0,1]$ with $S(x,0) = x$ for all $x \in [0,1]$, i.e., $([0,1],S)$ is a commutative, linearly ordered semigroup with neutral element 0 and annihilator 1.

There is a strong duality between t-norms and t-conorms. Let $N : [0,1] \to [0,1]$ be a *strong (fuzzy) negation*, i.e., an order-reversing involution. For a t-norm T, the function $S_{T,N} : [0,1]^2 \to [0,1]$ given by

$$S_{T,N}(x,y) = N(T(N(x),N(y)))$$

is a t-conorm, called the *N-dual of T*. Dually, for a t-conorm S, the function $T_{S,N} : [0,1]^2 \to [0,1]$ given by

$$T_{S,N}(x,y) = N(S(N(x),N(y)))$$

is a t-norm, called the *N-dual of S*. Moreover, we have $T_{S_{T,N},N} = T$ and $S_{T_{S,N},N} = S$.

If, in particular, we use the *standard (fuzzy) negation* $N_{\mathbf{s}} : [0,1] \to [0,1]$ defined by

$$N_{\mathbf{s}}(x) = 1 - x, \tag{3}$$

then the $N_{\mathbf{s}}$-duals of T and S are simply called *duals* thereof.

The duals of the three important t-norms are the maximum $S_{\mathbf{M}}$, the probabilistic sum $S_{\mathbf{P}}$ and the bounded sum $S_{\mathbf{L}}$ given, respectively, by $S_{\mathbf{M}}(x,y) = \max(x,y)$, $S_{\mathbf{P}}(x,y) = x + y - xy$ and $S_{\mathbf{L}}(x,y) = \min(1, x+y)$.

The family $(T_\lambda)_{\lambda \in [0,\infty]}$ of *Frank t-norms* is given by

$$T_\lambda(x,y) = \begin{cases} T_{\mathbf{M}}(x,y) & \text{if } \lambda = 0, \\ T_{\mathbf{P}}(x,y) & \text{if } \lambda = 1, \\ T_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty, \\ \log_\lambda \left(1 + \dfrac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right) & \text{otherwise.} \end{cases}$$

The family $(S_\lambda)_{\lambda\in[0,\infty]}$ of *Frank t-conorms* is given by

$$S_\lambda(x,y) = \begin{cases} S_{\mathbf{M}}(x,y) & \text{if } \lambda = 0, \\ S_{\mathbf{P}}(x,y) & \text{if } \lambda = 1, \\ S_{\mathbf{L}}(x,y) & \text{if } \lambda = \infty, \\ 1 - \log_\lambda \left(1 + \dfrac{(\lambda^{1-x}-1)(\lambda^{1-y}-1)}{\lambda - 1}\right) & \text{otherwise.} \end{cases}$$

In [18], the Frank t-norms and t-conorms, are denoted by $T_\lambda^{\mathbf{F}}$ and $S_\lambda^{\mathbf{F}}$, respectively. As we do not work here with other families of t-norms and t-conorms, we omit the upper index **F** throughout this paper.

An element $x \in \,]0,1]$ is called a *zero divisor* of a t-norm T if there is some $y \in \,]0,1]$ with $T(x,y) = 0$. A continuous t-norm T is called *Archimedean* if $T(x,x) < x$ for all $x \in \,]0,1[$. A continuous Archimedean t-norm is called *nilpotent* if it has at least one zero divisor $x > 0$, and *strict* otherwise. The minimum $T_{\mathbf{M}} = T_0$ is the only Frank t-norm which is not Archimedean, and the Łukasiewicz t-norm $T_{\mathbf{L}} = T_\infty$ is the only nilpotent Frank t-norm; all the other Frank t-norms are strict.

The family of Frank t-norms $(T_\lambda)_{\lambda\in[0,\infty]}$ is strictly decreasing, and the family of Frank t-conorms $(S_\lambda)_{\lambda\in[0,\infty]}$ is strictly increasing with respect to the parameter λ (see [1]). Both families are continuous with respect to λ, i.e., for all $\lambda_0 \in [0,\infty]$

$$\lim_{\lambda\to\lambda_0} T_\lambda = T_{\lambda_0},$$
$$\lim_{\lambda\to\lambda_0} S_\lambda = S_{\lambda_0}.$$

For each $\lambda \in [0,\infty]$, the Frank t-norm T_λ and the Frank t-conorm S_λ are dual to each other, and they solve the functional equation

$$T(x,y) + S(x,y) = x + y. \tag{4}$$

It was shown in [9] that, together with their ordinal sums (see [34]), these are the only pairs of continuous t-norms and t-conorms solving the functional equation (4). Extensive overviews on Frank and other t-norms can be found in [18, 34].

2 Propositional fuzzy logics based on Frank t-norms

A many-valued logic with a continuum of truth values modelled by the unit interval $[0,1]$ is quite often called a *fuzzy logic*. In such a logic, conjunction and disjunction are usually interpreted by a t-norm and its dual t-conorm, respectively. A way to construct propositional logics within this framework was presented in [2], where the full details and proofs of most of the theorems can be found. Further results are proved in [15].

2.1 Basic definitions

To start with, let T_λ be the Frank t-norm with index $\lambda \in [0, \infty]$, $N_\mathbf{s}$ the standard negation given by (3), and S_λ the Frank t-conorm dual to T_λ.

A *t-norm-based propositional fuzzy logic* [2] (*S-fuzzy logic* for short) is defined, for each $\lambda \in [0, \infty]$, as an ordered pair $\mathcal{S}_\lambda = (\mathcal{L}, \mathcal{Q}_\lambda)$ of a *language* (*syntax*) $\mathcal{L}$ and a *structure* (*semantics*) $\mathcal{Q}_\lambda$ described as follows:

(i) The language of $\mathcal{S}_\lambda$ is a pair $\mathcal{L} = (A, (\neg, \wedge))$, where A is an at most countable set of *atomic symbols* and $\neg$ and $\wedge$ are *connectives* which, as usual, are called *negation* and *conjunction*, respectively.

(ii) The structure of $\mathcal{S}_\lambda$ is a pair $\mathcal{Q}_\lambda = ([0,1], (N_\mathbf{s}, T_\lambda))$, where $[0,1]$ is the set of *truth values*, and $N_\mathbf{s}$ and T_λ are the *interpretations* of the negation $\neg$ and the conjunction $\wedge$, respectively.

For simplicity, we fix the set A of atomic symbols throughout this paper. All S-fuzzy logics $\mathcal{S}_\lambda$ have the same syntax, they differ only by the semantics, so there is no need to index the language $\mathcal{L}$ by the parameter λ.

The logics corresponding to the basic t-norms will play a special role. For $\lambda = 0$, we obtain the *min-max S-fuzzy logic*, $\mathcal{S}_0 = \mathcal{S}_\mathbf{M}$. For $\lambda = \infty$, we obtain the *Łukasiewicz S-fuzzy logic*, $\mathcal{S}_\infty = \mathcal{S}_\mathbf{L}$. In these cases, we use the indices **M** and **L** also for the corresponding structures, etc.

The class $\mathcal{F}_\mathcal{S}$ of *well-formed formulas* in an S-fuzzy logic ($\mathcal{S}$-*formulas* for short) is defined inductively as follows:

(i) Each atomic symbol $p \in A$ is an $\mathcal{S}$-formula.

(ii) If φ is an $\mathcal{S}$-formula, then $\neg\varphi$ is an $\mathcal{S}$-formula.

(iii) If φ and ψ are $\mathcal{S}$-formulas, then $\varphi \wedge \psi$ is an $\mathcal{S}$-formula.

Since the class $\mathcal{F}_\mathcal{S}$ of well-formed formulas in $\mathcal{S}_\lambda$ is independent of λ, we omit this index.

For each function $t : A \to [0,1]$, there exists always a unique *natural extension* of t to a *truth assignment* $\bar{t}_{\mathcal{S}_\lambda} : \mathcal{F}_\mathcal{S} \to [0,1]$ which, for all atomic symbols p and for all $\mathcal{S}$-formulas φ and ψ, is obtained by induction in the following canonical way:

$$\begin{aligned} \bar{t}_{\mathcal{S}_\lambda}(p) &= t(p) \\ \bar{t}_{\mathcal{S}_\lambda}(\neg\varphi) &= N_\mathbf{s}(\bar{t}_{\mathcal{S}_\lambda}(\varphi)), \\ \bar{t}_{\mathcal{S}_\lambda}(\varphi \wedge \psi) &= T_\lambda(\bar{t}_{\mathcal{S}_\lambda}(\varphi), \bar{t}_{\mathcal{S}_\lambda}(\psi)). \end{aligned}$$

2.2 Derived connectives and compactness

Starting with the basic logical connectives $\neg$ and $\wedge$, we can define additional logical connectives in an S-fuzzy logic $\mathcal{S}_\lambda$. The *disjunction* $\vee$ in $\mathcal{S}_\lambda$ is defined using the de Morgan formula

$$\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi).$$

For the interpretation of the disjunction, we obtain

$$\overline{t}_{\mathcal{S}_\lambda}(\varphi \vee \psi) = N_{\mathbf{s}}(T_\lambda(N_{\mathbf{s}}(\overline{t}_{\mathcal{S}_\lambda}(\varphi)), N_{\mathbf{s}}(\overline{t}_{\mathcal{S}_\lambda}(\psi)))) = S_\lambda(\overline{t}_{\mathcal{S}_\lambda}(\varphi), \overline{t}_{\mathcal{S}_\lambda}(\psi)),$$

so the disjunction $\vee$ is interpreted by the t-conorm S_λ dual to T_λ.

The *implication* $\rightarrow$ in $\mathcal{S}_\lambda$ is defined as

$$\varphi \rightarrow \psi = \neg(\varphi \wedge \neg\psi).$$

This is one of numerous formulas which are equivalent to the (unique) implication in the classical logic. In fuzzy logics, these formulas are not necessarily equivalent, hence the choice of implication becomes important. For the interpretation of the implication, we obtain

$$\overline{t}_{\mathcal{S}_\lambda}(\varphi \rightarrow \psi) = N_{\mathbf{s}}(T_\lambda(\overline{t}_{\mathcal{S}_\lambda}(\varphi), N_{\mathbf{s}}(\overline{t}_{\mathcal{S}_\lambda}(\psi)))) = S_\lambda(N_{\mathbf{s}}(\overline{t}_{\mathcal{S}_\lambda}(\varphi)), \overline{t}_{\mathcal{S}_\lambda}(\psi)).$$

Thus the logical implication $\rightarrow$ is interpreted by the binary operation $I_\lambda : [0,1]^2 \rightarrow [0,1]$ defined by

$$I_\lambda(x,y) = S_\lambda(N_{\mathbf{s}}(x), y),$$

which is often called the *S-implication* induced by the t-norm T_λ. This notion is the main reason why we call the corresponding logic an S-fuzzy logic. Notice that, for all $\lambda \in [0, \infty[$,

$$I_\lambda(x,y) = 1 \qquad \text{if and only if} \qquad (x = 0 \text{ or } y = 1). \tag{5}$$

Only for the Łukasiewicz S-implication $I_{\mathbf{L}} = I_\infty$ we have

$$I_{\mathbf{L}}(x,y) = 1 \qquad \text{if and only if} \qquad x \leq y. \tag{6}$$

An important feature of S-fuzzy logics $\mathcal{S}_\lambda$ is that they have the *compactness property*. In order to formulate this properly, the following notions are helpful. For $\Gamma \subseteq \mathcal{F}_{\mathcal{S}}$ and $K \subseteq [0,1]$, we say that Γ is *K-satisfiable* in $\mathcal{S}_\lambda$ if there exists a truth assignment $\overline{t}_{\mathcal{S}_\lambda}$ such that we have $\overline{t}_{\mathcal{S}_\lambda}(\varphi) \in K$ whenever $\varphi \in \Gamma$. The set Γ is said to be *finitely K-satisfiable* in $\mathcal{S}_\lambda$ if each finite subset of Γ is K-satisfiable in $\mathcal{S}_\lambda$.

We then get the following results (see [2, Theorem 3.3, Proposition 3.6]):

Theorem 1 *Let $\lambda \in [0, \infty]$ and let $\mathcal{S}_\lambda$ be an S-fuzzy logic. Then for each $\Gamma \subseteq \mathcal{F}_{\mathcal{S}}$, for each closed subset K of $[0,1]$ and for each $r \in [0,1]$ we have:*

(i) The set Γ is K-satisfiable in $\mathcal{S}_\lambda$ if and only if it is finitely K-satisfiable in $\mathcal{S}_\lambda$.

(ii) If Γ is $\{r\}$-satisfiable in $\mathcal{S}_\lambda$, then there exists a maximal number $r^* \in [0,1]$ such that Γ is $\{r^*\}$-satisfiable in $\mathcal{S}_\lambda$.

2.3 Axiomatization and deduction

In analogy to the classical two-valued logic, an $\mathcal{S}$-formula φ is said to be a *logical axiom* if, for some $\alpha, \beta, \gamma \in \mathcal{F}_{\mathcal{S}}$, φ has one of the following forms:

[C1] $\alpha \to (\beta \to \alpha)$,

[C2] $[\alpha \to (\beta \to \gamma)] \to [(\alpha \to \beta) \to (\alpha \to \gamma)]$,

[C3] $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$.

Remark 2 Since the implication is considered a derived logical connective in $\mathcal{S}_\lambda$, its use in the axioms should be avoided. To be precise, we should rewrite the axioms [C1]–[C3] using only the basic connectives $\neg$ and $\wedge$. However, this would lead to expressions which may be not familiar to the readers. The form of axioms makes no difference in the notions depending on the axiomatic system, so we use the standard form of axioms known from the classical logic, which was also used in [2].

A set $\Gamma \subseteq \mathcal{F}_{\mathcal{S}}$ is said to be *closed under modus ponens* if we have $\psi \in \Gamma$ whenever $\varphi \in \Gamma$ and $\varphi \to \psi \in \Gamma$ (where $\varphi \to \psi = \neg(\varphi \wedge \neg\psi)$). The *closure* of a set $\Gamma \subseteq \mathcal{F}_{\mathcal{S}}$ under modus ponens is then the smallest subset of $\mathcal{F}_{\mathcal{S}}$ containing Γ and being closed under modus ponens.

An $\mathcal{S}$-formula φ is called an $\mathcal{S}$-*theorem* if it belongs to the closure of the set of all logical axioms under modus ponens. This notion is the same in all S-fuzzy logics $\mathcal{S}_\lambda$, so it is not indexed by λ.

A *theory* $\mathcal{T}$ in an S-fuzzy logic $\mathcal{S}_\lambda$, $\lambda \in [0, \infty]$, is a set of $\mathcal{S}$-formulas. An $\mathcal{S}$-formula is called $\mathcal{S}$-*provable* in $\mathcal{T}$ (in symbols $\mathcal{T} \vdash_{\mathcal{S}} \varphi$) if it belongs to the closure of the union of $\mathcal{T}$ and the set of all axioms under modus ponens. This notion is independent of the choice of the particular S-fuzzy logic $\mathcal{S}_\lambda$. For all S-fuzzy logics, we have the classical deduction theorem:

Theorem 3 *Let $\lambda \in [0, \infty]$, $\mathcal{T}$ be a theory in the S-fuzzy logic $\mathcal{S}_\lambda$, and let φ, ψ be $\mathcal{S}$-formulas. Then we have*

$$\mathcal{T} \cup \{\varphi\} \vdash_{\mathcal{S}} \psi \qquad \textit{if and only if} \qquad \mathcal{T} \vdash_{\mathcal{S}} \varphi \to \psi.$$

2.4 Completeness theorems

In any of the S-fuzzy logics $\mathcal{S}_\lambda$, a truth assignment $\bar{t}_{\mathcal{S}_\lambda}$ evaluates some $\mathcal{S}$-theorems, even the axioms, by values less than 1. Therefore, in order to achieve soundness and completeness of an S-fuzzy logic, the notion of tautology has to be adopted accordingly. We say that an $\mathcal{S}$-formula φ is a *tautology in* $\mathcal{S}_\lambda$ if $\bar{t}_{\mathcal{S}_\lambda}(\varphi) > 0$ for all $t \in [0,1]^A$. Notice that this notion depends on the choice of λ.

Theorem 4 *Let $\lambda \in [0, \infty[$. Then the S-fuzzy logic $\mathcal{S}_\lambda$ is sound and complete, i.e., the set of $\mathcal{S}$-theorems and the set of tautologies in $\mathcal{S}_\lambda$ coincide.*

Remark 5 An analogue of Theorem 4 for the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ does not hold because it is not sound. This follows from the existence of zero divisors of $T_{\mathbf{L}}$. For instance, the formula $(p \wedge p) \vee (\neg p \wedge \neg p)$, where p is an atomic symbol, is an $\mathcal{S}$-theorem which is not a tautology in $\mathcal{S}_{\mathbf{L}}$. Indeed, if we choose $t(p) = 0.5$, we obtain $\bar{t}_{\mathcal{S}_{\mathbf{L}}}((p \wedge p) \vee (\neg p \wedge \neg p)) = 0$.

In the other S-fuzzy logics, even more can be said in terms of validation sets. The *validation set* $V_{\mathcal{S}_\lambda}(\varphi)$ of a given $\mathcal{S}$-formula φ in $\mathcal{S}_\lambda$ is defined as

$$V_{\mathcal{S}_\lambda}(\varphi) = \{\bar{t}_{\mathcal{S}_\lambda}(\varphi) \mid t \in [0,1]^A\}.$$

Proposition 6 *For each $\lambda \in [0,\infty]$, each S-fuzzy logic $\mathcal{S}_\lambda$ and for each $\mathcal{S}$-formula φ, the validation set $V_{\mathcal{S}_\lambda}(\varphi)$ is a closed subinterval of $[0,1]$ such that either $0 \in V_{\mathcal{S}_\lambda}(\varphi)$ or $1 \in V_{\mathcal{S}_\lambda}(\varphi)$.*

An $\mathcal{S}$-formula φ is called an $\mathcal{S}$-*contradiction* if $\neg\varphi$ is an $\mathcal{S}$-theorem, and φ is called an $\mathcal{S}$-*contingency* if it is neither an $\mathcal{S}$-theorem nor an $\mathcal{S}$-contradiction.

For S-fuzzy logics $\mathcal{S}_\lambda$ which are different from the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ we can give a more specific characterization of the validation sets:

Theorem 7 *Let $\lambda \in [0,\infty[$, $\mathcal{S}_\lambda$ be an S-fuzzy logic, φ an $\mathcal{S}$-formula, and $V_{\mathcal{S}_\lambda}(\varphi)$ its validation set. Then we have:*

(i) φ is an $\mathcal{S}$-theorem if and only if, for some $a \in\,]0,1[$, $V_{\mathcal{S}_\lambda}(\varphi) = [a,1]$;

(ii) φ is an $\mathcal{S}$-contradiction if and only if, for some $b \in\,]0,1[$, $V_{\mathcal{S}_\lambda}(\varphi) = [0,b]$;

(iii) φ is an $\mathcal{S}$-contingency if and only if $V_{\mathcal{S}_\lambda}(\varphi) = [0,1]$.

For the min-max S-fuzzy logic $\mathcal{S}_{\mathbf{M}}$, we have an even stronger result (see [2, Corollary 5.3]):

Theorem 8 *Let φ be an $\mathcal{S}$-formula in the min-max S-fuzzy logic $\mathcal{S}_{\mathbf{M}}$, and $V_{\mathcal{S}_{\mathbf{M}}}(\varphi)$ its validation set. Then we have:*

(i) φ is an $\mathcal{S}$-theorem if and only if $V_{\mathcal{S}_{\mathbf{M}}}(\varphi) = [0.5,1]$;

(ii) φ is an $\mathcal{S}$-contradiction if and only if $V_{\mathcal{S}_{\mathbf{M}}}(\varphi) = [0,0.5]$;

(iii) φ is an $\mathcal{S}$-contingency if and only if $V_{\mathcal{S}_{\mathbf{M}}}(\varphi) = [0,1]$.

No analogue of Theorem 7 holds for the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$. Also, for $\lambda \in\,]0,\infty[$ there is no chance for a strengthening of Theorem 7 since, in contrast to Theorem 8(i), we have the following result [15]:

Theorem 9 *Let $\lambda \in\,]0,\infty[$, $\mathcal{S}_\lambda$ be an S-fuzzy logic and let D be the set of all numbers $a \in [0,1]$ such that there exists an $\mathcal{S}$-theorem φ_a with $V_{\mathcal{S}_\lambda}(\varphi_a) = [a,1]$. Then D is a dense subset of $[0,1]$.*

2.5 Infinitary S-fuzzy logics

The defining properties of a t-norm allow us to extend it to an operation with an arbitrary finite arity, and also to an infinitary operation with countably many arguments. In order to use this operation in the interpretation of a fuzzy logic, we modify our definition of an S-fuzzy logic as follows.

For each $\lambda \in [0, \infty]$, the *infinitary t-norm-based propositional fuzzy logic* [2] (*infinitary S-fuzzy logic* for short) $\mathcal{S}_\lambda^*$ is defined analogously to $\mathcal{S}_\lambda$ with the following exceptions: We introduce one new connective, the infinitary conjunction $\bigwedge$ (with countable arity). For a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{S}_\lambda^*$-formulas,

$$\bigwedge_{n\in\mathbb{N}} \varphi_n$$

is an $\mathcal{S}_\lambda^*$-formula. The infinitary conjunction is interpreted by the Frank t-norm T_λ with countably many arguments. Infinitary S-fuzzy logics do not possess some of the properties studied in the previous sections, e.g., the compactness property (Theorem 1). Also the analogues of Proposition 6 and Theorems 7 do not hold for infinitary S-fuzzy logics. We even loose the soundness. Nevertheless, infinitary S-fuzzy logics allow us to produce an interesting comparison of the universality of various t-norms which will be specified in Section 4.3.

Remark 10 The original definition of an infinitary fuzzy logic, presented only for $\lambda \in]0, \infty[$ in [2], differs by introducing one more binary connective $\Rightarrow$ (the crisp implication) with the interpretation

$$I_{\mathbf{c}}(x, y) = \begin{cases} 1 & \text{if } x \le y, \\ 0 & \text{otherwise.} \end{cases}$$

It is proved in [15] that this makes no difference, because a connective with the interpretation $I_{\mathbf{c}}$ can be derived from the negation and the infinitary conjunction.

3 Propositional fuzzy logics with residual implications

In this section, we present another approach to fuzzy logics (described in detail in [13]). A reasonable way of constructing connectives in fuzzy logics is to start with a left continuous t-norm T and to use the *residuum* (*R-implication*, see [4, 8, 7, 12, 30, 31, 29]) defined by

$$R_T(x, y) = \sup \{z \in [0, 1] \mid T(x, z) \le y\}. \tag{7}$$

as the interpretation of the implication. It is immediate that we have, as in (6),

$$R_T(x, y) = 1 \qquad \text{if and only if} \qquad x \le y.$$

Since we restrict our attention again to Frank t-norms T_λ, $\lambda \in [0,\infty]$, we shall write briefly R_λ rather than R_{T_λ}. In contrast to Section 2, the residuum R_λ cannot be substituted by an expression in the t-norm T_λ and other basic fuzzy logical operations. This requires further changes in the definition of a fuzzy logic.

3.1 Basic definitions

A *residuum-based propositional fuzzy logic* (*R-fuzzy logic* for short) is defined, for each $\lambda \in [0,\infty]$, as an ordered pair $\mathcal{R}_\lambda = (\mathcal{L}, \mathcal{Q}_\lambda)$ of a language (syntax) $\mathcal{L}$ and a structure (semantics) $\mathcal{Q}_\lambda$ described as follows:

(i) The language of $\mathcal{R}_\lambda$ is a pair $\mathcal{L} = (A, (\wedge, \rightarrow, \mathbf{0}))$, where A is an at most countable set of atomic symbols and $\wedge, \rightarrow$ and $\mathbf{0}$ are connectives which represent the conjunction, the implication and the (nulary) false statement, respectively.

(ii) The structure of $\mathcal{R}_\lambda$ is a pair $\mathcal{Q}_\lambda = ([0,1], (T_\lambda, R_\lambda, 0))$, where $[0,1]$ is the set of truth values, and T_λ, R_λ and 0 (the latter is the zero constant function) are the interpretations of the conjunction, the implication and the false statement, respectively.

Again, we assume a fixed set A of atomic symbols. All R-fuzzy logics $\mathcal{R}_\lambda$ have the same syntax; we denote by $\mathcal{F}_\mathcal{R}$ the class of well-formed formulas in an R-fuzzy logic ($\mathcal{R}$-*formulas* for short) constructed using the binary connectives $\wedge$ and $\rightarrow$ and the nulary connective $\mathbf{0}$.

The logics corresponding to the t-norms $T_\mathbf{M}, T_\mathbf{L}$ and $T_\mathbf{P}$ are the *Gödel R-fuzzy logic* $\mathcal{R}_0 = \mathcal{R}_\mathbf{M}$, the *Lukasiewicz R-fuzzy logic* $\mathcal{R}_\infty = \mathcal{R}_\mathbf{L}$ and the *product R-fuzzy logic* $\mathcal{R}_1 = \mathcal{R}_\mathbf{P}$. In fact, only these three logics are studied in [13].

Each function $t : A \rightarrow [0,1]$ allows to be extended naturally to a unique truth assignment $\bar{t}_{\mathcal{R}_\lambda} : \mathcal{F}_\mathcal{R} \rightarrow [0,1]$ such that for all atomic symbols p and for all $\mathcal{R}$-formulas φ and ψ:

$$\begin{aligned}
\bar{t}_{\mathcal{R}_\lambda}(p) &= t(p),\\
\bar{t}_{\mathcal{R}_\lambda}(\mathbf{0}) &= 0,\\
\bar{t}_{\mathcal{R}_\lambda}(\varphi \wedge \psi) &= T_\lambda(\bar{t}_{\mathcal{R}_\lambda}(\varphi), \bar{t}_{\mathcal{R}_\lambda}(\psi)),\\
\bar{t}_{\mathcal{R}_\lambda}(\varphi \rightarrow \psi) &= R_\lambda(\bar{t}_{\mathcal{R}_\lambda}(\varphi), \bar{t}_{\mathcal{R}_\lambda}(\psi)).
\end{aligned}$$

3.2 Derived connectives

Using the basic logical connectives $\wedge, \rightarrow$ and $\mathbf{0}$, we can define additional logical connectives in an R-fuzzy logic $\mathcal{R}_\lambda$.

The negation $\neg$ in $\mathcal{R}_\lambda$ is defined as an implication with consequence $\mathbf{0}$, i.e.,

$$\neg\varphi = \varphi \rightarrow \mathbf{0}.$$

Its interpretation is the fuzzy negation N_λ given by

$$N_\lambda(x) = R_\lambda(x, 0).$$

For $\lambda = \infty$, i.e., in the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, we obtain the standard negation, i.e.,

$$N_\infty(x) = N_{\mathbf{s}}(x) = 1 - x.$$

In all the other cases, i.e., for all $\lambda \in [0, \infty[$, we obtain the *Gödel (fuzzy) negation*,

$$N_\lambda(x) = N_{\mathbf{G}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

The Gödel negation is neither continuous nor involutive, so it is not a strong negation. It attains only the crisp truth values 0 and 1. This causes problems in the interpretation of a disjunction.

A disjunction $\vee$ in an R-fuzzy logic $\mathcal{R}_\lambda$ may be defined using the de Morgan formula

$$\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi).$$

Its interpretation is the operation $D_\lambda : [0, 1]^2 \to [0, 1]$ defined by

$$D_\lambda(x, y) = N_\lambda(T_\lambda(N_\lambda(x), N_\lambda(y))).$$

For the Łukasiewicz R-fuzzy logic, we obtain the Łukasiewicz t-conorm, i.e., $D_\infty = S_{\mathbf{L}}$. In all the other cases, i.e., for all $\lambda \in [0, \infty[$, we have

$$D_\lambda(x, y) = N_{\mathbf{G}}(T_\lambda(N_{\mathbf{G}}(x), N_{\mathbf{G}}(y))) = \begin{cases} 0 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases}$$

This operation attains only the crisp truth values 0 and 1, and it does not satisfy the boundary conditions for a t-conorm. So it is not an ideal candidate for a reasonable interpretation of the disjunction. We shall discuss alternative possibilities to define a disjunction in an R-fuzzy logic in Section 4.2.

Satisfiability (as well as finite satisfiability) in R-fuzzy logics is defined analogously to S-fuzzy logics, i.e., for $\Gamma \subseteq \mathcal{F}_{\mathcal{R}}$ and $K \subseteq [0, 1]$ we say that Γ is *K-satisfiable* in $\mathcal{R}_\lambda$ if there exists a truth assignment $\bar{t}_{\mathcal{R}_\lambda}$ such that $\bar{t}_{\mathcal{R}_\lambda}(\varphi) \in K$ whenever $\varphi \in \Gamma$. The set Γ is said to be *finitely K-satisfiable* in $\mathcal{R}_\lambda$ if each finite subset of Γ is K-satisfiable in $\mathcal{R}_\lambda$. In R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in [0, \infty[$, the interpretation of the implication is not continuous, so we cannot prove the compactness property analogously to Theorem 1.

3.3 Deduction

In contrast to S-fuzzy logics, we use the standard definition of tautology (called 1-tautology in [13]) in R-fuzzy logics. We say that an $\mathcal{R}$-formula φ

is a *1-tautology in* $\mathcal{R}_\lambda$ if $\overline{t}_{\mathcal{R}_\lambda}(\varphi) = 1$ for all $t \in [0,1]^A$. As some theorems in the classical logic are not 1-tautologies in $\mathcal{R}_\lambda$, it is necessary to change the logical axioms in order to obtain a sound logic. The notion of 1-tautology in $\mathcal{R}_\lambda$ depends on the choice of λ, hence we need different axiomatizations for different R-fuzzy logics. We shall discuss them in detail in the following sections.

An $\mathcal{R}$-formula φ is called an $\mathcal{R}_\lambda$*-theorem* if it belongs to the closure of the set of axioms of $\mathcal{R}_\lambda$ under modus ponens. The notions of a theory $\mathcal{T}$ in an R-fuzzy logic $\mathcal{R}_\lambda$ and of a formula φ which is $\mathcal{R}_\lambda$-provable in $\mathcal{T}$ (in symbols $\mathcal{T} \vdash_{\mathcal{R}_\lambda} \varphi$) are defined analogously to S-fuzzy logics. The only significant difference is that the notions of $\mathcal{R}_\lambda$-theorem and $\mathcal{R}_\lambda$-provability depend on λ, because we use different axiomatic systems in R-fuzzy logics.

In all R-fuzzy logics $\mathcal{R}_\lambda$, $\lambda \in [0,\infty]$, the following deduction theorem holds:

Theorem 11 *Let $\lambda \in [0,\infty]$, $\mathcal{T}$ be a theory in the R-fuzzy logic $\mathcal{R}_\lambda$, and let φ, ψ be $\mathcal{R}$-formulas. Then we have $\mathcal{T} \cup \{\varphi\} \vdash_{\mathcal{R}_\lambda} \psi$ if and only if there is an $n \in \mathbb{N}$ such that $\mathcal{T} \vdash_{\mathcal{R}_\lambda} \varphi^n \to \psi$, where φ^n, $n \in \mathbb{N}$, is the $\mathcal{R}$-formula defined recursively as follows:*

$$\varphi^1 = \varphi,$$
$$\varphi^{n+1} = \varphi \wedge \varphi^n.$$

3.4 Axiomatization of the Łukasiewicz R-fuzzy logic

Choosing the Łukasiewicz t-norm $T_{\mathbf{L}}$ as the conjunction operator in the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, we obtain the interpretation $R_{\mathbf{L}}$ of the implication defined by

$$R_{\mathbf{L}}(x,y) = \min(1 - x + y, 1).$$

The fact that $R_{\mathbf{L}}$ is just the implication introduced in [21] justifies it to call $T_{\mathbf{L}}$ and $S_{\mathbf{L}}$ the Łukasiewicz t-norm and t-conorm, respectively, although these operations nowhere appear explicitly in the work of Łukasiewicz.

In this case, the R-implication $R_{\mathbf{L}}$ coincides with the S-implication $I_{\mathbf{L}}$, and also the corresponding fuzzy negation coincides with the standard one. So the interpretation of logical connectives in the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ and the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ is identical (although not the same connectives are considered as the basic ones).

Remark 12 There is one more difference between the two Łukasiewicz fuzzy logics $\mathcal{S}_{\mathbf{L}}$ and $\mathcal{R}_{\mathbf{L}}$. The nulary connective $\mathbf{0}$ was not considered an $\mathcal{S}$-formula. Nevertheless, it can be introduced as a derived logical connective putting, e.g., $\mathbf{0} = \neg\varphi \wedge \varphi$ for a fixed $\mathcal{S}$-formula φ. These formulas are semantically equivalent in the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$.

In the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, the compactness theorem may be proved analogously to Theorem 1 (see [13]).

Theorem 13 *Let $\Gamma \subseteq \mathcal{F}_{\mathcal{R}}$, let K be a closed subset of $[0,1]$ and $r \in [0,1]$. The Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ has the following properties:*

(i) The set Γ is K-satisfiable in $\mathcal{R}_{\mathbf{L}}$ if and only if it is finitely K-satisfiable in $\mathcal{R}_{\mathbf{L}}$.

(ii) If Γ is $\{r\}$-satisfiable in $\mathcal{R}_{\mathbf{L}}$, then there exists a maximal number $r^* \in [0,1]$ such that Γ is $\{r^*\}$-satisfiable in $\mathcal{R}_{\mathbf{L}}$.

The Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is axiomatizable; its set of axioms (see [13]) is given as follows:

[A1] $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$,

[A2] $(\alpha \wedge \beta) \to \alpha$,

[A3] $(\alpha \wedge \beta) \to (\beta \wedge \alpha)$,

[A4] $[\alpha \wedge (\alpha \to \beta)] \to [\beta \wedge (\beta \to \alpha)]$,

[A5a] $[\alpha \to (\beta \to \gamma)] \to [(\alpha \wedge \beta) \to \gamma]$,

[A5b] $[(\alpha \wedge \beta) \to \gamma] \to [\alpha \to (\beta \to \gamma)]$,

[A6] $[(\alpha \to \beta) \to \gamma] \to [((\beta \to \alpha) \to \gamma) \to \gamma]$,

[A7] $\mathbf{0} \to \alpha$,

[L4] $[(\alpha \to \beta) \to \beta] \to [(\beta \to \alpha) \to \alpha]$.

The classical deduction theorem (Theorem 3) does not hold in the Łukasiewicz R-fuzzy logic. It is replaced by Theorem 11 which is weaker. The Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is sound and complete, i.e., the set of $\mathcal{R}_{\mathbf{L}}$-theorems and the set of 1-tautologies in $\mathcal{R}_{\mathbf{L}}$ coincide.

Remark 14 The soundness of the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ seems to contradict the non-soundness of the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ (see Remark 5; in Theorem 23 we shall see that these two logics are even "semantically equivalent"). The reason is that the notions of a theorem are different (as a consequence of different axiomatizations) in these two logics, and that the tautologies in $\mathcal{S}_{\mathbf{L}}$ are not necessarily 1-tautologies in $\mathcal{R}_{\mathbf{L}}$.

Remark 15 There is an alternative formulation of the Łukasiewicz fuzzy logic, based only on the implication $\to$ and the false statement $\mathbf{0}$ as basic connectives. The conjunction $\wedge$ is then considered as a derived connective,

$$\varphi \wedge \psi = \neg(\varphi \to \neg\psi).$$

This conjunction is interpreted by the Łukasiewicz t-norm $T_{\mathbf{L}}$, so the interpretation remains the same. In this approach, there is an axiomatization with the following four axioms:

[L1] $\alpha \to (\beta \to \alpha)$,

[L2] $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$,

[L3] $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$,

[L4] $[(\alpha \to \beta) \to \beta] \to [(\beta \to \alpha) \to \alpha]$.

Notice that [L1] and [L3] are just the axioms [C1] and [C3] of the classical logic, respectively, and that [L2] (which is equal to [A1]) is weaker than [C2]. The closure of all axioms of the forms [L1]–[L4] under modus ponens gives exactly all $\mathcal{R}_{\mathbf{L}}$-theorems which do not contain the conjunction $\wedge$.

The corresponding algebraic model of the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is an MV-algebra [17].

3.5 Axiomatization of the Gödel R-fuzzy logic

Choosing the minimum t-norm $T_{\mathbf{M}}$ as the conjunction operator in the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$, we obtain the interpretation $R_{\mathbf{M}}$ of the implication defined by

$$R_{\mathbf{M}}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{M}}$ (called the *Gödel fuzzy implication*) is not continuous in the points (x,x) with $x \in [0,1[$. It gives rise to the Gödel negation $N_{\mathbf{G}}$.

In the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$, we have at least the following compactness theorem (see [13]):

Theorem 16 *A set $\Gamma \subseteq \mathcal{F}_{\mathcal{R}}$ is $\{1\}$-satisfiable in $\mathcal{R}_{\mathbf{G}}$ if and only if it is finitely $\{1\}$-satisfiable in $\mathcal{R}_{\mathbf{G}}$.*

The Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is axiomatizable (see [5, 10, 13]); its axioms are [A1]–[A7] together with

[G] $\alpha \to (\alpha \wedge \alpha)$.

The axioms [A2] and [G] imply that the conjunction must be interpreted by an idempotent operation. The minimum $T_{\mathbf{M}}$ is the only idempotent t-norm, and so it is the only t-norm for the interpretation of a logic with these axioms.

The Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is the only R-fuzzy logic in which the classical deduction theorem (Theorem 3) holds. It is a special case of Theorem 11 (which is also valid for the Gödel R-fuzzy logic), because the conjunction is interpreted by the minimum which is idempotent. The Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is sound and complete.

The corresponding algebraic model of the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ is a Heyting algebra satisfying one additional condition (see [12, 13] for details).

3.6 Axiomatization of the product R-fuzzy logic

Choosing the product t-norm $T_{\mathbf{P}}$ as the conjunction operator in the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$, we obtain the interpretation $R_{\mathbf{P}}$ of the implication defined by

$$R_{\mathbf{P}}(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \dfrac{y}{x} & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{P}}$ (called the *Goguen fuzzy implication*) is not continuous in the point $(0,0)$. It gives rise to the Gödel negation $N_{\mathbf{G}}$, the same as for the Gödel R-fuzzy logic.

It seems to be an open problem whether a compactness theorem analogous to Theorem 16 holds for the product R-fuzzy logic.

The product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ is axiomatizable [14]; its axioms are [A1]–[A7] together with

[P1] $\neg\neg\gamma \to [[(\alpha \wedge \gamma) \to (\beta \wedge \gamma)] \to (\alpha \to \beta)]$,

[P2] $\neg(\alpha \wedge \alpha) \to \neg\alpha$.

The axiom [P1] expresses the validity of the *cancellation law*. So only t-norms satisfying the cancellation law are acceptable candidates for the interpretation of a logic with the axioms [P1] and [P2] (observe that a continuous t-norm satisfies the cancellation law if and only if it is strict).

The product R-fuzzy logic does not satisfy the classical deduction theorem (Theorem 3), only Theorem 11. The product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ is sound and complete.

The corresponding algebraic model of the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ is called a product algebra (see [13, 14]).

3.7 Axiomatization of other R-fuzzy logics

What was said about the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}} = \mathcal{R}_1$, remains essentially valid also for all R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0,\infty[$. Due to the representation theorem for strict t-norms (see, e.g., [20, 25]), there is an automorphism (i.e., an order-preserving bijection) $h_\lambda : [0,1] \to [0,1]$ such that, for all $x, y \in [0,1]$,

$$h_\lambda(T_\lambda(x,y)) = T_{\mathbf{P}}(h_\lambda(x), h_\lambda(y)). \tag{8}$$

The automorphism h_λ represents a change of the scale of the unit interval which transforms T_λ into the product t-norm $T_{\mathbf{P}}$. It transforms the corresponding R-implication R_λ into the Goguen fuzzy implication $R_{\mathbf{P}}$. The Gödel negation $N_{\mathbf{G}}$, however, is preserved under the automorphism h_λ. The whole structure is (up to the change of scale represented by h_λ) exactly the same as in the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$.

Again the problem remains open whether a compactness theorem analogous to Theorem 16 holds for the R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0,\infty[$.

All R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0,\infty[$ are axiomatizable by the same axioms as the product R-fuzzy logic $\mathcal{R}_\mathbf{P}$, i.e., by [A1]–[A7] together with [P1] and [P2]. They do not satisfy the classical deduction theorem (Theorem 3), only Theorem 11, but they are sound and complete.

3.8 Properties of general R-fuzzy logics

Let us summarize some properties which are common for all R-fuzzy logics. The general form of the completeness theorem is as follows:

Theorem 17 *For each $\lambda \in [0,\infty]$, the R-fuzzy logic $\mathcal{R}_\lambda$ is sound and complete, i.e., the set of $\mathcal{R}_\lambda$-theorems and the set of 1-tautologies in $\mathcal{R}_\lambda$ coincide.*

Again, more can be said in terms of validation sets. The validation set $V_{\mathcal{R}_\lambda}(\varphi)$ of a given $\mathcal{R}$-formula φ in $\mathcal{R}_\lambda$ is defined as

$$V_{\mathcal{R}_\lambda}(\varphi) = \{\bar{t}_{\mathcal{R}_\lambda}(\varphi) \mid t \in [0,1]^A\}.$$

Notice that this notion depends on the choice of λ. An analogy of Proposition 6 holds only for the Łukasiewicz R-fuzzy logic. In any other R-fuzzy logic, the implication is interpreted by a noncontinuous operation, and validation sets are not necessarily intervals.

An $\mathcal{R}$-formula φ is called an $\mathcal{R}_\lambda$-contradiction if $\neg\varphi$ is an $\mathcal{R}_\lambda$-theorem, and φ is called an $\mathcal{R}_\lambda$-contingency if it is neither an $\mathcal{R}_\lambda$-theorem nor an $\mathcal{R}_\lambda$-contradiction. In contrast to S-fuzzy logics, these notions depend on λ (because of different axiomatizations). We have the following characterization by the valuation sets:

Theorem 18 *Let $\lambda \in [0,\infty]$, $\mathcal{R}_\lambda$ be an R-fuzzy logic, φ an $\mathcal{R}$-formula, and $V_{\mathcal{R}_\lambda}(\varphi)$ its validation set. Then we have:*

(i) φ is an $\mathcal{R}_\lambda$-theorem if and only if $V_{\mathcal{R}_\lambda}(\varphi) = \{1\}$;

(ii) φ is an $\mathcal{R}_\lambda$-contradiction if and only if $V_{\mathcal{R}_\lambda}(\varphi) = \{0\}$;

(iii) φ is an $\mathcal{R}_\lambda$-contingency if and only if $\{0\} \neq V_{\mathcal{R}_\lambda}(\varphi) \neq \{1\}$.

If, for an $\mathcal{R}$-formula φ and a truth assignment $\bar{t}_{\mathcal{R}_\lambda}$, we have $\bar{t}_{\mathcal{R}_\lambda}(\varphi) \in]0,1[$, then φ is an $\mathcal{R}_\lambda$-contingency. Observe that this condition is not necessary in R-fuzzy logics since there are $\mathcal{R}_\lambda$-contingencies φ with $\bar{t}_{\mathcal{R}_\lambda}(\varphi) \notin]0,1[$ for any truth assignment $\bar{t}_{\mathcal{R}_\lambda}$. For example, if we take $\lambda \in [0,\infty[$ and an arbitrary atomic symbol p, then for the $\mathcal{R}$-formula $\varphi = \neg p$ we obtain $V_{\mathcal{R}_\lambda}(\varphi) = \{0,1\}$.

4 Comparison of the two approaches

During the introduction and discussion of S- and R-fuzzy logics, we already mentioned some of their similarities and differences. We shall summarize this knowledge and add a comparison from other viewpoints.

Both approaches can be formalized in a way similar to the classical logic. They use different sets of logical connectives. The missing basic connectives cannot always be substituted by derived connectives. Different interpretations of the implication cause the main difference in semantics. Both approaches work with logics which are truth functional; the truth assignment is calculated for a compound formula uniquely from the evaluation of its subformulas. The two approaches use the same single deduction rule — modus ponens, but they are based on different axiomatizations and, therefore, they have to work with a different notion of tautology in order to achieve soundness and completeness.

4.1 Advantages and disadvantages of S-fuzzy logics

In an S-fuzzy logic, the basic connectives are the conjunction $\wedge$ and the negation $\neg$. We can derive an implication $\to$ and a disjunction $\vee$, as well as the other usual logical connectives, in analogy to the classical logic. However, the nulary operation $\mathbf{0}$, i.e., the false statement as a constant, can be obtained only in the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ (e.g., as $\neg\varphi \wedge \varphi$ for an arbitrary formula φ). In all other S-fuzzy logics, i.e., in $\mathcal{S}_\lambda$ for $\lambda \in [0,\infty[$, there is no formula which is evaluated to zero by any truth assignment. This disadvantage can be easily eliminated by adding $\mathbf{0}$ as a basic nulary connective. No serious problems arise, only the validation sets may become singletons $\{0\}$ or $\{1\}$ and the formulation of Theorems 7 and 8 has to be generalized to include this case.

In an infinitary S-fuzzy logic with $\lambda \in]0,\infty]$, i.e., which is based on an Archimedean Frank t-norm, the nulary connective $\mathbf{0}$ can be introduced as a derived connective, e.g., as

$$\bigwedge_{n\in\mathbb{N}} \neg p \wedge p,$$

where p is an arbitrary atomic symbol.

In Section 3.7, we argued that all R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0,\infty[$ are equivalent to the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}} = \mathcal{R}_1$ up to an automorphism $h_\lambda : [0,1] \to [0,1]$. This argument does not work in S-fuzzy logics. The automorphism h_λ satisfying (8) is the same. However, it need not preserve the standard negation $N_{\mathbf{s}}$, because the equality $h_\lambda(N_{\mathbf{s}}(x)) = N_{\mathbf{s}}(h_\lambda(x))$ does not hold in general. In this case, there is still a t-conorm S satisfying

$$h_\lambda(S_\lambda(x,y)) = S_{\mathbf{P}}(h_\lambda(x), h_\lambda(y)),$$

which is not the ($N_{\mathbf{s}}$-)dual of T_λ but the N-dual of T_λ, where the strong negation N is given by

$$N(x) = h_\lambda^{-1}(N_{\mathbf{s}}(h_\lambda(x))).$$

As a consequence, the S-fuzzy logics $\mathcal{S}_\lambda$ with $\lambda \in]0,\infty[$ have basically different semantics.

Proposition 6 works because all connectives in an S-fuzzy logic have continuous interpretations. On the other hand, the choice of an S-implication as the interpretation of the implication causes serious problems from the logical point of view.

The most important disadvantage of S-fuzzy logics seems to be that their syntax is essentially the syntax of the classical logic and it does not bring anything new. Using the standard system of axioms of the classical logic, we obtain as $\mathcal{S}$-theorems exactly the theorems of the classical logic. Only the semantics is different. Except for the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$, there seems to be no chance to find an axiomatization allowing a completeness theorem for *1-tautologies* in $\mathcal{S}_\lambda$ with $\lambda \in [0, \infty[$ (i.e., for formulas φ such that $\bar{t}_{\mathcal{S}_\lambda}(\varphi) = 1$ for all truth assignments $\bar{t}_{\mathcal{S}_\lambda}$) to be proven. The problem is in equation (5); the S-implication does not give 1 for arguments which are not crisp. In fact, without adding the nulary connective $\mathbf{0}$, there are even no 1-tautologies in $\mathcal{S}_\lambda$ with $\lambda \in [0, \infty[$.

4.2 Advantages and disadvantages of R-fuzzy logics

In an R-fuzzy logic, the basic connectives are the conjunction $\wedge$, the implication $\to$ and the false statement $\mathbf{0}$. We can derive the negation $\neg$. Except for the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, a connective dual to the conjunction (by the de Morgan formula) does not give a reasonable disjunction. We also have some kind of non-symmetry, because we have a conjunction (interpreted by a t-norm) without a corresponding disjunction (interpreted by the dual t-conorm).

There is one observation restricting the latter disadvantage: The formula $\varphi \wedge (\varphi \to \psi)$ (in any R-fuzzy logic) has many properties of an (idempotent) conjunction of φ and ψ, and it is interpreted by the minimum t-norm (due to the properties of R-implications). In the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ (and only in it), this formula is semantically equivalent to $\varphi \wedge \psi$. Further, the formula

$$[(\varphi \to \psi) \to \psi] \wedge [[(\varphi \to \psi) \to \psi] \to [(\psi \to \varphi) \to \varphi]]$$

has properties of an (idempotent) disjunction of φ and ψ, and it is interpreted by the maximum t-conorm. So we have a disjunction (interpreted by a t-conorm) in any R-fuzzy logic, but it is dual to the (basic) conjunction only in the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$. In the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, we have a disjunction dual to the basic conjunction. So the problems with a disjunction arise only in R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in]0, \infty[$. All these logics are (up to an automorphism of $[0, 1]$) equivalent to the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$.

The use of an R-implication for the interpretation of the implication causes some problems; except for the Łukasiewicz R-fuzzy logic, the corresponding R-implication is not continuous. In this case (i.e., in $\mathcal{R}_\lambda$ with $\lambda \in [0, \infty[$) the negation is interpreted by the Gödel negation $N_{\mathbf{G}}$ which is not strong and attains crisp values only. This seems to decrease the applicability of such a logic.

On the other hand, the use of an R-implication allows very nice and deep logical results. New (weaker) sets of axioms of the Gödel, Łukasiewicz and product R-fuzzy logics led to quite new axiomatizations. They enrich also the classical logic by giving equally developed alternatives.

Remark 19 The main disadvantage of R-fuzzy logics $\mathcal{R}_\lambda$ with $\lambda \in \,]0,\infty[$ — the absence of a disjunction dual to the conjunction — led recently to a new concept, an R-fuzzy logic with an involutive negation (see [6]). In this approach, the negation $\neg$ becomes an additional basic connective interpreted by a strong negation N. This negation can be used in the de Morgan formula defining a dual disjunction which is interpreted by the N-dual t-conorm. Then $\neg\varphi$ does not necessarily coincide with $\varphi \to \mathbf{0}$. Also this logic is axiomatizable, sound and complete. On the other hand, the system of axioms is more complicated, and also one new deduction rule has to be added.

The introduction of infinitary R-fuzzy logics seems to be an open field of research since we did not find any study of this subject in the literature.

4.3 Comparison of strength of S- and R-fuzzy logics

We want to investigate whether some logics are semantically "stronger" than others in the sense that they contain more classes of semantically equivalent formulas. The meaning of "stronger" in this context is that each reasoning in the "weaker" logic can be translated into a semantically equivalent reasoning in the "stronger" logic. Philosophically speaking, this means that the "stronger" logic provides a richer environment for approximate reasoning than the "weaker" one in the same context.

To make things precise, let $\mathcal{S}_\lambda$ and $\mathcal{S}_\mu$ be two S-fuzzy logics with the same set of atomic symbols A. Then $\mathcal{S}_\lambda$ is said to be *stronger* than $\mathcal{S}_\mu$ (see [2]) if there exists a mapping $f : \mathcal{F}_\mathcal{S} \to \mathcal{F}_\mathcal{S}$ such that for each formula $\varphi \in \mathcal{F}_\mathcal{S}$ and for each function $t : A \to [0,1]$ we have $\overline{t}_{\mathcal{S}_\lambda}(f(\varphi)) = \overline{t}_{\mathcal{S}_\mu}(\varphi)$. We say that S-fuzzy logics $\mathcal{S}_\lambda$, $\mathcal{S}_\mu$ are *equally strong* if $\mathcal{S}_\lambda$ is stronger than $\mathcal{S}_\mu$ and $\mathcal{S}_\mu$ is stronger than $\mathcal{S}_\lambda$. We say that $\mathcal{S}_\lambda$ is *strictly stronger* than $\mathcal{S}_\mu$ if $\mathcal{S}_\lambda$ is stronger than $\mathcal{S}_\mu$, but $\mathcal{S}_\mu$ is not stronger than $\mathcal{S}_\lambda$. These notions can be carried over to infinitary S-fuzzy logics and to R-fuzzy logics in a natural way. The only difference is that we need a mapping f between different sets of formulas. We always assume that all the logics have the same set of atomic symbols.

The notion of strength allows us to formulate the following results concerning the comparison of fuzzy logics. For each $\lambda \in [0,\infty]$, the infinitary S-fuzzy logic $\mathcal{S}^*_\lambda$ is obviously strictly stronger than the (finitary) S-fuzzy logic $\mathcal{S}_\lambda$. There are important relations between Łukasiewicz and min-max S-fuzzy logics (see [2, Proposition 6.2]):

Proposition 20 *(i) The Łukasiewicz S-fuzzy logic $\mathcal{S}_\mathbf{L}$ is strictly stronger than the min-max S-fuzzy logic $\mathcal{S}_\mathbf{M}$.*

(ii) The infinitary Łukasiewicz S-fuzzy logic $\mathcal{S}^*_\mathbf{L}$ is strictly stronger than the infinitary min-max S-fuzzy logic $\mathcal{S}^*_\mathbf{M}$.

The S-fuzzy logic $\mathcal{S}_\lambda$ for $\lambda \in]0, \infty[$ is not stronger than the Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ (nor vice versa).

Nevertheless, for infinitary S-fuzzy logics we have the following positive result based on rather advanced techniques of mathematical analysis (see [2, Theorem 6.5]):

Theorem 21 *Each infinitary S-fuzzy logic $\mathcal{S}_\lambda^*$ with $\lambda \in]0, \infty[$ is strictly stronger than the infinitary Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}^*$.*

The comparison of infinitary S-fuzzy logics $\mathcal{S}_\lambda^*$ for different $\lambda \in]0, \infty[$ was formulated as an open problem in [2]. Using results from [23, 24], the following result was proved in [15] (compare also [3]):

Theorem 22 *For all $\lambda, \mu \in]0, \infty[$, the infinitary S-fuzzy logics $\mathcal{S}_\lambda^*$, $\mathcal{S}_\mu^*$ are equally strong.*

So far we compared (infinitary) S-fuzzy logics according to their strength. Now we shall include also R-fuzzy logics in this system of relations.

Although the minimum t-norm can be expressed by the operations of any R-fuzzy logic, for the corresponding (Gödel) R-implication such an expression does not exist. Similar arguments can be applied to other R-fuzzy logics, too. Therefore, the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$, the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ and the product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ are incomparable in the sense that none of them is stronger than any other.

Because of the particular properties of the Łukasiewicz fuzzy operations, we obtain the following fact:

Theorem 23 *The Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ and the Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ are equally strong.*

For each $\lambda \in [0, \infty[$, the R-implication R_λ is not continuous, hence the S-fuzzy logic $\mathcal{S}_\lambda$ is not stronger than the R-fuzzy logic $\mathcal{R}_\lambda$. The discussion of strong negations (see Subsection 4.2) shows that, for each $\lambda \in]0, \infty[$, the R-fuzzy logic $\mathcal{R}_\lambda$ is not stronger than the S-fuzzy logic $\mathcal{S}_\lambda$. Also for $\lambda = 0$, the Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}} = \mathcal{R}_0$ is not stronger than the min-max S-fuzzy logic $\mathcal{S}_{\mathbf{M}} = \mathcal{S}_0$.

Nevertheless, the infinitary S-fuzzy logics $\mathcal{S}_\lambda^*$ with $\lambda \in]0, \infty[$ are so strong that they satisfy the following relation (see [15]):

Theorem 24 *For all $\lambda \in]0, \infty[$ and $\mu \in [0, \infty]$, the infinitary S-fuzzy logic $\mathcal{S}_\lambda^*$ is strictly stronger than the R-fuzzy logic $\mathcal{R}_\mu$.*

Concluding remarks

We have discussed two main approaches to propositional fuzzy logics based on Frank t-norms, the S-fuzzy logics (where negation and conjunction are basic connectives) and the R-fuzzy logics (where the basic connectives are conjunction, implication and the false statement). We have seen that the main difference is the interpretation of the implication (by an S-implication in S-fuzzy logics and by the residuum in R-fuzzy logics).

In both approaches we have studied the important issues of compactness, deduction, axiomatization, soundness and completeness.

Finally, we tried to compare S-fuzzy logics and R-fuzzy logics in a twofold way: on the one hand, by pointing out the advantages and disadvantages of the two concepts, on the other hand, by comparing their semantical strength.

It should be noted that there are many other approaches to $[0,1]$-valued logics starting from different points of view, some of which are described in detail in [16, 26, 28]. For a rather extensive overview, see [11].

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References

[1] D. Butnariu and E. P. Klement. *Triangular Norm-Based Measures and Games with Fuzzy Coalitions* . Kluwer, Dordrecht, 1993.

[2] D. Butnariu, E. P. Klement, and S. Zafrany. On triangular norm-based propositional fuzzy logics. *Fuzzy Sets and Systems*, 69:241–255, 1995.

[3] D. Butnariu, E.P. Klement, and M. Navara. All strict triangular norms are "equally strong". In *Proceedings FUZZY '97 — International Conference on Fuzzy Logic and Applications*, pages 116–124. Tel Aviv University, 1997.

[4] D. Dubois and H. Prade. A review of fuzzy set aggregation connectives. *Inform. Sci.*, 36:85–121, 1985.

[5] M. Dummett. A propositional calculus with denumerable matrix. *J. Symb. Logic*, 24:97–106, 1959.

[6] F. Esteva, L. Godo, P. Hájek and M. Navara. Residuated fuzzy logics with an involutive negation. Submitted for publication.

[7] J. C. Fodor. Fuzzy connectives via matrix logic. *Fuzzy Sets and Systems*, 56:67–77, 1993.

[8] J. C. Fodor. On fuzzy implication operators. *Fuzzy Sets and Systems*, 42:293–300, 1991.

[9] M. J. Frank. On the simultaneous associativity of $F(x, y)$ and $x + y - F(x, y)$. *Aequationes Math.*, 19:194–226, 1979.

[10] K. Gödel. Zum intuitionistischen Aussagenkalkül. *Anz. Österr. Akad. Wiss. Math.-Natur. Kl.*, 69:65–66, 1932.

[11] S. Gottwald. *Fuzzy Sets and Fuzzy Logic. Foundations of Application — from a Mathematical Point of View.* Vieweg, Braunschweig/Wiesbaden, 1993.

[12] S. Gottwald. *Mehrwertige Logik.* Akademie-Verlag, Berlin, 1989.

[13] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[14] P. Hájek, L. Godo, and F. Esteva. A complete many-valued logic with product-conjunction. *Arch. Math. Logic*, 35:191–208, 1996.

[15] J. Hekrdla, E.P. Klement, and M. Navara. Two approaches to fuzzy propositional logics. Submitted for publication.

[16] U. Höhle. Monoidal logic. In R. Kruse, J. Gebhardt, and R. Palm, editors, *Fuzzy Systems in Computer Science*, pages 233–243. Vieweg, Wiesbaden, 1994.

[17] C. C. Chang. Algebraic analysis of many valued logics. *Trans. Amer. Math. Soc.*, 88:467–490, 1958.

[18] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms.* In preparation.

[19] E. P. Klement and S. Weber. Generalized measures. *Fuzzy Sets and Systems*, 40:375–394, 1991.

[20] C. M. Ling. Representation of associative functions. *Publ. Math. Debrecen*, 12:189–212, 1965.

[21] J. Łukasiewicz. Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls. *Comptes Rendus Séances Société des Sciences et Lettres Varsovie cl. III*, 23:51–77, 1930.

[22] K. Menger. Statistical metrics. *Proc. Nat. Acad. Sci. U.S.A.*, 8:535–537, 1942.

[23] R. Mesiar. On the structure of T_s-tribes. *Tatra Mt. Math. Publ.*, 3:167–172, 1993.

[24] M. Navara. A characterization of triangular norm based tribes. *Tatra Mt. Math. Publ.*, 3:161–166, 1993.

[25] H. T. Nguyen and E. Walker. *A First Course in Fuzzy Logic.* CRC Press, Boca Raton, 1997.

[26] V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic . Part I — Syntactical aspects, Part II — Main results. *Kybernetika*, 26:47–66, 134–154, 1990.

[27] E. Pap. Decomposable measures and applications on nonlinear partial differential equations. *Rend. Circ. Mat. Palermo (2) Suppl.*, 28:387–403, 1992.

[28] J. Pavelka. On fuzzy logic I, II, III. *Math. Logik Grundlag. Math.*, 25:45–52, 119–134, 447–464, 1979.

[29] W. Pedrycz. *Fuzzy Control and Fuzzy Systems.* Research Stud. Press. Taunton and Wiley, New York, 1989.

[30] W. Pedrycz. Fuzzy control and fuzzy systems. Technical report, 1982. Delft University of Technology, Department of Mathematics, Report 82 14.

[31] W. Pedrycz. Fuzzy relational equations with generalized connectives and their applications. *Fuzzy Sets and Systems*, 10:185–201, 1983.

[32] B. Schweizer and A. Sklar. Associative functions and abstract semigroups. *Publ. Math. Debrecen*, 10:69–81, 1963.

[33] B. Schweizer and A. Sklar. Associative functions and statistical triangle inequalities. *Publ. Math. Debrecen*, 8:169–186, 1961.

[34] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces.* North-Holland, Amsterdam, 1983.

[35] L. A. Zadeh. Fuzzy sets. *Inform. Control*, 8:338–353, 1965.