# PROBLEMS

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## Proposals

*To be considered for publication, solutions should be mailed before July 1, 1981.*

**1112.** The function $\sin(1/t)$ is bounded and continuous everywhere except at 0, thus is (Riemann) integrable over any bounded interval. We may, therefore, define $F(x)$ to be $\int_0^x \sin(1/t)\,dt$, where $x$ is any real number. Is $F$ differentiable at 0? [*Richard Dowds, Fredonia State University College*.]

**1113.** On Christmas Eve, 1983, Dean Jixon, the famous seer who had made startling predictions of the events of the preceding year, declared that the volcanic and seismic activities of 1980 and 1981 were connected with mathematics. The diminishing of this geologic activity depended upon the existence of an elementary proof of the irreducibility of the polynomial $P(x)=x^{1981}+x^{1980}+12x^2+24x+1983$. Is there such a proof? [*William A. McWorter, Jr., The Ohio State University*.]

**1114.** Prove or disprove: There exists a function $f$ defined on $[-1,1]$ with $f'$ continuous such that $\sum_{n=1}^{\infty} f(1/n)$ converges but $\sum_{n=1}^{\infty}|f(1/n)|$ diverges. (This is a relaxing of the condition "$f''$ continuous" to "$f'$ continuous" in Problem 1060, this MAGAZINE, January 1979 and March 1980.) [*Robert Clark, student, Temple University*.]

**1115.** Let $D$ be the disc $x^2+y^2<1$. Let points $A$ and $B$ be selected at random in $D$. Find the probability that the open disc whose center is the midpoint of $\overline{AB}$ and whose radius is $AB/2$ is a subset of $D$. [*Roger L. Creech, East Carolina University*.]

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ASSISTANT EDITORS: DON BONAR, *Denison University;* WILLIAM A. MCWORTER, JR., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgment of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

# Solutions

**Pythagorean Triangles** **November 1979**

**1088.** (a) For each positive integer $m$, how many Pythagorean triangles are there which have an area equal to $m$ times the perimeter? How many of these are primitive?

(b*) Can this result be generalized to all triangles with integer sides and area equal to $m$ times the perimeter? [*Alan Wayne, Pasco-Hernando Community College.*]

*Solution* (a): It is well known that the legs of a Pythagorean triangle are $2kuv$ and $k(u^2-v^2)$, and the hypotenuse is $k(u^2+v^2)$, for some integers $k$, $u$ and $v$, and that this representation is unique provided $k>0$, $u>v>0$, and $u$ and $v$ are relatively prime and not both odd. Setting the area equal to $m$ times the perimeter yields $kv(u-v)=2m$.

Consider first the number $a_m$ of primitive ($k=1$) Pythagorean triangles corresponding to a fixed ratio $m$. Because $u$ and $v$ have opposite parity, $u-v$ is odd. Because $u$ and $v$ are relatively prime, so are $v$ and $u-v$. If $m=2^{k_0}p_1^{k_1}\cdots p_r^{k_r}$ is the canonical factorization of $m$ (where possibly $k_0=0$), any power $2^{k_0}$ (and the additional factor of 2) must divide $v$, while each of the $r$ powers $p_i^{k_i}$ must divide either $v$ or else $u-v$. Hence $a_m=2^r$.

Consider next the number $b_m$ of all Pythagorean triples corresponding to a fixed ratio $m$. In the equation $kv(u-v)=2m$, $k$ and $v$ can contribute the $k_0+1$ factors of 2 in $2m$ in $k_0+2$ ways. For each $i=1,2,\dots,r$: either $k$ can contribute all $k_i$ factors of $p_i$; or $v$ can contribute some, and $k$ the others, in $k_i$ ways; or else $u-v$ can contribute some, and $k$ the others, in $k_i$ ways. Hence $b_m=(k_0+2)(2k_1+1)\cdots(2k_r+1)$.

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***Part (a) also solved by Bern Problem Solving Group (Switzerland), Walter Bluger (Canada), P. J. Federico, Steven Kleiman & Henry Klostergaard, L. Kuipers (Switzerland), Scott Smith, Lawrence Somer, Dave Van Leeuwen, Michael Vowe (Switzerland), Ken Yocum, and the proposer. Federico gave a discussion of the equations which must be satisfied to solve part (b). Kleiman & Klostergaard have written an article for publication which discusses an algorithm for solving part (b).***

**Old Year Resolution** **January 1980**

**1089.** Determine the highest power of 1980 which divides

$$\frac{(1980n)!}{(n!)^{1980}}.$$

[*M. S. Klamkin, University of Alberta.*]

*Solution*: Let $V_m(x)$ be the exponent of the highest power of $m$ which divides $x$. If $p$ is a prime,

$$V_p\left(\frac{(mn)!}{(n!)^m}\right)=V_p((mn)!)-mV_p(n!)=\sum_{k>1}\left(\left[\frac{mn}{p^k}\right]-m\left[\frac{n}{p^k}\right]\right).$$

Thus, if $m$ has the prime factorization $m=\prod_{i=1}^{r}p_i^{e_i}$,

$$V_m\left(\frac{(mn)!}{(n!)^m}\right)=\min_i\left[\frac{1}{e_i}\sum_{k>1}\left(\left[\frac{mn}{p_i^k}\right]-m\left[\frac{n}{p_i^k}\right]\right)\right].$$

The brackets $[\cdot]$ in all cases denote the greatest integer, and we note that the summand is the $m$-residue of $[mn/p_i^k]$. In particular, since $1980=2^2 3^2\cdot 5\cdot 11$,

$$V_{1980}\left(\frac{(1980n)!}{(n!)^{1980}}\right)=\min_{1\leqslant i\leqslant 4}\left[\frac{1}{e_i}\sum_{k\geqslant 1}\left(\left[\frac{1980n}{p_i^k}\right]-1980\left[\frac{n}{p_i^k}\right]\right)\right],$$

where $e_1=e_2=2$, $e_3=e_4=1$, and $p_1$, $p_2$, $p_3$ and $p_4$ are 2, 3, 5 and 11, respectively. Depending upon $n$, the minimum may occur in any of the four terms. $V_p((mn)!/(n!)^m)$ is small when $n$ is a power of $p$, and for $p^{k-1}\leqslant n<p^k$, it grows with the sum of the digits of $n$ in base $p$. If $n=8$, the minimum in $V_{1980}$ occurs when $p=2$; if $n=9$, 5 or 11, it occurs when $p=3$, 5 or 11 respectively.

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*Also solved by William E. Gould and the proposer. Klamkin also shows that when p is a prime the relative maximum values of $V_p((pn)!/(n!)^p)$ as a function of n occur when $n=p^m-1$ and the value then is $m(p-1)$.*

**A GCD Problem** **January 1980**

**1090.** It is well known that if $n$ is prime, then for every pair of relatively prime integers $a$ and $b$ the gcd of $(a^n-b^n)/(a-b)$ and $(a-b)$ is 1 or $n$. Find a corresponding result valid for every integer $n\geqslant 1$ and every pair of distinct integers $a$ and $b$. [*Tom M. Apostol, California Institute of Technology*.]

*Solution*: Let $(x,y)$ denote the gcd of $x$ and $y$. We will prove the general formula

$$\left(\frac{a^n-b^n}{a-b},a-b\right)=\left(n(a,b)^{n-1},a-b\right).$$

When $(a,b)=1$ the right member is $(n,a-b)$, and when $n$ is prime this is 1 or $n$.

We use the identity

$$\frac{a^n-b^n}{a-b}=\sum_{k=0}^{n-1}a^k b^{n-1-k}=\sum_{k=1}^{n-1}(a^k-b^k)b^{n-1-k}+nb^{n-1}$$
$$=(a-b)Q(a,b)+nb^{n-1}, \tag{1}$$

where $Q(a,b)$ is a polynomial in $a$ and $b$ with integer coefficients.

Let

$$d=\left(\frac{a^n-b^n}{a-b},a-b\right),\quad e=\left(n(a,b)^{n-1},a-b\right).$$

From (1) we see that $d|nb^{n-1}$. By symmetry, $d|na^{n-1}$ so $d|n(a^{n-1},b^{n-1})=n(a,b)^{n-1}$, hence $d|e$. Also $e|nb^{n-1}$ and $e|(a-b)$ and (1) shows that $e|(a^n-b^n)/(a-b)$, hence $e|d$, so $e=d$.

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*Also solved by Gordon Fisher, L. Kuipers (Switzerland), Lawrence Somer, and Ken Yocom.*