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COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

S. PREMA, B. SRUTHA KEERTHI

Abstract. In this paper, we introduce a new subclass of the function class Σ of bi-univalent functions defined in the open unit disc. We find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclass.

1. Introduction and Definitions

Let Ω denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disc $U := \{z : |z| < 1\}$.

Further, by δ we shall denote the class of all functions in Ω which are univalent in U. It is well known that every function $f \in \delta$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad \{|w| < r_0(f); r_0(f) \geq 1/4\}$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

A function $f(z) \in \Omega$ is univalent whose image contains the unit disc (see [5]).

Let Σ denote the class of bi-univalent functions in U given by (1.1). Brannan and Taha [6] (see also [7]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\delta^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order α $(0 < \alpha \leq 1)$, respectively (see [8]). Thus, the following Brannan and Taha [6] (see also [7]), a function $f(z) \in \Omega$ is in the class $\delta^*_\Sigma(\alpha)$ of strongly bi-starlike functions of order α $(0 < \alpha \leq 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma, \left| arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, z \in U)$$

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and

$$\left|arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, w \in U)$$

where g is the extension of f^{-1} to U. The classes $\delta^*_\Sigma(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $\delta^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $\delta^*_\Sigma(\alpha)$ and $K^*_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6, 7]).

The object of the present paper is to introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclass of the function class Σ employing the techniques used earlier by Srivastava et al. [5] (see also [6, 7, 8]).

In order to derive our main results, we have to recall here the following lemma [12].

Lemma 1.1. *If $h \in P$, then $|c_k| \leq 2$ for each K, where P is the family of all functions h analytic in U for which Re $h(z) > 0$*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots \quad \textit{for } z \in U$$

2. Coefficient Bounds for the Function Class $P_\Sigma(\alpha, \lambda)$

Definition 2.1. *A function $f(z)$ given by (1.1) is said to be in the class $P_\Sigma(\alpha,\lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma, \ \left|arg\left(\frac{z^{1-\lambda}f'(z)}{f(z)^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 0, z \in U) \tag{2.1}$$

and

$$\left|arg\left(\frac{w^{1-\lambda}g'(w)}{g(w)^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1, \lambda \geq 0, w \in U) \tag{2.2}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \tag{2.3}$$

We note that for $\lambda = 0$, the class $P_\Sigma(\alpha,\lambda)$ reduces to the class $\delta^*_\Sigma(\alpha)$ which was given by Xiao-Fei Li and An-Ping Wang (see also [9, 10]*).*

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $P_\Sigma(\alpha,\lambda)$.

Theorem 2.2. *Let $f(z)$ given by (1.1) be in the class $P_\Sigma(\alpha,\lambda)$, $0 < \alpha \leq 1$ and $\lambda \geq 0$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1+\lambda)(\alpha+1+\lambda)}} \tag{2.4}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1+\lambda)^2} + \frac{2\alpha}{(2+\lambda)} \quad . \tag{2.5}$$

Proof. We can write the argument inequalities in (2.1) and (2.2) equivalently as follows:

$$\left[\frac{z^{1-\lambda}f'(z)}{f(z)^{1-\lambda}}\right] = [p(z)]^\alpha \tag{2.6}$$

$$\left[\frac{w^{1-\lambda}g'(z)}{g(z)^{1-\lambda}}\right] = [q(w)]^\alpha \tag{2.7}$$

respectively, where $p(z)$ and $q(w)$ satisfy the following inequalities

$$Re(p(z)) > 0 \quad (z \in U) \quad \text{and} \quad Re(q(w)) > 0 \quad (w \in U).$$

Furthermore, the functions $p(z)$ and $q(w)$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \tag{2.8}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$(1+\lambda)a_2 = p_1\alpha \tag{2.10}$$

$$(2+\lambda)a_3 = p_2\alpha + \frac{\alpha(\alpha-1)p_1^2}{2} + \frac{\left(1 - \frac{\lambda}{2} - \frac{\lambda^2}{2}\right) p_1^2\alpha^2}{(1+\lambda)^2} \tag{2.11}$$

and

$$-(1+\lambda)a_2 = q_1\alpha \tag{2.12}$$

$$(2+\lambda)(2a_2^2 - a_3) = q_2\alpha + \frac{\alpha(\alpha-1)q_1^2}{2} + \frac{\left(1 - \frac{\lambda}{2} - \frac{\lambda^2}{2}\right) q_1^2\alpha^2}{(1+\lambda)^2} \tag{2.13}$$

from (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$2(1+\lambda)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2) \quad . \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$2(2+\lambda)a_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)(p_1^2+q_1^2)}{2} + \frac{(1-\lambda)\left(1+\frac{\lambda}{2}\right)\alpha^2(p_1^2+q_1^2)}{(1+\lambda)^2} \quad . \tag{2.16}$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{(1+\lambda)(\alpha+1+\lambda)} \quad .$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately we have

$$|a_2| \le \frac{2\alpha}{\sqrt{(1+\lambda)(\alpha+1+\lambda)}} \quad .$$

Next, in order to find the bound on $|a_3|$, by subtracting (2.11) from (2.13), we get

$$(2+\lambda)(2a_3 - 2a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)(p_1^2-q_1^2)}{2} + \frac{(1-\lambda)\left(1+\frac{\lambda}{2}\right)\alpha^2(p_1^2-q_1^2)}{(1+\lambda)^2}$$

upon substituting the value at a_2^2 from (2.15) and observing that $p_1^2 = q_1^2$, it follows that

$$a_3 = a_2^2 + \frac{\alpha(p_2-q_2)}{2(2+\lambda)}$$
$$= \frac{\alpha^2(p_1^2+q_1^2)}{2(1+\lambda)^2} + \frac{\alpha(p_2-q_2)}{2(2+\lambda)}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3| \le \frac{4\alpha^2}{(1+\lambda)^2} + \frac{2\alpha}{(2+\lambda)} \quad .$$

This completes the proof of Theorem 2.2. □

Putting $\lambda = 0$ in Theorem 2.2, we have

Corollary 2.3. *Let $f(z)$ given by (1.1) be in the class $\delta^*_\Sigma(\alpha)$. Then*

$$|a_2| \le \frac{2\alpha}{\sqrt{\alpha+1}} \tag{2.17}$$

and

$$|a_3| \le 4\alpha^2 + \alpha. \tag{2.18}$$

3. Coefficient Bounds for the Function Class $P_\Sigma(\beta,\lambda)$

Definition 3.1. *A function $f(z)$ given by (1.1) is said to be in the class $P_\Sigma(\beta,\lambda)$ if the following conditions are satisfied:*

$$f \in \Sigma,\ Re\left[\frac{z^{1-\lambda}f'(z)}{f(z)^{1-\lambda}}\right] > \beta \quad (0 \le \beta < 1, \lambda \ge 0, z \in U) \tag{3.1}$$

and

$$Re\left[\frac{w^{1-\lambda}g'(w)}{g(w)^{1-\lambda}}\right] > \beta \quad (0 \le \beta < 1, \lambda \ge 0, w \in U) \tag{3.2}$$

where the function $g(w)$ is defined by (2.3).

We note that for $\lambda = 0$, the class $P_\Sigma(\beta,\lambda)$ reduces to the class $P_\Sigma(\beta)$.

Theorem 3.2. *Let $f(z)$ given by (1.1) be in the class $P_\Sigma(\beta,\lambda)$, $0 \le \beta < 1$ and $\lambda \ge 0$. Then*

$$|a_2| \le \sqrt{\frac{2(1-\beta)}{(1+\lambda)}} \tag{3.3}$$

and

$$|a_3| \le \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{(2+\lambda)}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist $p(z)$ and $q(w)$ such that

$$\left[\frac{z^{1-\lambda}f'(z)}{f(z)^{1-\lambda}}\right] = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\left[\frac{w^{1-\lambda}g'(w)}{g(w)^{1-\lambda}}\right] = \beta + (1-\beta)q(w) \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$(1+\lambda)a_2 = p_1(1-\beta) \tag{3.7}$$

$$(2+\lambda)a_3 = p_2(1-\beta) + \frac{(1-\lambda)}{(1+\lambda)^2}(1-\beta)^2p_1^2 \tag{3.8}$$

and

$$-(1+\lambda)a_2 = (1-\beta)q_1 \tag{3.9}$$

$$(2+\lambda)(2a_2^2 - a_3) = q_2(1-\beta) + \frac{(1-\lambda)}{(1+\lambda)^2}q_1^2(1-\beta)^2 \tag{3.10}$$

from (3.7) and (3.9) we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2(1+\lambda)^2a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

Now from (3.8), (3.10) and (3.12), we obtain

$$2(2+\lambda)a_2^2 = (2+\lambda)a_3 + (1-\beta)q_2 + \frac{(1-\lambda)(1-\beta)^2 q_1^2}{(1+\lambda)^2}$$
$$= (1-\beta)(p_2+q_2) + 2(1-\lambda)a_2^2 \quad .$$

Therefore, we have

$$a_2^2 = \frac{(p_2+q_2)(1-\beta)}{2(1-\lambda)} \quad .$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{(1+\lambda)}}.$$

Next, in order to find the bound on $|a_3|$, by subtracting (3.8) from (3.10), we get

$$(2+\lambda)(2a_3 - 2a_2^2) = (1-\beta)(p_2 - q_2).$$

It follows that

$$2(2+\lambda)a_3 = 2(2+\lambda)a_2^2 + (1-\beta)(p_2-q_2)$$
$$= \frac{(2+\lambda)(1-\beta)^2(p_1^2+q_1^2)}{(1+\lambda)^2} + (1-\beta)(p_2-q_2)$$

once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{(2+\lambda)}.$$

This completes the proof of Theorem 3.2. □

Putting $\lambda = 0$ in Theorem 3.2, we have the following corollary.

Corollary 3.3. *Let $f(z)$ given by (1.1) be in the class $P_\Sigma(\beta)$, $(0 \leq \beta < 1)$. Then*

$$|a_2| \leq \sqrt{2(1-\beta)} \tag{3.13}$$

and

$$|a_3| \leq 4(1-\beta)^2 + 1 - \beta \quad . \tag{3.14}$$

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S. Prema

Department of Mathematics SKR Engineering College Poonamalle, Chennai - 600 123

E-mail address: cs.prem@yahoo.co.in

B. Srutha Keerthi

Department of Applied Mathematics Sri Venkateswara College of Engineering Sriperumbudur, Chennai - 602105, India

E-mail address: sruthilaya06@yahoo.co.in