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Hausdorff measure of quasicircles ☆

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Abstract

S. Smirnov (2010) [10] proved recently that the Hausdorff dimension of any K-quasicircle is at most $1+k^2$, where $k=(K-1)/(K+1)$. In this paper we show that if Γ is such a quasicircle, then

$$H^{1+k^2}\big(B(x,r)\cap\Gamma\big)\leqslant C(k)r^{1+k^2}\quad \text{for all } x\in\mathbb{C},\ r>0,$$

where H^s stands for the s-Hausdorff measure. On a related note we derive a sharp weak-integrability of the derivative of the Riemann map of a quasidisk.

Keywords: Quasiconformal mappings in the plane; Hausdorff measure

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1. Introduction

A homeomorphism $f : \Omega \to \Omega'$ between planar domains is called *K-quasiconformal* if it preserves orientation, belongs to the Sobolev class $W^{1,2}_{\mathrm{loc}}(\Omega)$ and its directional derivatives satisfy the distortion inequality

$$\max_{\alpha} |\partial_\alpha f| \leqslant K \min_{\alpha} |\partial_\alpha f| \quad \text{a.e. in } \Omega.$$

This estimate is equivalent to saying that f satisfies the Beltrami equation

$$\bar{\partial} f(z) = \mu(z) \partial f(z)$$

for almost all $z \in \mathbb{C}$, where μ is the so-called Beltrami coefficient or dilatation, with $\|\mu\|_\infty \leqslant k = (K-1)/(K+1)$.

Infinitesimally, quasiconformal mappings carry circles to ellipses with eccentricity uniformly bounded by K. If $K = 1$ we recover conformal maps, while for $K > 1$ quasiconformal maps need not be smooth, in fact, they may distort the Hausdorff dimension of sets. The higher integrability result of Astala [1] provides precise estimates for this latter phenomenon. Very recently, these distortion bounds have been established even for the corresponding Hausdorff measures [7,11]. In the present note we consider quasicircles and their relation to Hausdorff measures.

A *K-quasicircle* is the image of the unit circle under a K-quasiconformal homeomorphism of the Riemann sphere $\hat{\mathbb{C}}$. Sometimes, it will be more convenient to specialize to quasilines, these are images of the real line under a quasiconformal homeomorphism of the finite plane $\mathbb{C}$. For many different characterizations of quasicircles and quasidisks (domains bounded by quasicircles), we refer the reader to [5]. Complex dynamics (Julia sets, limit sets of quasi-Fuchsian groups) provide a rich source of examples of quasicircles with Hausdorff dimension greater than one. Astala conjectured in [1] that $1 + k^2$ is the optimal bound on the dimension of K-quasicircles, where $k = (K-1)/(K+1)$. Smirnov [10] proved that indeed a K-quasicircle has Hausdorff dimension at most $1 + k^2$. The question of sharpness is an open problem with important connections to extremal behavior of harmonic measure [9]. Currently, the best known lower bound appears to be the computer-aided estimate $1 + 0.69k^2$ of [3].

Our main result is the following strengthening of Smirnov's theorem in terms of $1 + k^2$-dimensional Hausdorff measure H^{1+k^2}.

Theorem 1.1. *If Γ is a K-quasicircle in $\hat{\mathbb{C}}$, then*

$$H^{1+k^2}\big(\Gamma \cap B(z,r)\big) \leqslant C(K) r^{1+k^2} \quad \text{for all } z \in \mathbb{C} \text{ and with } k = \frac{K-1}{K+1}.$$

To prove this result we use a well-known factorization to "conformal inside" and "conformal outside" parts. The conformal inside part is taken care of by Smirnov's work; we recall the necessary estimates in Section 2. Section 3 handles the conformal outside part and we finally put together the estimates in Section 4. Section 3 contains most of the novelties. Here we adopt the technique of [7] and show the boundedness of the Beurling transform with respect to some weights arising from a special packing condition. To implement the techniques in [7] to our setting we have to overcome some difficulties. For instance, the arguments in [7] are well suited to estimate quasiconformal distortion in terms of Hausdorff contents, while we need to obtain estimates in terms of Hausdorff measures. Also, the arguments involving the factorization of quasiconformal maps and the use of packing conditions are much more delicate in our case.

In Section 5 of this paper we prove another related result which controls the expansion of the Riemann map $\phi : \mathbb{D} \to \Omega$ onto a bounded K-quasidisk Ω. In particular, we obtain the following precise integrability condition for the derivative:

$$\phi' \in \text{ weak-}L^p(\mathbb{D}), \quad \text{with } p = \frac{2(K^2+1)}{(K^2-1)}.$$

This is a strengthening of a result from [9] to the critical exponent p above and it is optimal. See Theorem 5.2 for the precise statement.

In the paper, as usual, the letter C denotes a constant that may change at different occurrences, while constants with subscript, such as C_1, retain their values. The notation $A \approx B$ means that there is a constant C (often allowed to depend on the quasiconformality constant K) such that $1/C \cdot A \leqslant B \leqslant C \cdot A$. The notation $A \lesssim B$ means that there is a constant C (often allowed to depend on the quasiconformality constant K) such that $A \leqslant C \cdot B$. For instance, we shall frequently use this notation in conjunction with the well-known quasisymmetry property (see e.g. [2]) of quasiconformal maps. Also, as usual, if B denotes a ball, $2B$ denotes the ball with the same center as B and twice the radius of B (and similarly for squares and other multiples).

2. Smirnov's theorem on quasicircles

Before stating Smirnov's theorem, let us introduce some definitions.

Definition 2.1. A quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ is called *principal* if it is conformal outside some compact set $K \subset \mathbb{C}$ and satisfies the normalization $f(z) = z + O(1/z)$ at infinity.

Definition 2.2. A quasiconformal mapping $f : \mathbb{C} \to \mathbb{C}$ is called *antisymmetric* (with respect to the real line) (or equivalently we say that it has antisymmetric dilatation μ) if its Beltrami coefficient μ satisfies

$$\mu(z) = -\overline{\mu(\bar{z})} \quad \text{for a.e. } z \in \mathbb{C}.$$

The significance of this definition is that by making use of a symmetrization procedure [10] shows that, any K-quasiline may be represented as the image of the real line under a K-quasiconformal antisymmetric map. As one of the referees pointed out to us, this symmetrization procedure (with respect to the unit circle) was in fact introduced by Kühnau in his work on quasiconformal reflections [6].

Now we state Smirnov's bound on the dimension of quasicircles.

Theorem 2.3. *(See [10].) The Hausdorff dimension of a $\frac{1+k}{1-k}$-quasicircle is at most $1 + k^2$, for any $k \in [0, 1)$.*

We shall need the following formulation. This is implicit in [10] and appears exactly as stated in [2, Theorem 13.3.6].

Theorem 2.4. *Let $f:\mathbb{C}\to\mathbb{C}$ be a $\frac{1+k}{1-k}$-quasiconformal map, with $0<k<1$. Suppose that f has antisymmetric dilatation and that it is principal and conformal outside the unit disk. Let $B_j=B(z_j,r_j)$, $1\leqslant j\leqslant n$, be a collection of pairwise disjoint disks contained in the unit disk such that $z_j\in\mathbb{R}$, so that f is conformal on each B_j. For $0<t\leqslant 2$, let $t(k)$ be such that*

$$\frac{1}{t(k)}-\frac{1}{2}=\frac{1-k^2}{1+k^2}\left(\frac{1}{t}-\frac{1}{2}\right).$$

Then,

$$\left(\sum_{j=1}^{n}\left(\left|f'(z_j)\right|r_j\right)^{t(k)}\right)^{\frac{1}{t(k)}}\leqslant 8\left(\sum_{j=1}^{n}r_j^t\right)^{\frac{1-k^2}{1+k^2}\frac{1}{t}}.$$

As an easy corollary we give the special case $t=1$ in a scale invariant form.

Corollary 2.5. *Let $f:\mathbb{C}\to\mathbb{C}$ be a $\frac{1+k}{1-k}$-quasiconformal map, with $0<k<1$, with antisymmetric dilatation. Let B_j, $1\leqslant j\leqslant n$, be a collection of pairwise disjoint disks contained in a disk B. Suppose that the disks B_j, B, are centered on the real line and that f is conformal on the disks B_j. Then,*

$$\frac{\sum_{j=1}^n \operatorname{diam}(f(B_j))^{1+k^2}}{\operatorname{diam}(f(B))^{1+k^2}}\leqslant C(k)\left(\frac{\sum_{j=1}^n \operatorname{diam}(B_j)}{\operatorname{diam}(B)}\right)^{1-k^2}.$$

In particular, notice that in the above situation,

$$\sum_{j=1}^{n}\operatorname{diam}\bigl(f(B_j)\bigr)^{1+k^2}\leqslant C(k)\operatorname{diam}\bigl(f(B)\bigr)^{1+k^2}.$$

Proof. We factorize $f=f_2\circ f_1$, where f_1, f_2 are both K-quasiconformal maps, with f_1 principal and conformal on $\mathbb{C}\setminus 2B$, and f_2 is conformal on $f_1(2B)$. If we denote by μ_f the dilatation of f, we assume that the one of f_1 is $\mu_{f_1}=\chi_{2B}\mu_f$. Let $g(z)=az+b$ be the mapping that maps the unit disk to $2B$. The map $h=g^{-1}\circ f_1\circ g$ with the collection of disks $\{g^{-1}(B_j)\}$ verifies the assumptions of Theorem 2.4, and so specializing to $t=1$ we obtain

$$\sum_{j=1}^{n}\operatorname{diam}\bigl(h\bigl(g^{-1}(B_j)\bigr)\bigr)^{1+k^2}\leqslant C(k)\left(\sum_{j=1}^{n}\operatorname{diam}\bigl(g^{-1}(B_j)\bigr)\right)^{1-k^2}.$$

Since $\operatorname{diam}(h(g^{-1}(B_j)))=\operatorname{diam}(g^{-1}(f_1(B_j)))=a^{-1}\operatorname{diam}(f_1(B_j))$, we deduce

$$\sum_{j=1}^{n}\operatorname{diam}\bigl(f_1(B_j)\bigr)^{1+k^2}\leqslant C(k)a^{2k^2}\left(\sum_{j=1}^{n}\operatorname{diam}(B_j)\right)^{1-k^2}.$$

On the other hand, since f_2 is conformal on $f_1(2B)$, by Koebe's distortion theorem and quasisymmetry,

$$\frac{\operatorname{diam}(f_2(f_1(B_j)))}{\operatorname{diam}(f_2(f_1(2B)))} \approx \frac{\operatorname{diam}(f_1(B_j))}{\operatorname{diam}(f_1(2B))}.$$

Therefore,

$$\sum_{j=1}^{n} \operatorname{diam}\big(f(B_j)\big)^{1+k^2} \leqslant C(k)\left(\sum_{j=1}^{n} \operatorname{diam}(B_j)\right)^{1-k^2} \left(\frac{\operatorname{diam}(f(2B))}{\operatorname{diam}(f_1(2B))}\right)^{1+k^2} a^{2k^2}.$$

Recalling that $a = \operatorname{diam}(B)$ and taking into account that $\operatorname{diam}(f_1(2B)) \approx \operatorname{diam}(2B)$ by Koebe's distortion theorem, the corollary follows. □

3. Estimates for the "conformal outside" map

3.1. *Smooth and packed families of dyadic squares*

Let us denote the family of dyadic squares by $\mathcal{D}$. Let $0 < t < 2$ be fixed. Let $\{Q\}_{Q\in\mathcal{J}}$ be a family of pairwise disjoint dyadic squares. Given $\tau \geqslant 1$, we say that $\mathcal{J}$ is a τ-*smooth* family if

(a) If $P, Q \in \mathcal{J}$ are such that $2P \cap 2Q \neq \emptyset$, then $\tau^{-1}\ell(P) \leqslant \ell(Q) \leqslant \tau\ell(P)$.
(b) $\sum_{Q\in\mathcal{J}} \chi_{2Q} \leqslant \tau$.

Actually the condition (b) follows from (a) with a worse constant than τ, and the arguments below would also work if we eliminated (b) from the previous definition. However, for the sake of clarity and simplicity we prefer the definition above.

We say that $\mathcal{J}$ is α-packed if for any dyadic square $R \in \mathcal{D}$ which contains at least two squares from $\mathcal{J}$,

$$\sum_{Q\in\mathcal{J}:\ Q\subset R} \ell(Q)^t \leqslant \alpha\ell(R)^t. \tag{3.1}$$

At first sight, the fact that we ask (3.1) only for squares R which contain at least two squares from $\mathcal{J}$ may seem strange. In fact, (3.1) holds with $\alpha = 1$ for any square R which contains a unique square Q. The advantage of the formulation above is that for α arbitrarily small, there exist α-packed families $\mathcal{J}$, which is not the case if we allow R to contain a unique square $Q \in \mathcal{J}$. This fact will play an important role below.

Recall that a quasisquare is the image of a square under a quasiconformal map. Given a quasiconformal map f, we say that Q is a (dyadic) f-quasisquare if it is the image of a (dyadic) square under f. If $Q = f(P)$ is an f-quasisquare, we denote $aQ := f(aP)$. If now $\{Q\}_{Q\in\mathcal{J}}$ is family of pairwise disjoint dyadic f-quasisquares, we say that it is τ-smooth and α-packed if it verifies the properties above, replacing $\ell(P)$, $\ell(Q)$, $\ell(R)$ by $\operatorname{diam}(Q)$, $\operatorname{diam}(P)$, $\operatorname{diam}(R)$, and (3.1) is required to hold for all dyadic f-quasisquares R containing at least two quasisquares from $\mathcal{J}$.

3.2. Boundedness of the Beurling transform with respect to some weights

In the following, given a non-negative measurable function (i.e. a weight) ω and a subset $A \subset \mathbb{C}$, we use the standard notation

$$\omega(A) = \int_A \omega \, dm.$$

Recall also that the Beurling transform of a function $f : \mathbb{C} \to \mathbb{C}$ is given by

$$\mathcal{S}f(z) = \frac{-1}{\pi} \text{ p.v.} \int_{\mathbb{C}} \frac{f(\xi)}{(z-\xi)^2} \, dm(\xi).$$

Proposition 3.1. *Let $0 < t < 2$. Let $\mathcal{P} = \{P_i\}_{i=1}^N$ be a finite τ-smooth α-packed family of pairwise disjoint dyadic squares. Denote $\overline{P} = \bigcup_{i=1}^N P_i$ and set*

$$\omega = \sum_{P \in \mathcal{P}} \ell(P)^{t-2} \chi_P.$$

Then, the Beurling transform is bounded in $L^2(\omega)$. That is to say,

$$\left\| \mathcal{S}(f \chi_{\overline{P}}) \right\|_{L^2(\omega)} \leqslant C_1 \|f\|_{L^2(\omega)} \quad \text{for all } f \in L^2(\omega). \tag{3.2}$$

The constant C_1 only depends on t, τ, and α.

Proof. We will follow the arguments of [7] quite closely. By interpolation, it is enough to show that

$$\int_G \left| \mathcal{S}(\chi_F) \right| \omega \, dm \leqslant C_p \omega(F)^{1/p} \omega(G)^{1/p'} \quad \text{for all } F, G \subset \overline{P} \text{ and } 1 < p < \infty.$$

To prove this estimate, for $f \in L^1_{\text{loc}}(\mathbb{C})$, we split $\chi_{\overline{P}} \mathcal{S}(f \chi_{\overline{P}})$ into a local and a non-local part as follows

$$\chi_{\overline{P}} \mathcal{S}(f \chi_{\overline{P}}) = \sum_{i=1}^N \chi_{\overline{P} \cap 2P_i} \mathcal{S}(f \chi_{P_i}) + \sum_{i=1}^N \chi_{\overline{P} \setminus 2P_i} \mathcal{S}(f \chi_{P_i}) =: \mathcal{S}_{\text{local}}(f) + \mathcal{S}_{\text{non}}(f).$$

For the local part we will use the boundedness of $\mathcal{S}$ in $L^p(\mathbb{C})$, the fact that ω is a constant times Lebesgue measure on each P_i, and Hölder's inequality:

$$\begin{aligned} \int_G \left| \mathcal{S}_{\text{local}}(\chi_F) \right| \omega \, dm &\leqslant \sum_i \int_{2P_i \cap G} \left| \mathcal{S}(\chi_{P_i \cap F}) \right| w \, dm \\ &\leqslant C_p \sum_i |P_i \cap F|^{1/p} |2P_i \cap G|^{1/p'} \omega|_{P_i}, \end{aligned}$$

where $|A|$ stands for the Lebesgue measure of A. Since $\omega \approx \omega_{|P_i}$ on $2P_i \cap \overline{P}$, because of the property (a) in Section 3.1, using again Hölder's inequality, we get

$$\int_G |\mathcal{S}_{\text{local}}(\chi_F)|\omega\,dm \lesssim C_p \sum_i \omega(P_i \cap F)^{1/p}\omega(2P_i \cap G)^{1/p'}$$
$$\leqslant C_p\Big(\sum_i \omega(P_i \cap F)\Big)^{1/p}\Big(\sum_i \omega(2P_i \cap G)\Big)^{1/p'}$$
$$\leqslant C\omega(F)^{1/p}\omega(G)^{1/p'},$$

where we used the property (b) of smooth families in the last inequality.

Consider now the non-local part. For any two squares P, Q, denote

$$D(P,Q) = \operatorname{dist}(P,Q) + \ell(P) + \ell(Q).$$

Since for all $P, Q \in \mathcal{P}$ such that $Q \setminus 2P \neq \emptyset$,

$$\operatorname{dist}(P, Q \setminus 2P) \approx D(P,Q),$$

for $x \in P \in \mathcal{P}$ we have

$$|\mathcal{S}_{\text{non}} f(x)| \lesssim \sum_{Q \in \mathcal{P}:\ Q \neq P} \frac{1}{D(P,Q)^2} \int_Q |f|\,dm =: Tf(x).$$

We will show that

$$\int_G |T(\chi_F)|\omega\,dm \leqslant C_p \omega(F)^{1/p}\omega(G)^{1/p'}. \tag{3.3}$$

Let M_ω be the following maximal operator:

$$M_\omega f(x) = \sup_{x \in Q} \frac{1}{\ell(Q)^t} \int_Q |f|\omega\,dm,$$

where the supremum is taken over all the squares containing x. Since $\omega(Q) \leqslant C\ell(Q)^t$ for any square Q (by the packing condition), it follows by standard arguments (using covering lemmas) that M_ω is bounded in $L^p(\omega)$, $1 < p \leqslant \infty$, and from $L^1(\omega)$ to $L^{1,\infty}(\omega)$. To prove (3.3), for a fixed $C_1 > 2$ we define

$$F' = \begin{cases} F & \text{if } 8\omega(F) \leqslant \omega(G), \\ F \cap \{M_\omega(\chi_G) \leqslant C_1\omega(G)/\omega(F)\} & \text{otherwise.} \end{cases}$$

From the weak $(1,1)$ inequality for M_ω, we have $\omega(F') \geqslant \omega(F)/2$ if C_1 is chosen big enough. As in [7], to prove (3.3) it is enough to show that

$$\int_G (T\chi_{F'})\omega\,dm \leqslant C\min\bigl(\omega(F),\omega(G)\bigr)$$

(iterating this estimate, (3.3) follows). We have

$$\begin{aligned}\int_G (T\chi_{F'})\omega\,dm &= \sum_{P\in\mathcal{P}}\sum_{Q\in\mathcal{P}:\ P\neq Q}\frac{|F'\cap Q|}{D(P,Q)^2}\omega(G\cap P)\\ &= \sum_{Q\in\mathcal{P}}\bigl|F'\cap Q\bigr|\sum_{P\in\mathcal{P}:\ P\neq Q}\frac{1}{D(P,Q)^2}\omega(G\cap P).\end{aligned}$$

Denote by $A_k(Q)$ the family of squares $P\in\mathcal{P}$ such that

$$2^k\ell(Q)\leqslant D(P,Q)<2^{k+1}\ell(Q).$$

Then,

$$\begin{aligned}\sum_{P\in\mathcal{P}:\ P\neq Q}\frac{1}{D(P,Q)^2}\omega(G\cap P) &\leqslant C\sum_{k\geqslant 0}\frac{1}{\ell(2^kQ)^2}\sum_{P\in A_k(Q)}\omega(G\cap P)\\ &\leqslant C\sum_{k\geqslant 0}\frac{1}{\ell(2^kQ)^2}\omega\bigl(G\cap 2^{k+3}Q\bigr),\end{aligned}$$

because the squares $P\in A_k(Q)$ are contained $2^{k+3}Q$. By the packing condition (3.1), we have

$$\omega\bigl(G\cap 2^{k+3}Q\bigr)\leqslant C\ell\bigl(2^{k+3}Q\bigr)^t\leqslant C\ell\bigl(2^kQ\bigr)^t.$$

On the other hand, assuming $Q\cap F'\neq\emptyset$, by the definition of F',

$$\omega\bigl(G\cap 2^{k+3}Q\bigr)\leqslant C\ell\bigl(2^{k+3}Q\bigr)^t\frac{\omega(G)}{\omega(F)}\leqslant C\ell\bigl(2^kQ\bigr)^t\frac{\omega(G)}{\omega(F)}.$$

So we get,

$$\omega\bigl(G\cap 2^{k+3}Q\bigr)\leqslant C\min\left(1,\frac{\omega(G)}{\omega(F)}\right)\ell\bigl(2^kQ\bigr)^t =: CA\ell\bigl(2^kQ\bigr)^t.$$

Therefore, since $t-2<0$,

$$\begin{aligned}\int_G (T\chi_{F'})\omega\,dm &\leqslant CA\sum_{Q\in\mathcal{P}}\bigl|F'\cap Q\bigr|\sum_{k\geqslant 0}\ell\bigl(2^kQ\bigr)^{t-2}\leqslant CA\sum_{Q\in\mathcal{P}}\bigl|F'\cap Q\bigr|\ell(Q)^{t-2}\\ &= CA\omega\bigl(F'\bigr)\leqslant C\min\bigl(\omega(F),\omega(G)\bigr). \qquad\square\end{aligned}$$

Remark 3.2. Given a K-quasiconformal map $f:\mathbb{C}\to\mathbb{C}$, the proposition above also holds (by the same proof) if we consider a family of dyadic f-quasisquares instead of dyadic squares, with

$$\omega=\sum_{P\in\mathcal{P}}\operatorname{diam}(P)^{t-2}\chi_P.$$

Then, the constant C_1 in (3.2) depends on K, besides α, τ and t.

3.3. Distortion when f is conformal outside a family of quasisquares

Lemma 3.3. *Let $0<t<2$. Let $\{Q\}_{Q\in\mathcal{J}}$ be a finite τ-smooth α-packed family of pairwise disjoint dyadic g-quasisquares, where g is some K_0-quasiconformal map with $K_0\leqslant M_0$, $\alpha\leqslant 1$. Denote $F=\bigcup_{Q\in\mathcal{J}}Q$ and let $f:\mathbb{C}\to\mathbb{C}$ be a principal K-quasiconformal map, conformal on $\mathbb{C}\setminus F$. There exists $\delta_0=\delta_0(t,\tau,M_0)>1$ such that if $K\leqslant\delta_0$, then*

$$\sum_{Q\in\mathcal{J}}\operatorname{diam}\bigl(f(Q)\bigr)^t\leqslant C_2\sum_{Q\in\mathcal{J}}\operatorname{diam}(Q)^t,$$

with C_2 depending on only on t, τ, M_0.

The proof is analogous to the one in [7, Lemma 5.6], using the fact the Beurling transform is bounded in $L^2(\omega)$, by Proposition 3.1.

We wish to extend the preceding result to K-quasiconformal maps with K arbitrarily large. As usual, we will do this by an appropriate factorization of f. First we need the following technical result.

Lemma 3.4. *Let $0<t<2$. Let $\{Q\}_{Q\in\mathcal{J}}$ be a finite τ-smooth α-packed family of pairwise disjoint dyadic g-quasisquares, where g is some K_0-quasiconformal map with $K_0\leqslant M_0$, $\alpha\leqslant 1$. Denote $F=\bigcup_{Q\in\mathcal{J}}Q$ and let $f:\mathbb{C}\to\mathbb{C}$ be a principal K-quasiconformal map, conformal on $\mathbb{C}\setminus F$. There exists $\delta_0=\delta_0(t,\tau,M_0)>1$ such that if $K\leqslant\delta_0$, then for any g-dyadic quasisquare R,*

$$\sum_{Q\in\mathcal{J}:\ Q\subset R}\operatorname{diam}\bigl(f(Q)\bigr)^t\leqslant C_3\frac{\sum_{Q\in\mathcal{J}:\ Q\subset 3R}\operatorname{diam}(Q)^t}{\operatorname{diam}(R)^t}\operatorname{diam}\bigl(f(R)\bigr)^t,$$

with $C_3=C_3(t,\tau,M_0)$. In particular, the family of $(f\circ g)$-quasisquares $\{f(Q)\}_{Q\in\mathcal{J}}$ is $(9C_3\alpha)$-packed.

Proof. We factorize $f=f_2\circ f_1$, where f_1, f_2 are K-quasiconformal maps, with f_1 conformal in $(\mathbb{C}\setminus\bigcup_{Q\in\mathcal{J}}Q)\cup(\mathbb{C}\setminus 3R)$, and f_2 is conformal on $f_1(3R)$. By Lemma 3.3, we have

$$\sum_{Q\in\mathcal{J}:\ Q\subset 3R}\operatorname{diam}\bigl(f_1(Q)\bigr)^t\leqslant C_2\sum_{Q\in\mathcal{J}:\ Q\subset 3R}\operatorname{diam}(Q)^t. \tag{3.4}$$

By Koebe's distortion theorem and quasisymmetry, since f_2 is conformal in $f_1(3R)$, for every $Q\subset R$,

$$\frac{\operatorname{diam}(f_2(f_1(Q)))}{\operatorname{diam}(f_2(f_1(3R)))} \approx \frac{\operatorname{diam}(f_1(Q))}{\operatorname{diam}(f_1(3R))}.$$

Thus,

$$\sum_{Q\in\mathcal{J}:\ Q\subset R} \frac{\operatorname{diam}(f(Q))^t}{\operatorname{diam}(f(3R))^t} \approx \sum_{Q\in\mathcal{J}:\ Q\subset R} \frac{\operatorname{diam}(f_1(Q))^t}{\operatorname{diam}(f_1(3R))^t}.$$

The lemma follows from this estimate, (3.4), and the fact that $\operatorname{diam}(f_1(3R)) \approx \operatorname{diam}(3R)$, since f_1 is principal and conformal on $\mathbb{C} \setminus 3R$. □

Lemma 3.5. *Let $0 < t < 2$. Let $\{Q\}_{Q\in\mathcal{J}}$ be a finite τ-smooth α-packed family of pairwise disjoint dyadic g-quasisquares, where g is some K_0-quasiconformal map with $K_0 \leqslant M_0$. Denote $F = \bigcup_{Q\in\mathcal{J}} Q$ and let $f : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal map conformal on $\mathbb{C} \setminus F$ with $K \leqslant M_0$. There exists $\delta_1 > 0$ small enough depending only on t, τ, M_0 such that if $\alpha \leqslant \delta_1$, then*

$$\sum_{Q\in\mathcal{J}} \operatorname{diam}\bigl(f(Q)\bigr)^t \leqslant C_4 \sum_{Q\in\mathcal{J}} \operatorname{diam}(Q)^t,$$

with C_4 depending on only on t, α, τ, M_0.

Proof. Notice that for any K'-quasiconformal map h with $K' \leqslant M_0$, the family $\{h(Q)\}_{Q\in\mathcal{J}}$ is τ'-smooth, with τ' depending only on τ and M_0. Let n be big enough so that $M_0^{1/n} \leqslant \delta_0'$, where $\delta_0' = \delta_0(t, \tau', M_0)$ (with δ_0 from Lemma 3.3).

We factorize $f = f_n \circ \cdots \circ f_1$ so that each f_i is $K^{1/n}$-quasiconformal on $\mathbb{C}$, and moreover f_i is conformal in $\mathbb{C} \setminus \bigcup_{Q\in\mathcal{J}} f_{i-1} \circ \cdots \circ f_1(Q)$, for $i \geqslant 1$ (with $f_0 = id$). Notice that for all i, the quasisquares $f_i \circ \cdots \circ f_1(Q)$ are τ'-smooth (since $f_i \circ \cdots \circ f_1$ is K_i'-quasiconformal with $K_i' \leqslant M_0$). Suppose that α is small enough so that

$$\bigl(C_3'\bigr)^n \alpha \leqslant 1,$$

where $C_3' = 9C_3(t, \tau', M_0)$. Then, Lemma 3.4 can be applied repeatedly to deduce that for each $i \leqslant n$ the family of quasisquares $\{f_i \circ \cdots \circ f_1(Q)\}_{Q\in\mathcal{J}}$ is $(C_3')^i\alpha$-packed, and thus 1-packed (without loss of generality, we assume $C_3' \geqslant 1$). As a consequence, Lemma 3.3 can also be applied repeatedly to get

$$\sum_{Q\in\mathcal{J}} \operatorname{diam}\bigl(f(Q)\bigr)^t \leqslant C_2 \sum_{Q\in\mathcal{J}} \operatorname{diam}\bigl(f_{n-1} \circ \cdots \circ f_1(Q)\bigr)^t \leqslant \cdots \leqslant C_2^n \sum_{Q\in\mathcal{J}} \operatorname{diam}(Q)^t. \qquad \square$$

4. Gluing "conformal inside" and "conformal outside"

In the following lemma we make use of the conformal inside vs. outside decomposition.

Lemma 4.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a $\frac{1+k}{1-k}$-quasiconformal map with antisymmetric dilatation. Let $\{Q\}_{Q\in\mathcal{J}}$ be a finite family of pairwise disjoint squares with equal side length centered on $\mathbb{R}$,* *which are contained in another square R centered on $\mathbb{R}$. Then,*

$$\sum_{Q\in\mathcal{J}} \operatorname{diam}\bigl(f(Q)\bigr)^{1+k^2} \leqslant C(k)\left(\frac{\sum_{Q\in\mathcal{J}}\ell(Q)}{\ell(R)}\right)^{1-k^2} \operatorname{diam}\bigl(f(R)\bigr)^{1+k^2}. \tag{4.1}$$

Proof. First, let us make the assumption that f is principal. We are going to relax this assumption at the end of the proof. Using quasisymmetry if necessary, we may assume that the squares $Q \in \mathcal{J}$ belong to a dyadic lattice (a translation of the usual dyadic lattice, say). Take a small constant $0 < \alpha < 1$ to be fixed below. We will prove (4.1) assuming that $\mathcal{J}$ is α-packed. The general statement follows easily from this particular case: since the squares $Q \in \mathcal{J}$ have equal side length and are centered on $\mathbb{R}$, we can easily split $\mathcal{J} = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_m$ so that $m \approx 1/\alpha$ and each $\mathcal{J}_i$ is α-packed (recall that the constant α in the definition of α-packings only involves squares R which contain at least two squares $Q \in \mathcal{J}$). Then we apply (4.1) to each family $\mathcal{J}_i$ and we add the resulting estimates. This introduces an additional multiplicative constant m on the right-hand side of (4.1). We note here that $m \approx 1/\alpha$ and we will later choose α depending only on $K = \frac{1+k}{1-k}$.

So assume that the squares $Q \in \mathcal{J}$ are α-packed and denote $V = \bigcup_i Q_i$. Take a decomposition $f = f_2 \circ f_1$, where f_1, f_2 are principal K-quasiconformal mappings. We require f_1 to be conformal in V and f_2 outside $f_1(\overline{V})$. Further, we suppose that the dilatation of f_1 is $\mu_{f_1} = \chi_{\mathbb{C}\setminus V}\mu_f$ so that μ_{f_1} is also antisymmetric.

By Corollary 2.5, for any square P centered on $\mathbb{R}$,

$$\sum_{Q\in\mathcal{J}:\ Q\subset P} \operatorname{diam}\bigl(f_1(Q)\bigr)^{1+k^2} \leqslant C(k)\left(\frac{\sum_{Q\in\mathcal{J}:\ Q\subset P}\ell(Q)}{\ell(P)}\right)^{1-k^2} \operatorname{diam}\bigl(f_1(P)\bigr)^{1+k^2}. \tag{4.2}$$

In particular, the family of quasisquares $\{f_1(Q)\}_{Q\in\mathcal{J}}$ is $C(k)\alpha^{1-k^2}$-packed. It is also clear that they form a τ-smooth family, with τ depending only on K. Therefore, if α has been chosen small enough (depending only on K), from Lemma 3.5 we deduce that

$$\sum_{Q\in\mathcal{J}} \operatorname{diam}\bigl(f_2\bigl(f_1(Q)\bigr)\bigr)^{1+k^2} \leqslant C \sum_{Q\in\mathcal{J}} \operatorname{diam}\bigl(f_1(Q)\bigr)^{1+k^2},$$

and so by (4.2) with $P = R$,

$$\sum_{Q\in\mathcal{J}} \operatorname{diam}\bigl(f(Q)\bigr)^{1+k^2} \leqslant C(k)\left(\frac{\sum_{Q\in\mathcal{J}}\ell(Q)}{\ell(R)}\right)^{1-k^2} \operatorname{diam}\bigl(f_1(R)\bigr)^{1+k^2}.$$

Since f_2 is principal and conformal outside $f_1(R)$, by Koebe's distortion theorem we deduce $\operatorname{diam}(f(R)) \approx \operatorname{diam}(f_1(R))$, and thus (4.1) follows for a principal mapping. We reduce the general case to this one. Suppose that f is not necessarily principal antisymmetric map. Take a decomposition $f = g_2 \circ g_1$, where g_1 is principal antisymmetric K-quasiconformal map which is conformal outside $3R$ and g_2 is a K-quasiconformal map which is conformal on $g_1(3R)$. This decomposition is analogous to the one used in Corollary 2.5 and again by Koebe's distortion theorem and quasisymmetry we have for every $Q \in \mathcal{J}$

$$\frac{\operatorname{diam}(f(Q))}{\operatorname{diam}(f(3R))} \approx \frac{\operatorname{diam}(g_1(Q))}{\operatorname{diam}(g_1(3R))}.$$

Now the lemma follows from (4.1) applied to the principal map g_1. □

Theorem 4.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be an antisymmetric $\frac{1+k}{1-k}$-quasiconformal map. Then, for any compact subset $E \subset \mathbb{R}$ and any ball $B \subset \mathbb{C}$ centered on $\mathbb{R}$ which contains E,*

$$H^{1+k^2}\big(f(E)\big) \leqslant C(k)\operatorname{diam}\big(f(B)\big)^{1+k^2}\left(\frac{H^1(E)}{\operatorname{diam}(B)}\right)^{1-k^2}.$$

Proof. Consider an arbitrary covering $E \subset \bigcup_i I_i$ for a finite number of pairwise different dyadic intervals of length $\varepsilon \operatorname{diam}(B)$. Consider squares concentric with I_i with $\ell(Q_i) = \operatorname{diam}(I_i)$. By Lemma 4.1 and quasisymmetry we deduce that

$$\sum_i \operatorname{diam}\big(f(I_i)\big)^{1+k^2} \leqslant C(k)\left(\frac{\sum_i \operatorname{diam}(I_i)}{\ell(B)}\right)^{1-k^2} \operatorname{diam}\big(f(B)\big)^{1+k^2}.$$

Because of the Hölder continuity of quasiconformal maps (see e.g. [2]), for each i, with a constant C_5 depending on k,

$$\frac{\operatorname{diam}(f(I_i))}{\operatorname{diam}(f(B))} \leqslant C_5\left(\frac{\operatorname{diam}(I_i)}{\operatorname{diam}(B)}\right)^{\frac{1-k}{1+k}} = C_5\varepsilon^{\frac{1-k}{1+k}}.$$

Therefore, with $\delta = C_5\varepsilon^{\frac{1-k}{1+k}}$

$$H_\delta^{1+k^2}\big(f(E)\big) \leqslant C(k)\left(\frac{\sum_i \operatorname{diam}(I_i)}{\operatorname{diam}(B)}\right)^{1-k^2} \operatorname{diam}\big(f(B)\big)^{1+k^2}.$$

By the definition of length on $\mathbb{R}$, we have $\sum_i \operatorname{diam}(I_i) \leqslant H^1(U_\varepsilon(E) \cap \mathbb{R})$, where U_ε stands for the ε-neighborhood, and we assume that $I_i \cap E \neq \emptyset$. Thus,

$$H_\delta^{1+k^2}\big(f(E)\big) \leqslant C(k)\left(\frac{H^1(U_{\varepsilon \operatorname{diam}(B)}(E) \cap \mathbb{R})}{\operatorname{diam}(B)}\right)^{1-k^2} \operatorname{diam}\big(f(B)\big)^{1+k^2}.$$

Letting $\varepsilon \to 0$, the theorem follows. □

Now we are ready to prove Theorem 1.1 stated in the introduction.

Proof of Theorem 1.1. If Γ is a K-quasiline, then $\Gamma = f(\mathbb{R})$ with some K-quasiconformal map $f : \mathbb{C} \to \mathbb{C}$. As we remarked in Section 2, we may further suppose that f is antisymmetric. Our goal is to show

$$H^{1+k^2}\big(\Gamma \cap B(z,r)\big) \leqslant C(k)r^{1+k^2} \quad \text{for all } z \in \mathbb{C}. \tag{4.3}$$

First of all, we may assume that $z \in \Gamma$. Indeed, if $\Gamma \cap B(z,r) = \emptyset$ then there is nothing to prove. Otherwise, we can find $z_0 \in \Gamma$ such that $B(z,r) \subset B(z_0, 2r)$ and hence by replacing $B(z,r)$ with a twice larger disk we may assume that the center lies on Γ. Let us set $E = f^{-1}(\bar{B} \cap \Gamma) \subset \mathbb{R}$. Using quasisymmetry we can easily find a disk B_0 centered on $\mathbb{R}$ which contains E and such that $\operatorname{diam} f(B_0) \approx r$. We apply now Theorem 4.2 with E and B_0 as above and find that (4.3) holds true.

The case where Γ is a quasicircle in $\mathbb{C}$ can be reduced to the one of a quasiline. Indeed, with the help of a Möbius transformation we may pass to a quasiline and see that (4.3) holds, for instance, with $r \leqslant \operatorname{diam} \Gamma/10$. For $r > \operatorname{diam} \Gamma/10$ we just use (4.3) for a finite number of disks of radius $\operatorname{diam} \Gamma/10$. □

5. Boundary expansion of the Riemann map

The Riemann map $\phi : \mathbb{D} \to \Omega$ onto a quasidisk is Hölder continuous up to the boundary, in short, quasidisks are Hölder domains. For a map with a K-quasiconformal extension the sharp Hölder exponent is $1 - k$ [8], where $k = (K-1)/(K+1)$, as usual. Very recently, the following counterpart was established in terms of the integrability of the derivative.

Theorem 5.1. *(See [9, Corollary 3.9].) If $\phi : \mathbb{D} \to \mathbb{C}$ is a conformal map with K-quasiconformal extension then*

$$\phi' \in L^p(\mathbb{D}) \quad \text{for all } 2 \leqslant p < \frac{2(K+1)}{K-1}.$$

The upper bound for the exponent is the best possible.

In the next theorem we prove the weak-integrability of ϕ' in the borderline case $p = \frac{2}{k} = \frac{2(K+1)}{K-1}$. In terms of area distortion for subsets of the unit disk, the exponent $1/K$ from Astala's theorem improves to $1-k$, just as the Hölder continuity exponent does.

Theorem 5.2. *If $\phi : \mathbb{D} \to \mathbb{C}$ is a conformal map with K-quasiconformal extension to $\mathbb{C}$, then $\phi' \in$ weak-$L^{2/k}(\mathbb{D})$ with $k = (K-1)/(K+1)$. More precisely,*

$$\left|\left\{z \in \mathbb{D}\colon \left|\phi'(z)\right| > \rho\right\}\right| \leqslant C(K)\left|\phi'(0)\right|^{2/k} \rho^{-2/k} \quad \text{for any } \rho > 0. \tag{5.1}$$

In terms of area distortion, for any Borel set $E \subset \mathbb{D}$,

$$\left|\phi(E)\right| \leqslant C(K)\left|\phi'(0)\right|^2 |E|^{1-k}.$$

Remark 5.3. The power map $z \mapsto z^{1-k}$ maps conformally the upper half-plane to a sector domain of angle $(1-k)\pi$ and admits a $\frac{1+k}{1-k}$-quasiconformal extension to $\mathbb{C}$ [4]. This example shows that Theorem 5.2 is sharp up to the numerical value of the constant terms involved.

First we will prove the following lemma, as an application of the theorems of Smirnov and Astala.

Lemma 5.4. *Let $\psi : \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal map which is conformal in $\mathbb{C}_+ = \{z\colon \operatorname{Im} z > 0\}$ and outside $\mathbb{D}$. Let $B_j = B(z_j, r_j)$, $1 \leqslant j \leqslant n$, be a collection of pairwise disjoint disks contained in the unit disk such that $z_j \in \mathbb{R}$. We set $E = \bigcup B_j$. Then we have the following estimate for area expansion*

$$\bigl|\psi(E)\bigr| \leqslant C(K)|E|^{1-k}, \tag{5.2}$$

with $k = (K-1)/(K+1)$.

Proof. In order to deduce (5.2) it is sufficient to assume that ψ is conformal in E. Otherwise, we use the usual decomposition to principal K-quasiconformal mappings $\psi = \psi_2 \circ \psi_1$ where ψ_1 is conformal on E and ψ_2 conformal outside $\psi_1(E)$. Now invoking the fact $|\psi_2(\psi_1(E))| \leqslant K|\psi_1(E)|$ from [1] matters are reduced to ψ_1. In the rest of the proof we assume that ψ is conformal in E and for notational convenience replace K by K^2, that is, ψ assumed to be globally K^2-quasiconformal. The exponent $1-k$ in (5.2) then takes the form

$$1 - \frac{K^2-1}{K^2+1} = \frac{2}{K^2+1} = \frac{1}{K} \cdot \frac{1-k^2}{1+k^2}, \tag{5.3}$$

where $k = (K-1)/(K+1)$.

We use the symmetrization result of [10]: ψ can be written as a superposition $\psi = f \circ g$ of a K-quasiconformal map g symmetric with respect to $\mathbb{R}$ followed by a K-quasiconformal antisymmetric map f. Both of these maps are normalized to be principal. We observe that under the conformality assumptions on ψ, the map g is conformal in the disks B_j and outside $\mathbb{D}$ while f is conformal in the quasidisks $g(B_j)$ and outside $g(\mathbb{D})$. In view of Koebe's $1/4$ theorem,

$$\hat{B}_j := B\biggl(g(z_j), \frac{1}{4}\bigl|g'(z_j)\bigr|r_j\biggr) \subset g(B_j) \quad \text{and} \quad g(\mathbb{D}) \subset B(0,2).$$

We apply Theorem 2.4 with $t = 2$ for the map f and disks $\hat{B}_j \subset B(0,2)$,

$$\sum_{j=1}^{n} \bigl(\bigl|f'\bigl(g(z_j)\bigr)\bigr|\bigl|g'(z_j)\bigr|r_j\bigr)^2 \leqslant C\Biggl(\sum_{j=1}^{n}\bigl(\bigl|g'(z_j)\bigr|r_j\bigr)^2\Biggr)^{\frac{1-k^2}{1+k^2}}.$$

As $\psi'(z_j) = f'(g(z_j))g'(z_j)$, we may write the previous inequality as the comparison of area

$$\bigl|\psi(E)\bigr| \approx \sum_{j=1}^{n}\bigl(\bigl|\psi'(z_j)\bigr|r_j\bigr)^2 \leqslant C(K)\bigl|g(E)\bigr|^{\frac{1-k^2}{1+k^2}}.$$

For the map g we use the area distortion inequality

$$\bigl|g(E)\bigr| \leqslant C(K)|E|^{\frac{1}{K}}$$

from [1] and conclude the proof by (5.3). □

Next, we sketch the reduction of Theorem 5.2 to Lemma 5.4.

Proof of Theorem 5.2. By our assumption $\phi(\infty)=\infty$ and we may also require $\phi(0)=0$. Since $\phi(\mathbb{D})$ is a quasidisk $|\phi'(0)|\approx|\phi(1)|$, so we may use the third normalization $\phi(1)=1$. For an arc $I\subset\partial\mathbb{D}$, consider the Carleson "square" with base I given by $Q_I=\{z\colon z/|z|\in I$ and $1-\ell(I)\leqslant|z|\leqslant 1\}$. The top of Q_I is $z_I=(1-\ell(I))\zeta_I$ where ζ_I is the center of I. We are going to use the following property of a conformal map to quasidisk target: for any arc I on $\partial\mathbb{D}$ we have $|\phi'(z_I)|\cdot\operatorname{diam} I\approx\operatorname{diam}\phi(I)$. Furthermore for the top half of the Carleson square $|\phi'(z)|\approx|\phi'(z_I)|$. We are left to estimate the area of disjoint Carleson squares such that $|\phi'(z_I)|>\rho$. We will only do this in a fixed sector S about -1. Let us transfer the situation from the disk to the upper half-plane $\mathbb{C}_+=\{\operatorname{Im} z>0\}$. We denote by T the following Möbius transformation

$$T(z)=\frac{z-i}{z+i}.$$

Then $T(\mathbb{C}_+)=\mathbb{D}$ and $T(i)=0$, $T(0)=-1$ and $T(\infty)=1$. The conjugated map $\psi=T^{-1}\circ\phi\circ T$ is conformal in $\mathbb{C}_+$, globally K-quasiconformal and satisfies $\psi(i)=i$, $\psi(-i)=-i$ and $\psi(\infty)=\infty$. We choose the inscribed disk B_j inside the image of Q_I under T^{-1} and its reflection along $\mathbb{R}$. Then $\operatorname{diam}(\psi B_j)\gtrsim\rho\operatorname{diam} B_j$ because $|\phi'(z_I)|>\rho$. For the set $E=\bigcup B_j$ we have

$$\rho^2|E|\approx\sum_{j=1}^{n}(\rho\operatorname{diam} B_j)^2\leqslant\sum_{j=1}^{n}(\operatorname{diam}\psi B_j)^2\approx\bigl|\psi(E)\bigr|.$$

With an appropriate choice of the sector S, we may assume $E\subset B(0,1/2)$ and apply Lemma 5.4 for the principal map ψ_1 with dilatation $\chi_{\mathbb{D}}\mu_\psi$,

$$\bigl|\psi(E)\bigr|\lesssim\bigl|\psi_1(E)\bigr|\leqslant C(K)|E|^{1-k}.$$

Combining the last two inequalities we obtain the desired estimate

$$\bigl|\bigl\{|\phi'|>\rho\bigr\}\bigr|\lesssim\left|\bigcup_{|\phi'(z_I)|>\rho}Q_I\right|\approx|E|\lesssim\rho^{-\frac{2}{k}}.$$

This proves the first part of the theorem.

In order to prove the second part, we proceed as follows. Consider now an arbitrary Borel set $E\subset\mathbb{D}$,

$$\bigl|\phi(E)\bigr|=\int_E\bigl|\phi'(z)\bigr|^2\,dm(z)=2\int_0^\infty\rho\bigl|\bigl\{z\in E\colon\bigl|\phi'(z)\bigr|>\rho\bigr\}\bigr|\,d\rho.$$

We split the integral to two parts at $T=|E|^{-k/2}$. On the interval $[0,T]$ we use the trivial estimate $T|E|$ for the integrand and on $[T,\infty]$ we use the weak-integrability (5.1). The claimed area distortion inequality now follows

$$\bigl|\phi(E)\bigr|\leqslant 2|E|T^2+C(K)T^{2(1-1/k)}\leqslant C(K)|E|^{1-k}. \qquad\square$$

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