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SOME PROPERTIES OF GENERALIZED CONVOLUTION OF HARMONIC UNIVALENT FUNCTIONS

Abstract. The purpose of the present paper is to investigate some interesting properties on generalized convolutions of functions for the classes $HP^*(\alpha)$, $HS(\alpha)$ and $HC(\alpha)$. Further, an application of the convolution on certain integral operator are mentioned.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D , if both u and v are real harmonic in D . In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$, (see Clunie and Sheil-Small [7]). For more basic results on harmonic functions one may refer following standard introductory text book by Duren [14], (see also Ahuja [1] and Ponnusamy and Rasila [?]).

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in S_H$, we may express the analytic functions h and g as

$$(1.1) \quad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$

A function $f(z)$ of the form (1.1) in S_H is said to be harmonic starlike of order α , ($0 \leq \alpha < 1$) in U , if and only if

$$(1.2) \quad \frac{\partial}{\partial \theta} \{\arg f(z)\} > \alpha, \quad z \in U,$$

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and is said to be harmonic convex of order α , ($0 \leq \alpha < 1$) in U , if and only if

$$(1.3) \quad \frac{\partial}{\partial \theta} \left\{ \arg \left(\frac{\partial}{\partial \theta} f(z) \right) \right\} > \alpha, \quad z \in U.$$

The classes of all harmonic starlike functions of order α and harmonic convex functions of order α are denoted by $S_H^*(\alpha)$ and $K_H(\alpha)$, respectively. These classes have been extensively studied by Jahangiri [17].

For $\alpha = 0$, the classes $S_H^*(\alpha)$ and $K_H(\alpha)$ are denoted by S_H^* and K_H , respectively. These classes were studied in detail by Silverman [34] and Silverman and Silvia [35], (see also Avci and Zlotkiewicz [2]).

A necessary and sufficient condition for a function $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are of the form

$$(1.4) \quad h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \text{ and } g(z) = - \sum_{n=1}^{\infty} |b_n| z^n,$$

belong to the classes S_H^* and K_H

$$(1.5) \quad \sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1$$

and

$$(1.6) \quad \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \leq 1,$$

respectively, (see [34]).

Recently, Öztürk and Yalcin [27] defined two new subclasses $HS(\alpha) \subset S_H^*$ and $HC(\alpha) \subset K_H$ of the form (1.1) satisfying the condition

$$(1.7) \quad \sum_{n=2}^{\infty} (n - \alpha)(|a_n| + |b_n|) \leq (1 - \alpha)(1 - |b_1|)$$

and

$$(1.8) \quad \sum_{n=2}^{\infty} n(n - \alpha)(|a_n| + |b_n|) \leq (1 - \alpha)(1 - |b_1|),$$

respectively.

The results of the classes $HS(\alpha)$ and $HC(\alpha)$ were extended by Dixit and Porwal in [10].

A function $f(z)$ of the form (1.1) in S_H is said to be in the class $HP(\alpha)$, if and only if

$$(1.9) \quad \operatorname{Re} \{h'(z) + g'(z)\} > \alpha, \quad z \in U,$$

for some α ($0 \leq \alpha < 1$).

We further denote by $HP^*(\alpha)$ the subclass of $HP(\alpha)$ such that, the functions h and g in $f = h + \bar{g}$ are of the form (1.4).

The classes $HP(\alpha)$ and $HP^*(\alpha)$ have been extensively studied by Karpuzoğullari et al. [19].

Let $f_j(z)$ ($j = 1, 2$) in S_H be given by

$$(1.10) \quad f_j(z) = z - \sum_{n=2}^{\infty} |a_{n,j}|z^n - \sum_{n=1}^{\infty} |b_{n,j}|\bar{z}^n.$$

Then the convolution $f_1 * f_2$ is defined by

$$(1.11) \quad (f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} |a_{n,1}a_{n,2}|z^n - \sum_{n=1}^{\infty} |b_{n,1}b_{n,2}|\bar{z}^n.$$

Furthermore, for any real number p and q , we define the generalized convolution $(f_1 \Delta f_2)(p, q; z)$ by

$$(1.12) \quad (f_1 \Delta f_2)(p, q; z) = z - \sum_{n=2}^{\infty} |a_{n,1}|^p |a_{n,2}|^q z^n - \sum_{n=1}^{\infty} |b_{n,1}|^p |b_{n,2}|^q \bar{z}^n.$$

In the special case, if we take $p = q = 1$, then we have

$$(1.13) \quad (f_1 \Delta f_2)(1, 1; z) = (f_1 * f_2)(z), \quad (z \in U).$$

Studies of convolution play an important role in Geometric Function Theory. It has attracted large number of researchers of the field. By making use of convolution, several new and interesting subclasses have been defined and studied in the direction of subordination, partial sums, neighborhood, argument problems, integral mean inequalities and some other related interesting properties. For detailed study see the excellent text book by Ruscheweyh [32], (see also [3], [11], [12], [15], [16], [18], [24], [30], [36], [37]). In 1975, Schild and Silverman [33] studied the various interesting results on the convolution of analytic functions. Later on, Choi et al. [6], Darwish [8], Darwish and Aouf [9], Domokos [13], Nishiwaki and Owa [22], Nishiwaki et al. [23], Owa [25] and Srivastava et al. [38] studied the generalized convolution for analytic functions only. Although, analogous to these results on harmonic functions have not been explored so far in the literature. In the present paper, an attempt has been made to study systematically on the generalized convolution of harmonic univalent functions analogous to analytic univalent functions. Finally, we give an appealing application of convolution on the certain integral operator in the class $HP^*(\beta)$.

2. Main results

In order to prove our results for functions to the class $HP^*(\alpha)$, we shall need the following lemma given by Karpuzoğullari et al. [19].

LEMMA 2.1. *Let the function $f(z)$ be defined by (1.4). Then $f \in HP^*(\alpha)$, if and only if*

$$(2.1) \quad \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1 - \alpha,$$

where $0 \leq \alpha < 1$.

THEOREM 2.1. *If the functions $f_j(z)$ ($j = 1, 2$) defined by (1.10) with $b_{1,j} = 0$ ($j = 1, 2$) are in the classes $HP^*(\alpha_j)$ for each j and the condition*

$$(2.2) \quad |a_{n,1}|^{\frac{1}{p}}|a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}}|b_{n,2}|^{\frac{1}{q}} \leq (|a_{n,1}| + |b_{n,1}|)^{\frac{1}{p}}(|a_{n,2}| + |b_{n,2}|)^{\frac{1}{q}}$$

for ($n = 2, 3, \dots$) is satisfied then

$$(2.3) \quad (f_1 \triangle f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in HP^*(\alpha),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha = 1 - (1 - \alpha_1)^{\frac{1}{p}}(1 - \alpha_2)^{\frac{1}{q}}$.

Proof. Since $f_j(z) \in HP^*(\alpha_j)$, by using Lemma 2.1, we have

$$(2.4) \quad \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_j} \right) (|a_{n,j}| + |b_{n,j}|) \leq 1, \quad (j = 1, 2).$$

From (2.4) we have

$$(2.5) \quad \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_1} \right) (|a_{n,1}| + |b_{n,1}|) \right\}^{\frac{1}{p}} \leq 1,$$

and

$$(2.6) \quad \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_2} \right) (|a_{n,2}| + |b_{n,2}|) \right\}^{\frac{1}{q}} \leq 1.$$

Now

$$\begin{aligned} & \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_1} \right)^{\frac{1}{p}} \left(\frac{n}{1 - \alpha_2} \right)^{\frac{1}{q}} \left(|a_{n,1}|^{\frac{1}{p}}|a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}}|b_{n,2}|^{\frac{1}{q}} \right) \\ & \leq \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_1} \right)^{\frac{1}{p}} \left(\frac{n}{1 - \alpha_2} \right)^{\frac{1}{q}} (|a_{n,1}| + |b_{n,1}|)^{\frac{1}{p}} (|a_{n,2}| + |b_{n,2}|)^{\frac{1}{q}} \quad (\text{using (2.2)}) \\ & \leq \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_1} \right) (|a_{n,1}| + |b_{n,1}|) \right\}^{\frac{1}{p}} \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_2} \right) (|a_{n,2}| + |b_{n,2}|) \right\}^{\frac{1}{q}} \\ & \hspace{15em} (\text{using Hölder's Inequality}) \\ & \leq 1, \quad (\text{using (2.5) and (2.6)}). \end{aligned}$$

Since

$$(2.7) \quad (f_1 \triangle f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) = z - \sum_{n=2}^{\infty} |a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} z^n - \sum_{n=2}^{\infty} |b_{n,1}|^{\frac{1}{p}} |b_{n,2}|^{\frac{1}{q}} \bar{z}^n$$

it suffices to show that $(f_1 \triangle f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in HP^*(\alpha)$ if

$$(2.8) \quad \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha} \right) \left(|a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}} |b_{n,2}|^{\frac{1}{q}} \right) \leq 1.$$

For this we have to show that L.H.S. of (2.8) is bounded above by

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha_1} \right)^{\frac{1}{p}} \left(\frac{n}{1-\alpha_2} \right)^{\frac{1}{q}} \left(|a_{n,1}|^{\frac{1}{p}} |a_{n,2}|^{\frac{1}{q}} + |b_{n,1}|^{\frac{1}{p}} |b_{n,2}|^{\frac{1}{q}} \right),$$

which is equivalent to $\alpha \leq 1 - (1 - \alpha_1)^{\frac{1}{p}} (1 - \alpha_2)^{\frac{1}{q}}$. ■

COROLLARY 2.2. *If the functions $f_j(z)$ ($j = 1, 2$) given by (1.10) with $b_{1,j} = 0$ ($j = 1, 2$) are in the class $HP^*(\alpha)$ and condition (2.2) is satisfied, then*

$$(f_1 \triangle f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in HP^*(\alpha), \quad (p > 1).$$

Proof. In view of Lemma 2.1, Corollary 2.2 follows readily from Theorem 2.1 in special case when $\alpha_j = \alpha$. ■

The proof of Theorems 2.3, 2.4 and 2.5 are much akin to that of Theorem 2.1, so we only state these theorems.

THEOREM 2.3. *If the functions $f_j(z)$ ($j = 1, 2$) defined by (1.10) with $b_{1,j} = 0$ ($j = 1, 2$) are in the classes $HS(\alpha_j)$ for each j and the condition (2.2) is satisfied then*

$$(2.9) \quad (f_1 \triangle f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in HP^*(\alpha),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$(2.10) \quad \alpha = 1 - \frac{2}{\left(\frac{2-\alpha_1}{1-\alpha_1} \right)^{\frac{1}{p}} \left(\frac{2-\alpha_2}{1-\alpha_2} \right)^{\frac{1}{q}}}.$$

THEOREM 2.4. *If the functions $f_j(z)$ ($j = 1, 2$) defined by (1.10) with $b_{1,j} = 0$ ($j = 1, 2$) are in the classes $HC(\alpha_j)$ for each j and the condition (2.2) is satisfied then*

$$(2.11) \quad (f_1 \triangle f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in HP^*(\alpha),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$(2.12) \quad \alpha = 1 - \frac{1}{\left(\frac{2-\alpha_1}{1-\alpha_1}\right)^{\frac{1}{p}} \left(\frac{2-\alpha_2}{1-\alpha_2}\right)^{\frac{1}{q}}}.$$

THEOREM 2.5. *If the functions $f_1(z)$ and $f_2(z)$ defined by (1.10) with $b_{1,j} = 0$ ($j = 1, 2$) are in the classes $HS(\alpha_1)$ and $HC(\alpha_2)$, respectively, and the condition (2.2) is satisfied then*

$$(2.13) \quad (f_1 \triangle f_2) \left(\frac{1}{p}, \frac{1}{q}; z \right) \in H P^*(\alpha),$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$(2.14) \quad \alpha = 1 - \frac{2^{\frac{1}{p}}}{\left(\frac{2-\alpha_1}{1-\alpha_1}\right)^{\frac{1}{p}} \left(\frac{2-\alpha_2}{1-\alpha_2}\right)^{\frac{1}{q}}}.$$

THEOREM 2.6. *Let the functions $f_j(z)$ ($j = 1, 2, \dots, m$) defined by (1.10) with $b_{1,j} = 0$ ($j = 1, 2, \dots, m$), be in the classes $HP^*(\alpha_j)$ for each j and let $F_m(z)$ be defined by*

$$(2.15) \quad F_m(z) = z - \sum_{n=2}^{\infty} \left(\sum_{j=1}^m |a_{n,j}|^p \right) z^n - \sum_{n=2}^{\infty} \left(\sum_{j=1}^m |b_{n,j}|^p \right) \bar{z}^n, \quad (p \geq 1).$$

Then $F_m(z) \in HP^*(\alpha_m)$, where $\alpha_m = 1 - \frac{m(1-\alpha)^p}{2^{p-1}}$, $\alpha = \min_{1 \leq j \leq m} \alpha_j$ and $m(1-\alpha)^p \leq 2^{p-1}$.

Proof. Since $f_j(z) \in HP^*(\alpha_j)$, using Lemma 2.1, we observe that

$$(2.16) \quad \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha_j} \right) (|a_{n,j}| + |b_{n,j}|) \leq 1, \quad (j = 1, 2, \dots, m)$$

and

$$(2.17) \quad \begin{aligned} \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha_j} \right)^p (|a_{n,j}|^p + |b_{n,j}|^p) &\leq \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha_j} \right)^p (|a_{n,j}| + |b_{n,j}|)^p \\ &\leq \left\{ \sum_{n=2}^{\infty} \left(\frac{n}{1-\alpha_j} \right) (|a_{n,j}| + |b_{n,j}|) \right\}^p \leq 1. \quad (\text{Using (2.16)}). \end{aligned}$$

It follows from (2.17) that

$$(2.18) \quad \sum_{n=2}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^m \left(\frac{n}{1-\alpha_j} \right)^p (|a_{n,j}|^p + |b_{n,j}|^p) \right\} \leq 1.$$

Putting $\alpha = \min_{1 \leq j \leq m} \alpha_j$, and by virtue of Lemma 2.1, we find that

$$\begin{aligned} \sum_{n=2}^{\infty} \left(\frac{n}{1 - \alpha_m} \right) \left(\sum_{j=1}^m |a_{n,j}|^p + |b_{n,j}|^p \right) \\ \leq \sum_{n=2}^{\infty} \frac{1}{m} \left(\frac{n}{1 - \alpha} \right)^p \left(\sum_{j=1}^m (|a_{n,j}|^p + |b_{n,j}|^p) \right) \leq 1, \\ \text{if } \alpha_m \leq 1 - \frac{m(1 - \alpha)^p}{n^{p-1}}, \quad (n \geq 2). \end{aligned}$$

Now let $g(n) = 1 - \frac{m(1 - \alpha)^p}{n^{p-1}}$.

It is easy to verify that $g(n)$ is an increasing function of n for $p \geq 1$. Therefore,

$$\alpha_m = \inf_{n \geq 2} g(n) = g(2) = 1 - \frac{m(1 - \alpha)^p}{2^{p-1}}.$$

By $m(1 - \alpha)^p \leq 2^{p-1}$, we see that $0 \leq \alpha_m < 1$. This completes the proof of Theorem 2.6. ■

The proof of Theorems 2.7 and 2.8 are similar to that of Theorem 2.6, so we only state these theorems.

THEOREM 2.7. *Let the functions $f_j(z)$ ($j = 1, 2, \dots, m$) defined by (1.10) with $b_{1,j} = 0$ ($j = 1, 2, \dots, m$), be in the classes $HS(\alpha_j)$ for each j and let $F_m(z)$ be defined by (2.15). Then $F_m(z) \in HP^*(\alpha_m)$, where $\alpha_m = 1 - \frac{2m(1 - \alpha)^p}{(2 - \alpha)^p}$, $\alpha = \min_{1 \leq j \leq m} \alpha_j$ and $2m(1 - \alpha)^p \leq (2 - \alpha)^p$.*

THEOREM 2.8. *Let the functions $f_j(z)$ ($j = 1, 2, \dots, m$) defined by (1.10) with $b_{1,j} = 0$ ($j = 1, 2, \dots, m$), be in the classes $HC(\alpha_j)$ for each j and let $F_m(z)$ be defined by (2.15). Then $F_m(z) \in HP^*(\alpha_m)$, where $\alpha_m = 1 - \frac{m(1 - \alpha)^p}{2^{p-1}(2 - \alpha)^p}$, $\alpha = \min_{1 \leq j \leq m} \alpha_j$ and $m(1 - \alpha)^p \leq 2^{p-1}(2 - \alpha)^p$.*

3. Application of the convolution on certain integral operator

In this section, we study mapping properties of the integral operator $J_c(f)$ on the class $HP^*(\alpha)$ in which we show that $J_c(f) \in HP^*(\beta)$ if $f \in HP^*(\alpha)$, the result is sharp.

The convolution of the two functions of the form

$$(3.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$(3.2) \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

is given by

$$(3.3) \quad (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Now we define the function $\phi(a, c; z)$ by

$$(3.4) \quad \phi(a, c; z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n, \quad (c \neq 0, -1, -2, \dots),$$

where $(\lambda)_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$\begin{aligned} (\lambda)_n &= \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)(\lambda + 2) \dots (\lambda + n - 1) & (n \in N = \{1, 2, 3, \dots\}). \end{cases} \end{aligned}$$

The function $\phi(a, c; z)$ is an incomplete function related to the Gauss hypergeometric function by

$$(3.5) \quad \phi(a, c; z) = zF(1, a; c; z).$$

Carlson and Shaffer [5] defined a linear operator $L(a, c)$, corresponding to the function $\phi(a, c; z)$ on A via the convolution, where A is the class of functions of the form (3.1) which are analytic in the open unit disc U .

$$(3.6) \quad L(a, c)h(z) = \phi(a, c; z) * h(z), \quad (h(z) \in A).$$

If $c > a > 0$, $L(a, c)$ has the integral representation

$$(3.7) \quad L(a, c)h(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-2}(1-u)^{c-a-1} h(uz) du.$$

Clearly, $L(a, a)$ is the identity operator and

$$L(a, c) = L(a, b).L(b, c) = L(b, c).L(a, b), \quad (b, c \neq 0, -1, -2, \dots).$$

Moreover if $a \neq 0, -1, -2, \dots$, then $L(a, c)$ has an inverse $L(c, a)$ and is a one-one mapping of A onto itself (see Owa and Srivastava [26]).

Bernardi [4] defined the integral operator J_c by

$$\begin{aligned}
 J_c(f) &= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (c > -1) \\
 (3.8) \quad &= z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n = L(c+1, c+2) f(z)
 \end{aligned}$$

or

$$(3.9) \quad J_c(f) = \phi(c+1, c+2; z) * f(z).$$

The operator J_c was studied earlier by Libera [20] and Livingston [21].

Now, we define the Bernardi integral operator $J_c(f)$ on the class S_H of harmonic univalent functions of the form (1.1) as follows:

$$\begin{aligned}
 J_c(f) &= J_c(h) + \overline{J_c(g)} \\
 (3.10) \quad &= z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n + \overline{\sum_{n=1}^{\infty} \frac{c+1}{c+n} b_n z^n}
 \end{aligned}$$

$$(3.11) \quad = L(c+1, c+2)h(z) + \overline{L(c+1, c+2)g(z)}.$$

$$(3.12) \quad = \phi(c+1, c+2; z) * h(z) + \overline{\phi(c+1, c+2; z) * g(z)}.$$

THEOREM 3.1. *If the function $f(z)$ defined by (1.4), with $b_1 = 0$, is in the class $HP^*(\alpha)$, then $J_c(f)$ defined by (3.10) is in the class $HP^*(\beta)$, where*

$$(3.13) \quad \beta = (2\alpha - 1) + 2(1 - \alpha)(c + 1) \sum_{n=1}^{\infty} \frac{(-1)^n}{c + n + 1}.$$

The result is sharp.

Proof. By using (3.12), we have

$$z[J_c(h) + J_c(g)]' = \phi(c+1, c+2; z) * (z(h'(z) + g'(z))).$$

A simple calculation shows that

$$[J_c(h) + J_c(g)]' = \frac{1}{z} [L(c+1, c+2)(zh'(z) + zg'(z))].$$

Using (3.7), we obtain that

$$(3.14) \quad Re\{J_c(h) + J_c(g)\}' = (c+1) \int_0^1 u^c Re\{h'(zu) + g'(zu)\} du.$$

Since $f(z) \in HP^*(\alpha)$, we put

$$h'(z) + g'(z) = G(z),$$

then $G(z) = 1 + p_1z + p_2z^2 + \dots$ is analytic in U and $Re\{G(z)\} > \alpha$. It is known that [31] if $q(z) = 1 + c_1z + c_2z^2 + \dots$ is analytic in U and

$Re\{q(z)\} > \gamma$, ($0 \leq \gamma < 1$), then

$$(3.15) \quad Re\{q(z)\} \geq \frac{1 + (2\gamma - 1)r}{1 + r}, \quad (|z| = r < 1).$$

Hence by using (3.14) and (3.15), we have

$$\begin{aligned} Re\{J_c(h) + J_c(g)\}' &\geq (c + 1) \int_0^1 u^c \frac{1 + (2\alpha - 1)u}{1 + u} du \\ &= (2\alpha - 1) + 2(1 - \alpha)(c + 1) \int_0^1 \frac{u^c}{1 + u} du \\ &= (2\alpha - 1) + 2(1 - \alpha)(c + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{c + n + 1}, \end{aligned}$$

that is $J_c(f) \in H P^*(\beta)$, where β is defined by (3.13). Further, to show that the result is sharp, we consider the function $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are connected by the relation

$$(3.16) \quad h'(z) + g'(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}. \quad \blacksquare$$

REMARK. It should be of interest to note that, analogously to these results, one may obtain results on generalized convolutions of n functions ($n \geq 2$).

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