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CONNECTIONS BETWEEN CERTAIN CLASSES OF HARMONIC UNIVALENT MAPPINGS INVOLVING GENERALIZED BESSEL FUNCTIONS

SAURABH PORWAL[1*] AND DILSHAD AHAMAD[2]

Abstract. In the present paper, we obtain some sufficient conditions of certain convolution operator involving generalized Bessel functions of first kind belonging to various subclasses of harmonic univalent functions. To be more precise, we investigate such connections with harmonic γ-uniformly convex and harmonic γ-uniformly starlike mappings in the plane.

1. Introduction and preliminaries

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$.

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply-connected domain D if both u and v are real harmonic in D. In any simply-connected domain we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|, z \in D$. See Clunie and Sheil-Small [7], for more basic results on harmonic functions one may refer to the following standard introductory text book by Duren [8].

Let H be the family of all harmonic functions of the form $f = h + \overline{g}$, where

$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n,\ g(z) = \sum_{n=1}^{\infty} B_n z^n,\ |B_1| < 1, \tag{1.2}$$

are in the class A. Let S_H denote the subclass of H consisting of functions $f = h + \overline{g}$ of the form (1.2) that are harmonic univalent and sense-preserving in the open unit disk U for which $f(0) = f_z(0) - 1 = 0$.

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* Corresponding author.
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Note that S_H reduces to class S of normalized analytic univalent functions if the co-analytic part of its member is zero.

The classes S_H^0 and S_H were first studied in [7]. Also, we let K_H^0 denote the subclasses of S_H^0 of harmonic functions which are convex in U. For definitions and properties of this class, one may refer to [7] or [8].

For $0 \leq \beta < 1$, let

$$N_H(\beta) = \left\{ f \in H : \Re\left(\frac{f'(z)}{z'}\right) \geq \beta,\ z = re^{i\theta} \in U \right\},$$

$$R_H(\beta) = \left\{ f \in H : \Re\left(\frac{f''(z)}{z''}\right) \geq \beta,\ z = re^{i\theta} \in U \right\},$$

where

$$z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right),\ z'' = \frac{\partial}{\partial\theta}(z'),\ f'(z) = \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right), f''(z) = \frac{\partial}{\partial\theta}\left(f'(z)\right).$$

Define

$$TN_H(\beta) \equiv N_H(\beta) \cap T \text{ and } TR_H(\beta) \equiv R_H(\beta) \cap T,$$

where T consists of the functions $f = h + \overline{g}$ in S_H so that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n,\ g(z) = \sum_{n=1}^{\infty} |B_n| z^n. \tag{1.3}$$

The classes $N_H(\beta)$ and $R_H(\beta)$ were initially introduced and studied, respectively, in [3] and [4].

Let $HUC(\gamma, \alpha)$ be a subclass of the functions $f = h + \overline{g}$ in H which satisfy the condition

$$\Re\left\{1 + (1 + \gamma e^{i\eta})\frac{z^2h''(z) + \overline{2zg'(z) + z^2g''(z)}}{zh'(z) - \overline{zg'(z)}}\right\} \geq \alpha,$$

for some γ $(0 \leq \gamma < \infty)$, α $(0 \leq \alpha < 1)$ and $z \in U$.

Define $THUC(\gamma, \alpha) \equiv HUC(\gamma, \alpha) \cap T$.

A mapping in $HUC(\gamma, \alpha)$ or $THUC(\gamma, \alpha)$ is called γ-uniformly harmonic convex in U. These classes were studied in by Kim *et al.* in [10]. For $g \equiv 0$, $\gamma = 1$ and $\alpha = 0$ the class $HUC(\gamma, \alpha)$ reduces to the class UCV of analytic uniformly convex functions studied by Goodman [9].

Analogues to $HUC(\gamma, \alpha)$ is the class $HUS^*(\gamma, \alpha)$ consisting of harmonic functions $f = h + \overline{g}$ in H which satisfy the condition

$$\Re\left\{\frac{zf'(z)}{z'f(z)} - \alpha\right\} \geq \gamma\left|\frac{zf'(z)}{z'f(z)} - 1\right|$$

for some γ $(0 \leq \gamma < \infty)$, $\alpha (0 \leq \alpha < 1)$ and $z \in U$. Also define $THUS^*(\gamma, \alpha) \equiv HUS^*(\gamma, \alpha) \cap T$. The mappings in $HUS^*(\gamma, \alpha)$ or $THUS^*(\gamma, \alpha)$ are called γ-harmonic uniformly starlike in U. For $\alpha = 0$, these classes were studied in [20]. For $g \equiv 0$, $\gamma = 1$ and $\alpha = 0$, $HUS^*(\gamma, \alpha)$ reduces to the family US^* of analytic uniformly starlike functions defined by Rønning [11].

Now, we recall that the generalized Bessel function of the first kind of order p is given by the power series

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}, \ z \in C. \tag{1.4}$$

where $b, p, c \in C$. The equation (1.4) is a generalization of Bessel, modified Bessel, spherical Bessel and modified spherical Bessel functions. It is worth mentioning that, in particular, when $b = c = 1$, we reobtain the Bessel function $\omega_{p,1,1} = J_p$, and for $c = -1, b = 1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{-p/2} \omega_{p,b,c}(z^{1/2}).$$

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a \neq 0, -1, -2, \ldots$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)\ldots(a+n-1), & \text{if } n = 1, 2, 3, \ldots, \end{cases},$$

we obtain for the function $u_{p,b,c}$ the following representation

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p + \frac{(b+1)}{2}\right)_n} \frac{z^n}{n!}, \tag{1.5}$$

where $p + (b+1)/2 \neq 0, -1, -2, \ldots$. This function is analytic on C and satisfies the second-order linear differential equation

$$4z^2 u''(z) + 2\left(2p + b + 1\right) z u'(z) + czu(z) = 0.$$

For convenience throughout in the sequel, we use the following notations:

$$u_{p,b,c} = u_p, \ k = p + \frac{b+1}{2}.$$

For complex parameters c_1, k_1, c_2, k_2 ($k_1, k_2 \neq 0, -1, -2, \ldots$), Porwal [12] define the functions $\phi_1(z) = zu_{p_1}(z)$ and $\phi_2(z) = zu_{p_2}(z)$.

Corresponding to these functions, they introduce the following convolution operator

$$\Omega \equiv \Omega \begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} : H \to H$$

defined by

$$\Omega \begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f = f * (\phi_1 + \overline{\phi_2}) = h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)}$$

for any function $f = h + \overline{g}$ in H.

Letting

$$\Omega \begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} A_n z^n,\ G(z) = \sum_{n=1}^{\infty} \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} B_n z^n. \quad (1.6)$$

In the present paper, we generalize the convolution operator $\Omega\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix}$ in to $\Omega^\lambda\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix}$ as follows

$$\Omega^\lambda\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f = h(z)*\Big(\lambda z\phi_1'(z) + (1-\lambda)\phi_1(z)\Big) + \overline{g(z) * \left(\lambda z\phi_2'(z) + (1-\lambda)\phi_2(z)\right)}$$

or equivalently

$$\Omega^\lambda\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f(z) = H(z) + \overline{G(z)},$$

where

$$H(z) = z + \sum_{n=2}^{\infty} \frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \left(\lambda n + (1-\lambda)\right) A_n z^n,\ G(z) = \sum_{n=1}^{\infty} \frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!} \left(\lambda n + (1-\lambda)\right) B_n z^n. \quad (1.7)$$

Throughout this paper, we will frequently use the notation

$$\Omega^\lambda(f) = \Omega^\lambda\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f.$$

The generalized Bessel function is a recent topic of study in Geometric Function Theory (e.g. see the work of [5], [12]-[17]). Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [1], [6], [18]-[19] [21]), we establish a number of connections between the classes $HUC(\gamma,\alpha)$, $HUS^*(\gamma,\alpha)$, $TN_H(\beta)$ and $TR_H(\beta)$ by applying the convolution operator Ω^λ.

2. Connections with harmonic uniformly convex mappings

In order to establish connections between harmonic convex mappings and harmonic γ-uniformly convex mappings, we need following results in Lemma 2.1 [7], Lemma 2.2 [10] and Lemma 2.4 [5].

Lemma 2.1. *If $f = h + \overline{g} \in K_H^0$, where h and g are given by (1.2) with $B_1 = 0$, then*

$$|A_n| \le \frac{n+1}{2},\ |B_n| \le \frac{n-1}{2}.$$

Lemma 2.2. *Let $f = h + \overline{g}$ be given by (1.2). If $0 \le \gamma < \infty$, $0 \le \alpha < 1$ and*

$$\sum_{n=2}^{\infty} n\left\{n(\gamma+1) - (\gamma+\alpha)\right\}|A_n| + \sum_{n=1}^{\infty} n\left\{n(\gamma+1) + (\gamma+\alpha)\right\}|B_n| \le 1-\alpha, \quad (2.1)$$

then f is harmonic, sense-preserving univalent functions in U and $f \in HUC(\gamma,\alpha)$.

Remark 2.3. In [10], it is also shown that $f = h + \overline{g}$ given by (1.3) is in the family $THUC(\gamma,\alpha)$, if and only if the coefficient condition (2.1) holds. Moreover, if $f \in THUC(\gamma,\alpha)$, then

$$|A_n| \leq \frac{1-\alpha}{n\left\{n(\gamma+1)-(\gamma+\alpha)\right\}},\ n \geq 2,$$

$$|B_n| \leq \frac{1-\alpha}{n\left\{n(\gamma+1)+(\gamma+\alpha)\right\}},\ n \geq 1.$$

Lemma 2.4. *If $b, p, c \in C$ and $k \neq 0, -1, -2, \ldots$ then the function u_p satisfies the recursive relation $4ku_p'(z) = -cu_{p+1}(z)$ for all $z \in C$.*

Theorem 2.5. *If $0 \leq \alpha < 1$, $0 \leq \gamma < \infty$, $0 \leq \lambda \leq 1$, $c_1, c_2 < 0$ and $k_1, k_2 > 0$ and the inequality*

$$\begin{aligned}
&\lambda(\gamma+1)\left\{u_{p_1}^{iv}(1)+(10\gamma+11-\alpha)u_{p_1}'''(1)+(24\gamma+31-7\alpha)u_{p_1}''(1)+(12\gamma+22-10\alpha)u_{p_1}'(1)\right.\\
&\quad\left.+2(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\\
&\quad+(1-\lambda)\left\{(\gamma+1)u_{p_1}'''(1)+(6\gamma+7-\alpha)u_{p_1}''(1)+(6\gamma+10-4\alpha)u_{p_1}'(1)+2(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\\
&\quad+\lambda\left\{(\gamma+1)u_{p_2}^{iv}(1)+(10\gamma+9+\alpha)u_{p_2}'''(1)+(24\gamma+19+5\alpha)u_{p_2}''(1)+(12\gamma+8+4\alpha)u_{p_2}'(1)\right\}\\
&\quad+(1-\lambda)\left\{(\gamma+1)u_{p_2}'''(1)+(6\gamma+5+\alpha)u_{p_2}''(1)+(6\gamma+4+2\alpha)u_{p_2}'(1)\right\}\Big]\\
&\quad\leq 2(1-\alpha),
\end{aligned}$$

is satisfied then $\Omega^{\lambda}\left(K_H^0\right) \subset HUC(\gamma,\alpha)$.

Proof. Let $f = h + \overline{g} \in K_H^0$ where h and g are of the form (1.2) with $B_1 = 0$. We need to show that $\Omega^{\lambda}(f) = H + \overline{G} \in HUC(\gamma,\alpha)$, where H and G defined by (1.7) with $B_1 = 0$ are analytic functions in U.

In view of Lemma 2.2, we need to prove that

$$P_1 \leq 1-\alpha,$$

where

$$
\begin{aligned}
P_1 =& \sum_{n=2}^{\infty} n\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\left|\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)A_n\right| \\
&+\sum_{n=2}^{\infty} n\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\left|\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)B_n\right| \\
&\leq \frac{1}{2}\left[\sum_{n=2}^{\infty} n(n+1)\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)\right. \\
&\left.+\sum_{n=2}^{\infty} n(n-1)\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)\right] \\
&= \frac{1}{2}\left[\lambda\sum_{n=2}^{\infty}\left\{(\gamma+1)(n-1)(n-2)(n-3)(n-4)+(10\gamma+11-\alpha)(n-1)(n-2)(n-3)\right.\right. \\
&\left.+(24\gamma+31-7\alpha)(n-1)(n-2)+(12\gamma+22-10\alpha)(n-1)+2(1-\alpha)\right\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \\
&+(1-\lambda)\sum_{n=2}^{\infty}\left\{(\gamma+1)(n-1)(n-2)(n-3)+(6\gamma+7-\alpha)(n-1)(n-2)\right. \\
&\left.+(6\gamma+10-4\alpha)(n-1)+2(1-\alpha)\right\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} \\
&+\lambda\sum_{n=2}^{\infty}\left\{(\gamma+1)(n-2)(n-3)(n-4)+(10\gamma+9+\alpha)(n-2)(n-3)\right. \\
&\left.+(24\gamma+19+5\alpha)(n-2)+(12\gamma+8+4\alpha)\right\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!} \\
&+(1-\lambda)\sum_{n=2}^{\infty}\left\{(\gamma+1)(n-2)(n-3)+(6\gamma+5+\alpha)(n-2)\right. \\
&\left.\left.+(6\gamma+4+2\alpha)\right\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-2)!}\right] \\
&\frac{1}{2}\left[\lambda\left\{(\gamma+1)u_{p_1}^{iv}(1)+(10\gamma+11-\alpha)u_{p_1}'''(1)+(24\gamma+31-7\alpha)u_{p_1}''(1)+(12\gamma+22-10\alpha)u_{p_1}'(1)\right.\right. \\
&\left.+2(1-\alpha)\left(u_{p_1}(1)-1\right)\right\} \\
&+(1-\lambda)\left\{(\gamma+1)u_{p_1}'''(1)+(6\gamma+7-\alpha)u_{p_1}''(1)+(6\gamma+10-4\alpha)u_{p_1}'(1)+2(1-\alpha)\left(u_{p_1}(1)-1\right)\right\} \\
&+\lambda\left\{(\gamma+1)u_{p_2}^{iv}(1)+(10\gamma+9+\alpha)u_{p_2}'''(1)+(24\gamma+19+5\alpha)u_{p_2}''(1)+(12\gamma+8+4\alpha)u_{p_2}'(1)\right\} \\
&\left.+(1-\lambda)\left\{(\gamma+1)u_{p_2}'''(1)+(6\gamma+5+\alpha)u_{p_2}''(1)+(6\gamma+4+2\alpha)u_{p_2}'(1)\right\}\right] \\
&\leq 1-\alpha,
\end{aligned}
$$

by given hypothesis.

This completes the proof of Theorem 2.5. □

If we put $\lambda = 0$, then we obtain the following result of Porwal [12].

Corollary 2.6. *If $0 \leq \alpha < 1$, $0 \leq \gamma < \infty$, $c_1, c_2 < 0$ and $k_1, k_2 > 0$ and the inequality*

$$
\begin{aligned}
(\gamma+1)&\left\{u_{p_1}'''(1)+7u_{p_1}''(1)+10u_{p_1}'(1)+2u_{p_1}(1)+u_{p_2}'''(1)+5u_{p_2}''(1)+10u_{p_2}'(1)\right\} \\
&-(\gamma+\alpha)\left\{u_{p_1}''(1)+4u_{p_1}'(1)+2u_{p_1}(1)-u_{p_2}''(1)-2u_{p_2}'(1)\right\} \leq 4(1-\alpha),
\end{aligned}
$$

is satisfied then $\Omega\left(K_H^0\right) \subset HUC(\gamma, \alpha)$.

In order to determine connection between $TN_H(\beta)$ and $HUC(\gamma,\alpha)$, we require the following result obtained in [3].

Lemma 2.7. *Let $f = h + \overline{g}$ where h and g as given by (1.3) and suppose that $0 \le \beta < 1$. Then*

$$f \in TN_H(\beta) \Leftrightarrow \sum_{n=2}^{\infty} n\,|A_n| + \sum_{n=1}^{\infty} n\,|B_n| \le 1-\beta.$$

Remark 2.8. If $f \in TN_H(\beta)$, then

$$|A_n| \le \frac{1-\beta}{n},\ n \ge 2,$$

$$|B_n| \le \frac{1-\beta}{n},\ n \ge 1.$$

Theorem 2.9. *If $0 \le \beta < 1$, $0 \le \alpha < 1$, $0 \le \gamma < \infty$, $0 \le \lambda \le 1$, $c_1, c_2 < 0$, $k_1, k_2 > 0$ and the inequality*

$$\begin{aligned}(1-\beta)&\left[\lambda\left\{(\gamma+1)u''_{p_1}(1)+(2\gamma+3-\alpha)u'_{p_1}(1)+(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\right.\\&+(1-\lambda)\left\{(\gamma+1)u'_{p_1}(1)+(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\\&+\lambda\left\{(\gamma+1)u''_{p_2}(1)+(4\gamma+\alpha+3)u'_{p_2}(1)+(2\gamma+\alpha+1)u_{p_2}(1)\right\}\\&\left.+(1-\lambda)\left\{(\gamma+1)u'_{p_2}(1)+(2\gamma+\alpha+1)u_{p_2}(1)\right\}\right]\\&\le 1-\alpha\end{aligned}$$

is satisfied, then

$$\Omega^{\lambda}(TN_H(\beta)) \subset HUC(\gamma,\alpha).$$

Proof. Let $f = h + \overline{g} \in TN_H(\beta)$ where h and g are given by (1.3). In view of Lemma 2.2, it is enough to show that $P_2 \le 1-\alpha$, where

$$\begin{aligned}P_2 =& \sum_{n=2}^{\infty} n\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\left|\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)A_n\right|\\&+\sum_{n=1}^{\infty} n\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\left|\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)B_n\right|.\end{aligned}$$

Using Remark 2.8, it follows that

$$\begin{aligned}
P_2 &\leq (1-\beta)\left[\sum_{n=2}^{\infty}\{n(\gamma+1)-(\gamma+\alpha)\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}(\lambda n+(1-\lambda))\right.\\
&\left.+\sum_{n=1}^{\infty}\{n(\gamma+1)+(\gamma+\alpha)\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}(\lambda n+(1-\lambda))\right]\\
&=(1-\beta)\left[\lambda\sum_{n=0}^{\infty}(n+2)\left[(n+2)(\gamma+1)-(\gamma+\alpha)\right]\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}\right.\\
&+(1-\lambda)\sum_{n=0}^{\infty}\left[(n+2)(\gamma+1)-(\gamma+\alpha)\right]\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}\\
&+\lambda\sum_{n=0}^{\infty}(n+1)\left[(n+1)(\gamma+1)+(\gamma+\alpha)\right]\frac{(-c_2/4)^{n}}{(k_2)_{n}n!}\\
&\left.+(1-\lambda)\sum_{n=0}^{\infty}\left[(n+1)(\gamma+1)-(\gamma+\alpha)\right]\frac{(-c_2/4)^{n}}{(k_2)_{n}n!}\right]\\
&=(1-\beta)\left[\lambda\left\{(\gamma+1)u''_{p_1}(1)+(2\gamma+3-\alpha)u'_{p_1}(1)+(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\right.\\
&+(1-\lambda)\left\{(\gamma+1)u'_{p_1}(1)+(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\\
&+\lambda\left\{(\gamma+1)u''_{p_2}(1)+(4\gamma+\alpha+3)u'_{p_2}(1)+(2\gamma+\alpha+1)u_{p_2}(1)\right\}\\
&\left.+(1-\lambda)\left\{(\gamma+1)u'_{p_2}(1)+(2\gamma+\alpha+1)u_{p_2}(1)\right\}\right]\\
&\leq 1-\alpha
\end{aligned}$$

by the given hypothesis, this completes the proof of Theorem 2.9. ☐

For the relationship between the classes $TR_H(\beta)$ and $HUC(\gamma,\alpha)$, we shall require the following lemma which is due to [4].

Lemma 2.10. *Let $f=h+\overline{g}$ where h and g are given by (1.3), and suppose that $0\leq\beta<1$. Then*

$$f\in TR_H(\beta)\Leftrightarrow\sum_{n=2}^{\infty}n^2|A_n|+\sum_{n=1}^{\infty}n^2|B_n|\leq 1-\beta.$$

Remark 2.11. If $f=h+\overline{g}\in TR_H(\beta)$ where h and g are given by (1.3), then

$$|A_n|\leq\frac{1-\beta}{n^2},\ n\geq 2$$

and

$$|B_n|\leq\frac{1-\beta}{n^2},\ n\geq 1.$$

Lemma 2.12. *([12]) If $c<0$ and $k>1$, then*

$$\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(k)_n(n+1)!}=\frac{-4(k-1)}{c}\left[u_{p-1}(1)-1\right].$$

Theorem 2.13. *If* $c_1, c_2 < 0,\ k_1, k_2 > 1,\ 0 \leq \alpha < 1,\ 0 \leq \lambda \leq 1,\ 0 \leq \beta < 1, \gamma \geq 0$ *and the inequality*

$$\begin{aligned}
(1-\beta)\,&\Big[\lambda\left\{(\gamma+1)u'_{p_1}(1)+(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\\
&+(1-\lambda)\left\{(\gamma+1)\left(u_{p_1}(1)-1\right)+(\gamma+\alpha)\left(\frac{4(k_1-1)}{c_1}\left(u_{p_1-1}(1)-1\right)+1\right)\right\}\\
&+\lambda\left\{(\gamma+1)u'_{p_2}(1)+(2\gamma+\alpha+1)u_{p_2}(1)\right\}\\
&+(1-\lambda)\left\{(\gamma+1)u_{p_2}(1)++(\gamma+\alpha)\left(\frac{-4(k_2-1)}{c_2}\left(u_{p_2-1}(1)-1\right)\right)\right\}\Big]\\
&\leq 1-\alpha
\end{aligned}$$

is satisfied then

$$\Omega^{\lambda}\left(TR_H(\beta)\right) \subset HUC(\gamma,\alpha).$$

Proof. Making use of Lemma 2.2 and the definition of P_2 in Theorem 2.9, we need only to prove that $P_2 \leq 1-\alpha$. Using Remark 2.11 and Lemma 2.12, it follows that

$$
\begin{aligned}
P_2 &= \sum_{n=2}^{\infty} n\left\{n(\gamma+1)-(\gamma+\alpha)\right\}\left|\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)A_n\right| \\
&+\sum_{n=1}^{\infty} n\left\{n(\gamma+1)+(\gamma+\alpha)\right\}\left|\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)B_n\right| \\
&\leq (1-\beta)\left[\sum_{n=2}^{\infty}\frac{\{n(\gamma+1)-(\gamma+\alpha)\}}{n}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)\right. \\
&\left.+\sum_{n=1}^{\infty}\frac{\{n(\gamma+1)+(\gamma+\alpha)\}}{n}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\left(\lambda n+(1-\lambda)\right)\right] \\
&= (1-\beta)\left[\lambda\left(\sum_{n=0}^{\infty}\{(n+2)(\gamma+1)-(\gamma+\alpha)\}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}\right)\right. \\
&+(1-\lambda)\left(\sum_{n=0}^{\infty}\left\{(\gamma+1)-\frac{(\gamma+\alpha)}{n+2}\right\}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}\right) \\
&+\lambda\left(\sum_{n=0}^{\infty}\{(n+1)(\gamma+1)+(\gamma+\alpha)\}\frac{(-c_2/4)^{n}}{(k_2)_{n}(n)!}\right) \\
&\left.+(1-\lambda)\left(\sum_{n=0}^{\infty}\left\{(\gamma+1)+\frac{(\gamma+\alpha)}{n+1}\right\}\frac{(-c_2/4)^{n}}{(k_2)_{n}(n)!}\right)\right] \\
&= (1-\beta)\left[\lambda\left\{(\gamma+1)u'_{p_1}(1)+(1-\alpha)\left(u_{p_1}(1)-1\right)\right\}\right. \\
&+(1-\lambda)\left\{(\gamma+1)\left(u_{p_1}(1)-1\right)+(\gamma+\alpha)\left(\frac{4(k_1-1)}{c_1}\left(u_{p_1-1}(1)-1\right)+1\right)\right\} \\
&+\lambda\left\{(\gamma+1)u'_{p_2}(1)+(2\gamma+\alpha+1)u_{p_2}(1)\right\} \\
&\left.+(1-\lambda)\left\{(\gamma+1)u_{p_2}(1)++(\gamma+\alpha)\left(\frac{-4(k_2-1)}{c_2}\left(u_{p_2-1}(1)-1\right)\right)\right\}\right] \\
&\leq 1-\alpha,
\end{aligned}
$$

by the given hypothesis. □

Theorem 2.14. *If* $c_1, c_2 < 0,\ k_1, k_2 > 0,\ 0 \leq \alpha < 1,\ 0 \leq \lambda \leq 1,\ \gamma \geq 0$ *and the inequality*

$$\lambda\left(u'_{p_1}(1)+u'_{p_2}(1)\right)+u_{p_1}(1)+u_{p_2}(1)\leq 2 \tag{2.2}$$

is satisfied, then $\Omega^{\lambda}\left(THUC(\gamma,\alpha)\right)\subset HUC(\gamma,\alpha)$.

Proof. By adopting the technique of the proof of Theorem 2.9, Lemma 2.2 and Remark 2.3, we obtain

$$\begin{aligned}
P_2 &\leq (1-\alpha)\left[\sum_{n=2}^{\infty}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}(\lambda n+(1-\lambda))+\sum_{n=1}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}(\lambda n+(1-\lambda))\right]\\
&= (1-\alpha)\left[\lambda\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}(n+2)}{(k_1)_{n+1}(n+1)!}+(1-\lambda)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}\right.\\
&\left.+\lambda\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n}(n+1)}{(k_2)_{n}n!}+(1-\lambda)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n}}{(k_2)_{n}n!}\right]\\
&= (1-\alpha)\left[\lambda u'_{p_1}(1)+u_{p_1}(1)-1+\lambda u'_{p_2}(1)+u_{p_2}(1)\right]\\
&\leq 1-\alpha,
\end{aligned}$$

by the given condition and this completes the proof of the theorem. □

Theorem 2.15. *If $c_1, c_2 < 0$, $k_1, k_2 > 0$, $0 \leq \alpha < 1$, $\gamma \geq 0$ and the inequality (2.2) is satisfied, then $\Omega^{\lambda}(THUC(\gamma,\alpha)) \subset THUC(\gamma,\alpha)$.*

Proof. The proof of this theorem is much akin to that of Theorem 2.14. Therefore we omits the details involved. □

3. Connections with harmonic uniformly starlike mappings

In this section we obtain the analogous results involving between various classes of planar harmonic mappings and $HUS^*(\gamma,\alpha)$ by applying the convolution operator Ω^{λ}.

Lemma 3.1. *Let $f = h + \overline{g} \in H$ be given by (1.2). If $0 \leq \gamma < \infty$, $0 \leq \alpha < 1$ and*

$$\sum_{n=2}^{\infty}\{n(\gamma+1)-(\gamma+\alpha)\}|A_n|+\sum_{n=1}^{\infty}\{n(\gamma+1)+(\gamma+\alpha)\}|B_n| \leq 1-\alpha, \quad (3.1)$$

then f is harmonic, sense-preserving univalent functions in U and $f \in HUS^*(\gamma,\alpha)$.

Remark 3.2. The result in Lemma 3.1 is a special case of the corresponding result proved in [2]. However, for $\gamma = 1$, Lemma 3.1 reduces to the result found in [20].

Remark 3.3. In [2], it is also shown that $f = h + \overline{g}$ given by (1.3) is in the family $THUS^*(\gamma,\alpha)$, if and only if the coefficient condition (3.1) holds. Moreover, if $f \in THUS^*(\gamma,\alpha)$, then

$$|A_n| \leq \frac{1-\alpha}{\{n(\gamma+1)-(\gamma+\alpha)\}}, \ n \geq 2,$$

$$|B_n| \leq \frac{1-\alpha}{\{n(\gamma+1)+(\gamma+\alpha)\}}, \ n \geq 1.$$

Applying the Lemma 3.1 and using the techniques of the proof of Theorem 2.5 so we only state the results of following theorems.

Theorem 3.4. *If* $0 \leq \alpha < 1$, $0 \leq \gamma < \infty$, $0 \leq \lambda \leq 1$, $c_1, c_2 < 0$ *and* $k_1, k_2 > 0, (k_1, k_2 \neq 0, -1, -2, \ldots)$ *and the inequality*

$$\begin{aligned}
&\lambda \left\{ (\gamma+1)u'''_{p_1}(1) + (6\gamma+7-\alpha)u''_{p_1}(1) + (6\gamma+10-4\alpha)u'_{p_1}(1) + 2(1-\alpha)\left(u_{p_1}(1)-1\right) \right\} \\
&\quad + (1-\lambda)\left\{ (\gamma+1)u''_{p_1}(1) + (3\gamma+4-\alpha)u'_{p_1}(1) + 2(1-\alpha)\left(u_{p_1}(1)-1\right) \right\} \\
&\quad + \lambda \left\{ ((\gamma+1)u'''_{p_2}(1) + (6\gamma+5+\alpha)u''_{p_2}(1) + (6\gamma+4+2\alpha)u'_{p_2}(1) \right\} \\
&\quad + (1-\lambda)\left\{ (\gamma+1)u''_{p_2}(1) + (3\gamma+2+\alpha)u'_{p_2}(1) \right\}] \\
&\quad \leq 2(1-\alpha),
\end{aligned}$$

is satisfied then $\Omega^\lambda \left(K_H^0\right) \subset HUS^*(\gamma, \alpha)$.

Theorem 3.5. *If all the restrictions and coefficient condition in Theorem 2.13 are satisfied then* $\Omega^\lambda \left(TN_H(\beta)\right) \subset HUS^*(\gamma, \alpha)$.

Theorem 3.6. *If all the restrictions and coefficient condition in Theorem 2.14 are satisfied then* $\Omega^\lambda \left(THUS^*(\gamma, \alpha)\right) \subset HUS^*(\gamma, \alpha)$.

Remark 3.7. If we put $\lambda = 0$ in Theorems 2.9-3.6 then we obtain the corresponding results of Porwal [12].

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[1] Department of Mathematics, U.I.E.T., C.S.J.M. University, Kanpur-208024, (U.P.), India

E-mail address: saurabhjcb@rediffmail.com

[2] Department of Mathematics, U.I.E.T., C.S.J.M. University, Kanpur-208024, (U.P.), India

E-mail address: dlshdhmd4@gmail.com